# Directed polymer in a heavy-tail random environment

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CIRM, September 11, 2019

Université Paris Nanterre

- Q. Berger and N. Torri, *Directed polymers in heavy-tail random environment*, Ann. Probab., 2019
- Q. Berger and N. Torri, *Entropy-controlled Last-Passage Percolation*, Ann. Appl. Probab., 2019

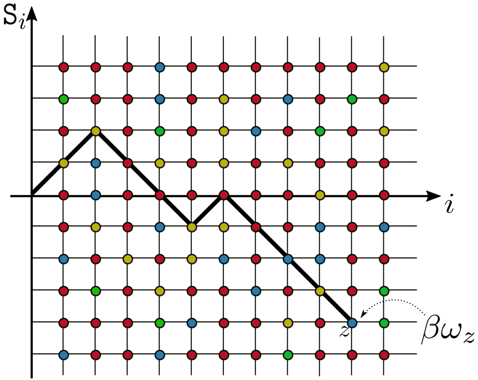
1. **Directed polymer model**

2. **Results (old and news)**
   - Finite Exponential moments
   - Heavy-tail

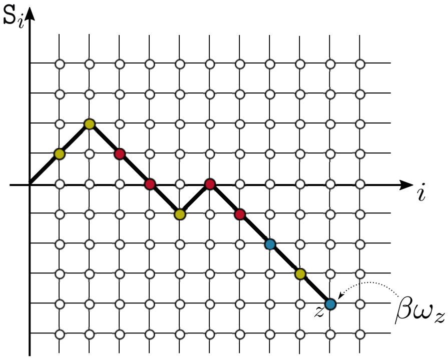
3. **Sketch of the proof**

# Directed polymer model

- $(S = (S_i)_{i \leq n}, \mathbf{P})$ simple symmetric random walk on $\mathbb{Z}^d$, $d \geq 1$
- $(\omega = (\omega_z)_{z \in \mathbb{N} \times \mathbb{Z}}, \mathbb{P})$ Environment



# Directed polymer model




**Energy of a trajectory**

$$H_n^{\beta,\omega}(s) := \sum_{i=1}^{n} \beta\omega_{i,s_i}$$

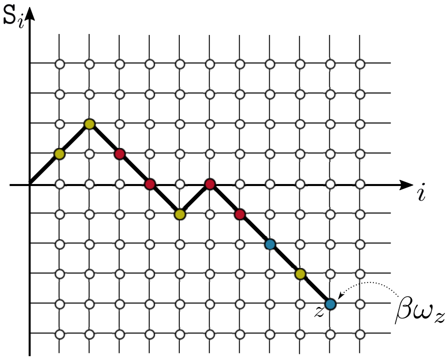
**Directed polymer**

$$\mathbf{P}_{n,\beta}^{\omega}(s) := \frac{1}{\mathbf{Z}_{n,\beta}^{\omega}} \exp\left(\sum_{i=1}^{n} \beta\omega_{i,s_i}\right)$$

- $\beta$ fixed
- $\beta = \beta_n \xrightarrow{n\to\infty} 0$

weak coupling limit

# Directed polymer model

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# Main question

Knowing the structure of $S$ under $\mathbf{P}_{n,\beta}^{\omega}$ when $n$ gets large

1. Localization phenomena
2. Fluctuations
3. Phase transition: $\beta_c$

Answer : it depends on $\mathbb{P}$, the law of the environment $\omega$

# Finite Exponential Moments

*Reference: Comets, Saint-Flour lecture notes – 2016.*

Environment with Finite Exponential Moments :

$$\lambda(\beta) := \log \mathbb{E}\left[e^{\beta \omega_{1,1}}\right] < \infty, \quad \forall \beta \geq 0 \text{ (and } \omega \text{ i.i.d.)}$$

Then $\exists\, \beta_c, \bar{\beta}_c \geq 0$ :

- $\beta < \beta_c$ weak disorder : trajectories are diffusive
- $\beta > \bar{\beta}_c$ strong disorder : localization phenomena, super-diffusivity

**Remark**

$\beta_c$ and $\bar{\beta}_c$ are "deterministic" : they depend on $\mathbb{P}$.

- $d = 1, 2$ : $\beta_c = \bar{\beta}_c = 0$.
- $d \geq 3$ open problem: $+\infty > \bar{\beta}_c \geq \beta_c > 0$.

In the rest of the talk $d = 1$

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# Localization

For any $\beta > 0$ there exists a corridor where the polymer path likes to be.

**Theorem (Comets)**

*Let $\omega \sim \mathcal{N}(0,1)$, then for any $\beta > 0$ there exists $(x_n^{\omega,\beta})_n$ such that*

$$\liminf_{n\to\infty} \mathbb{E}\mathbf{E}^{\omega}_{n,\beta}\left[\frac{1}{n}\sum_{k=1}^{n} 1_{\{S_k = x_n^{\omega,\beta}\}}\right] > 0$$

# Fluctuations

Find $\xi$ and $\chi$ such that

$$\mathbb{E}\mathbf{E}^{\omega}_{n,\beta}\Big[S_n\Big] \approx n^{\xi}\,, \qquad\qquad \frac{\log \mathbf{Z}^{\omega}_{n,\beta} - nf(\beta)}{n^{\chi}} \xrightarrow[n\to\infty]{(d)} \mathbf{X}\,.$$

## Conjecture

$\xi = 2/3$, $\chi = 1/3$ for all $\beta > 0$.

Complementary approach : Weak Coupling $\beta_n = \beta n^{-\gamma}$, $\quad \gamma > 0$.

## Question

$\xi = \xi(\gamma)$ and $\chi = \chi(\gamma)$?.

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# Weak Coupling Limit

$\beta_n = \beta n^{-\gamma}, \quad \gamma > 0.$

**Theorem (Albert, Kanin and Quastel, 2014)**

*For $\gamma = 1/4$ we have that*

- $\xi = 1/2$ *(Brownian fluctuations)*
- *Scaling limit*

$$\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\lambda(\beta) \xrightarrow[n\to\infty]{(d)} \log \mathcal{Z}_{\sqrt{2}\beta}$$

**Conjecture**

The same result holds true if the environment has at least 5 finite moments.

Random environment with Heavy-tail $\mathbb{P}(\omega_{1,1} \geq t) \sim t^{-\alpha}$, $\alpha > 0$

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# Weak Coupling Limit, Heavy-Tail

$(\omega, \mathbb{P})$ i.i.d. and $\mathbb{P}(\omega_{1,1} \geq t) \sim t^{-\alpha}$, $\alpha > 0$, $\beta_n = \beta n^{-\gamma}$, $\gamma \geq 0$

## Theorem (Dey and Zygouras, 2016)

*For $\gamma = \frac{1}{4}$ and $\alpha \geq 6$ or $\gamma = \frac{3}{2\alpha}$ and $\alpha \in (\frac{1}{2}, 6)$ we have that*

- $\xi = 1/2$ *(Brownian fluctuations)*
- *Scaling limit*
  - $\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\bar{\lambda}(\beta_n) \xrightarrow[n\to\infty]{(d)} \log \mathcal{Z}_{\sqrt{2}\beta}\,, \alpha \geq 6\,.$
  - $\frac{1}{n^{\chi(\gamma,\alpha)}}\left(\log \mathbf{Z}^{\omega}_{n,\beta_n} - nc_{\beta_n}\right) \xrightarrow[n\to\infty]{(d)} \begin{cases} \mathcal{N}(0, \sigma^2_{\beta})\,, & \alpha \in (2,6)\,, \\ \mathcal{W}^{(\alpha)}_{\beta}\,, & \alpha \in (\frac{1}{2}, 2)\,. \end{cases}$

$$\mathcal{W}^{(\alpha)}_{\beta} = \mathcal{W}^{(\alpha)}_{\beta}(\mathrm{PPP}(\mu_\alpha))$$

# Weak Coupling Limit, Heavy-Tail / 2

$(\omega, \mathbb{P})$ i.i.d. and $\mathbb{P}(\omega_{1,1} \geq t) \sim t^{-\alpha}$, $\alpha > 0$, $\beta_n = \beta n^{-\gamma}$, $\gamma \geq 0$

**Theorem (Auffinger and Louidor, 2011)**

*For* $\gamma = \frac{2}{\alpha} - 1$ *and* $\alpha \in (0, 2)$ *we have that*

- $\xi = 1$
- *Scaling limit*

$$\frac{1}{n} \log \mathbf{Z}^{\omega}_{n,\beta_n} \xrightarrow[n\to\infty]{(d)} \begin{cases} \tilde{\mathcal{T}}\,, & \beta > \beta^{\omega}_c\,, \\ 0\,, & \beta \leq \beta^{\omega}_c\,. \end{cases}$$

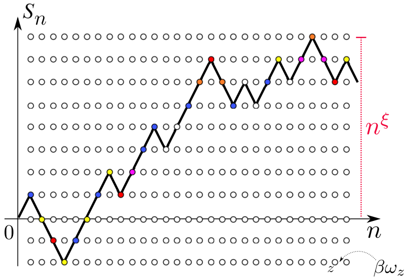
**Theorem ((Auffinger and Louidor, 2011 ; T. 2016)**

$\beta^{\omega}_c > 0$ *a.s. iff* $\alpha \in (0, \frac{1}{2})$

# Heavy-tail: Heuristics

$(\omega, \mathbb{P})$ i.i.d. and $\mathbb{P}(\omega_{1,1} \geq t) \sim t^{-\alpha}$, $\alpha > 0$, $\beta_n = \beta n^{-\gamma}$, $\gamma \geq 0$

How to get the transversal fluctuations exponent $\xi$?




**Flory Argument** :

Energetic gain = Entropy cost

Energy gain :

$\sum_{i=1}^n \beta_n \omega_{i,s_i} \approx \beta_n \max_{z \in \Box} \omega_z \approx \beta_n n^{\frac{1+\xi}{\alpha}}$

Entropy cost :

$\log \mathbf{P}(S_n = n^\xi) \approx n^{2\xi-1}$

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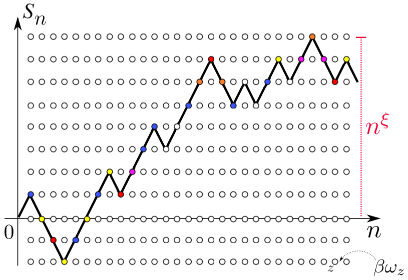


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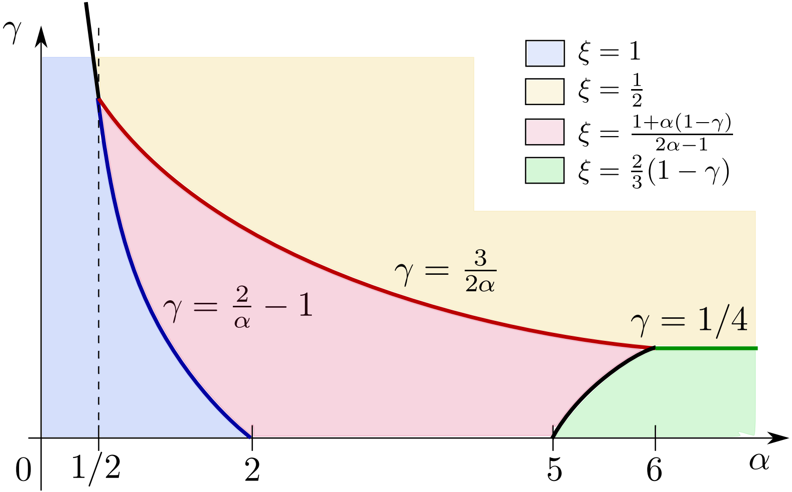
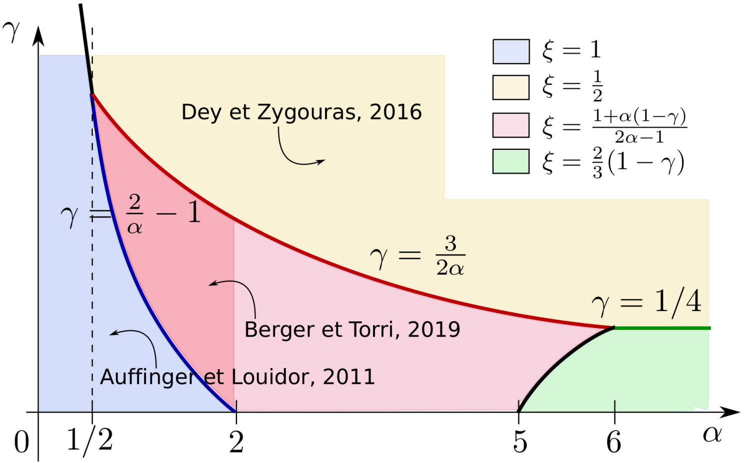
# Phase Diagram

$\mathbb{P}(\omega_{1,1} \geq t) \sim t^{-\alpha}$, $\beta_n = \beta n^{-\gamma}$



# Phase Diagram, results

$\mathbb{P}(\omega_{1,1} \geq t) \sim t^{-\alpha}$, $\beta_n = \beta n^{-\gamma}$



# Results

$\mathbb{P}(\omega_{1,1} \geq t) \sim t^{-\alpha}$, $\beta_n = \beta n^{-\gamma}$

**Theorem (Berger, T. - Ann. Probab., 2019 )**

*If* $\alpha \in (\frac{1}{2}, 2)$ *and* $\gamma \in (\frac{2}{\alpha} - 1, \frac{3}{2\alpha})$, *then*

$$\xi = \frac{1 + \alpha(1-\gamma)}{2\alpha - 1} \in (\frac{1}{2}, 1)\,.$$

*More precisely*

$$\mathbb{P}\Big( \mathbf{P}^{\omega}_{n,\beta_n}\Big( \max_{1 \geq i \geq n} |S_n| \asymp n^{\xi} \Big) \geq 1 - \varepsilon \Big) \geq 1 - \varepsilon\,.$$

*Scaling limit*

$$\lim_{n\to\infty} \frac{1}{n^{2\xi-1}} \Big( \log \mathbf{Z}^{\omega}_{n,\,\beta_n} - n\mathbb{E}\big[\omega 1_{\{\alpha>1\}}\big]\Big) = \mathcal{T} \qquad \textit{variational problem}$$

# Results

$\mathbb{P}(\omega_{1,1} \geq t) \sim t^{-\alpha}$, $\beta_n = \beta n^{-\gamma}$

**Theorem (Berger, T. - Ann. Probab., 2019 )**

*If* $\alpha \in (0, \frac{1}{2})$ *and* $\gamma = \frac{2}{\alpha} - 1$, *then*

$$\xi = \begin{cases} 1\,, & \beta > \beta_c^{\omega} \\ \dfrac{1}{2}\,, & \beta \leq \beta_c^{\omega} \end{cases}$$

*(and scaling limit for the partition function).*

# Sketch of the proof

- From the partition function to the variational problem
- The variational problem $\mathcal{T}$