The two-dimensional KPZ equation in the entire subcritical regime

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Collaborators

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The Kardar-Parisi-Zhang (KPZ) Equation:

$$\frac{\partial h}{\partial t}(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}|\nabla h|^2 + \beta\xi(t,x),$$

where ξ is space-time white noise on $[0,\infty) \times \mathbb{R}^d$.

Remark.

1. Models the height of a randomly growing d-dimensional surface.
2. Ill-defined due to $|\nabla h|^2$ (product of distributions).
3. Vast literature in $d = 1$ for models in the KPZ universality class (Integrable Probability).

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Defining the Solution of KPZ for $d = 1$

Step 1. Replace $\xi(t, x)$ by its mollification

$$\xi^\epsilon(t, x) := \int j^\epsilon(x - y)\xi(t, y)\mathrm{d}y,$$

where $\epsilon > 0$, $j \in C_c^\infty(\mathbb{R}^d)$ is a probability density, $j^\epsilon(x) = \epsilon^{-d} j(\frac{x}{\epsilon})$.

Step 2. Take the solution of the mollified equation

$$\frac{\partial h^\epsilon}{\partial t}(t, x) = \frac{1}{2}\Delta h^\epsilon(t, x) + \frac{1}{2}|\nabla h^\epsilon|^2 + \beta\xi^\epsilon(t, x),$$

check if $\exists\, A(\epsilon, t), B(\epsilon, t)$ s.t. $A(\epsilon, t)(h^\epsilon - B(\epsilon, t))$ converges as $\epsilon \downarrow 0$.

In dimension 1, $h^\epsilon(t, x) = \log u^\epsilon(t, x) + C_{\epsilon, j} t$ (Cole-Hopf), where

$$\frac{\partial u^\epsilon}{\partial t}(t, x) = \frac{1}{2}\Delta u^\epsilon(t, x) + \beta u^\epsilon(t, x)\xi^\epsilon(t, x)$$

solves the mutiplicative Stochastic Heat Equation (SHE), and $u^\epsilon \Rightarrow u$.

General solution theory for sub-critical singular SPDEs by Hairer (regularity structure), Imkeller-Gubinelli-Perkowski (paracontrolled distribution. Not applicable for $d = 2$ (critical).

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KPZ Universality Class ($d = 1$)

Solution theory of singular SPDEs identifies regularity properties of the solution h on small scales.

On large scales, h exhibits universal fluctuations:

$$\frac{h(t, t^{\frac{2}{3}}x) - \mathbb{E}[h(t, t^{\frac{2}{3}}x)]}{t^{\frac{1}{3}}} \Rightarrow F_{TW} \quad \text{as } t \to \infty,$$

where F_{TW} (depending on $h(0, \cdot)$) denotes one of the Tracy-Widom distributions from random matrix theory.

Models in KPZ universality class: KPZ equation, length of the longest increasing subsequence in a random permutation, first/last passage percolation, polynuclear growth, one-dimensional directed polymer ...

Major Challenge: Prove Tracy-Widom fluctuations for non-integrable models.

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KPZ in Dimension 2 (Existence of a Transition)

Dimension 2 is critical for KPZ (marginal in the language of disordered systems, super-renormalizable in the language of renormalization group theory). No solution theory.

For the solution h^ϵ of the mollified equation

$$\frac{\partial h^\epsilon}{\partial t}(t,x) = \frac{1}{2}\Delta h^\epsilon(t,x) + \frac{1}{2}|\nabla h^\epsilon|^2 + \beta_\epsilon \xi^\epsilon(t,x)$$

to converge, we need to choose an intermediate disorder scale

$$\beta_\epsilon = \hat{\beta}\sqrt{\frac{2\pi}{\log \epsilon^{-1}}}, \qquad \hat{\beta} > 0.$$

Theorem [Caravenna-S-Zygouras'17] Assume $h^\epsilon(0,\cdot) = 0$. There is a critical value $\hat{\beta}_c = 1$, s.t. for $\hat{\beta} \in (0, \hat{\beta}_c)$,

$$\tilde{h}^\epsilon(t,x) := h^\epsilon(t,x) - C_{\epsilon,j} t \Rightarrow N\Big(-\frac{1}{2}\sigma^2_{\hat{\beta}}, \sigma^2_{\hat{\beta}}\Big) \text{ with } \sigma^2_{\hat{\beta}} = \log\frac{1}{1-\hat{\beta}^2},$$

limit is indep. for different $x \in \mathbb{R}^2$. For $\hat{\beta} \geq \hat{\beta}_c$, $\tilde{h}^\epsilon(t,x) \Rightarrow -\infty$.

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Solution of 2-dim KPZ as a Random Distribution

[CSZ'17] identified the transition in the fdd of the 2-dim KPZ. Can also regard the solution as a random field acting on test functions.

[Chatterjee-Dunlap'18] For $\hat{\beta} > 0$ sufficiently small, as $\epsilon \downarrow 0$,

$$\sqrt{\log \epsilon^{-1}}\big(h^\epsilon(t,\cdot) - \mathbb{E}[h^\epsilon(t,\cdot)]\big)$$

is a tight sequence of random variables taking values in a negative Hölder space.

Our subsequent work identifies the limit for all $\hat{\beta} \in (0, \hat{\beta}_c)$.

Theorem [CSZ'18a] For each $\hat{\beta} \in (0, \hat{\beta}_c)$,

$$\sqrt{\log \epsilon^{-1}}\big(h^\epsilon(t,\cdot) - \mathbb{E}[h^\epsilon(t,\cdot)]\big) \Longrightarrow \sqrt{2\pi}\hat{\beta} v(\cdot,\cdot),$$

where $v(t,x)$ is a Gaussian generalized field solving the additive SHE

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v + \sqrt{\frac{1}{1-\hat{\beta}^2}}\,\xi(t,x), \quad v(0,\cdot) = 0,$$

also known as Edwards-Wilkinson equation. (Independently, [Gu'18] proved the same result for $\hat{\beta} > 0$ sufficiently small).

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Proof Strategy

Cole-Hopf transformation $\tilde{h}^\epsilon = h^\epsilon - C_{\epsilon,j} t = \log u^\epsilon$ relates to SHE:

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the solution of the same additive SHE.

Proof based on chaos expansions and fourth-moment theorem for convergence to normal.

Naive Strategy: Taylor expand $\tilde{h}^\epsilon = \log(1 + u^\epsilon - 1) \approx u^\epsilon - 1$?

Fails because $u^\epsilon(t,x) - 1 \not\approx 0$!

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