Gaussian random permutation and the boson point process

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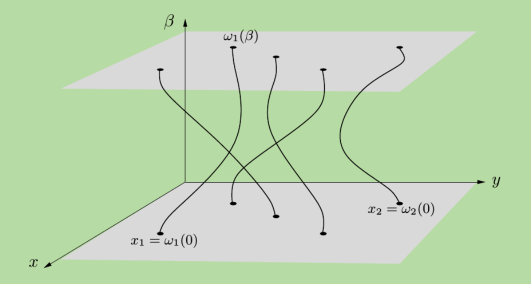


Picture by Daniel Ueltschi

C.I.R.M, September 2019

arXiv:1906.11120

Free Bose gas and spatial random permutations

Feynman 1953:

Partition function of the free Bose Gas:

$$Z_{\Lambda,N} = \frac{1}{N!} \sum_{\sigma \in S_N} \int_{\Lambda^N} e^{-\alpha \sum_i \|x_i - x_{\sigma(i)}\|^2} \, dx_1 \dots dx_N, \tag{1}$$

$\alpha > 0$ temperature.

$S_N :=$ set of permutations of $\{1, \dots, N\}$.

$\Lambda :=$ bounded subset of $\mathbb{R}^d$.

see also Ginibre 1970.

We start from (1).

Finite-volume spatial random permutation:

Configuration space:

$$(\underline{x}, \sigma) \in \Lambda^N \times S_N$$

$S_N :=$ set of permutations of $\{1, \ldots, N\}$.

$\underline{x} = (x_1, \ldots, x_N)$.

Canonical measure $G_{\Lambda,N}$:

$$G_{\Lambda,N} g := \frac{1}{Z_{\Lambda,N}} \frac{1}{N!} \sum_{\sigma \in S_N} \int_{\Lambda^N} g(\underline{x}, \sigma)\, e^{-\alpha \sum_{i=1}^N \|\sigma(x_i) - x_i\|^2} dx_1 \ldots dx_N.$$

$Z_{\Lambda,N} =$ normalization (partition function).

g test function

Incomplete Background

Deterministic points

Gandolfo, Ruiz, Ueltschi 2007 Points $= \mathbb{Z}^d$

Biskup Ritchammer 2015 Points $= \mathbb{Z}$ or PP in $d = 1$ quenched.

Armendáriz, F., Groisman, Leonardi 2015 Subcritical $\mathbb{Z}^d$

Point marginal: Boson point process

Macchi 1975

Fichtner 1991

Suto 1993 2002 Loop independence.

Shirai Takahashi 2003 Subcritical. Permanental point process.

Tamura Ito 2006 2007 Supercritical: superposition of 2 independent point processes

Eisenbaum 2008 Cox process, infinite divisibility

Permutation marginal:

Ginivre 1971 cycle distribution

Benfatto Cassandro Merola Presutti 2005 macro cycle convergence

Betz Ueltschi 2006 2009 2010 2011 macro cycle Poisson-Dirichtlet

Elboim-Peled 2017 more Poisson-Dirichtlet

Infinite-volume spatial random permutations

Goal today: construct

$$(\chi, \sigma) \text{ with law denoted } \mu_\rho$$

χ: point process (random locally finite subset of $\mathbb{R}^d$)

$\sigma : \chi \to \chi$ permutation (a bijection)

μ_ρ is translation-invariant, has *point density* ρ and

$$\mu_\rho \text{ is Gibbs for (the specifications induced by) } G_{\Lambda,N}$$

Construction gives new proofs of previous results on point marginals.

Unrooted loops

$\gamma = [x_1, \ldots, x_k]$ meaning $[x_2, \ldots, x_k, x_1] = [x_1, \ldots, x_k]$.

Support of loop $\{\gamma\} := \{x_1, \ldots, x_k\}$ and $\gamma(x_i) = x_{i+1}$.

Bijection: random permutation $\leftrightarrow$ loop configuration

$(\chi, \sigma) \mapsto \Gamma$, the set of loops given by

$\gamma \in \Gamma$ if $\gamma = [x_1, \ldots, x_k]$, $\{\gamma\} \subset \chi$ and $\gamma(x) = \sigma(x)$ for all $x \in \{\gamma\}$.

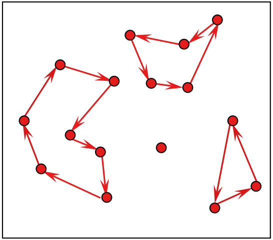
$\Gamma \mapsto (\chi, \sigma)$, the spatial permutation defined by

$\chi = \cup_{\gamma \in \Gamma} \{\gamma\}$; $\sigma(x) = \gamma(x)$ for $x \in \{\gamma\}$, $\gamma \in \Gamma$.

Loop density factorization: for finite χ:

$$e^{-\alpha \sum_{x\in\chi} \|\sigma(x)-x\|^2} = \prod_{\gamma\in\sigma} e^{-\alpha \sum_{x\in\{\gamma\}} \|\gamma(x)-x\|^2}$$

Sütő 1993.



Loops induced by a spatial random permutation in a box.
Arrow from x to y means $y = \sigma(x)$.
Isolated dots correspond to points $x = \sigma(x)$, loops of length 1.

Gaussian loop soup in $\mathbb{R}^d$

Unrooted loops: $D := \Big(\cup_{k\geq 1}(\mathbb{R}^d)^k\Big)/_\sim$,
where $(x_1,\ldots,x_k) \sim (x_2,\ldots,x_k,x_1)$.

Fix $\lambda \in (0,1]$ and define loop soup intensity measure on D:

$$Q_\lambda^{\mathrm{ls}}(d[x_1,\ldots,x_k]) := \tfrac{\lambda^k}{k}\big(\tfrac{\alpha}{\pi}\big)^{kd/2} e^{-\alpha\sum_{i=1}^k \|x_i-\gamma(x_i)\|^2}\, dx_1\ldots dx_k,$$

Gaussian loop soup:

$\Gamma_\lambda^{\mathrm{ls}} :=$ Poisson process on D with intensity Q_λ^{ls},
$\mu_\lambda^{\mathrm{ls}} :=$ Law of $\Gamma_\lambda^{\mathrm{ls}}$.

Analogous to Brownian loop soup. Lawler and Werner 2004, Lawler and Trujillo Ferreras 2007, Le Jan 2017.

Loop soup point density

$$\rho(\lambda) = \left(\frac{\alpha}{\pi}\right)^{d/2} \sum_{k\geq 1} \frac{\lambda^k}{k^{d/2}}$$

Critical density: $\rho_c := \sup_\lambda \rho(\lambda) = \rho(1)$

$$\rho_c = \left(\frac{\alpha}{\pi}\right)^{d/2} \sum_{k\geq 1} \frac{1}{k^{d/2}} \quad \begin{cases} = \infty & \text{if } d \leq 2 \\ < \infty & \text{if } d \geq 3. \end{cases}$$

In dimension $d \geq 3$:

Loop soup cannot have point density bigger than ρ_c.

$\rho : [0,1] \to [0, \rho_c]$ is invertible; denote $\lambda(\rho)$ its inverse.

Gaussian interlacements $d \geq 3$.

Doubly infinite trajectories:

$$W := \left\{ w : \mathbb{Z} \to \mathbb{R}^d, \lim_{n \to \pm\infty} \|w(n)\| = \infty \right\}$$

Define $X_n(w) := w(n)$.

Denote

$P^x :=$ double infinite random walk with
Normal$(0, \frac{1}{2\alpha})$ increments
starting at x.

Intensity via capacity (Sznitman):

Define entrance time in compact set A:

$$T_A(w) := \inf \left\{ n \in \mathbb{Z},\ X_n(w) \in A \right\} \in\ (-\infty, \infty].$$

Define

$$Q_A^{\text{cap}} g := \int_A E^x \left[g\, \mathbf{1}_{\{T_A = 0\}} \right] dx.$$

$g : W \to \mathbb{R}$ is a test function.

$e_A(x) := P^x[T_A = 0]$ *equilibrium measure.*

$\int_A e_A(x)\, dx$ *Capacity* of A.

Intensity via visit debiasing:

Denote the number of visits to A by

$$n_A(w) := \sum_{n \in \mathbb{Z}} \mathbf{1}_A(X_n(w))$$

Define:

$$Q_A^{\mathrm{unif}} g := \int_A E^x \left[\frac{g}{n_A} \right] dx.$$

Weight inversely proportional to the number of visits to A.

Time shift θ: $[\theta w](k) := w(k+1)$. $(\theta g)(w) := g(\theta w)$.

For $g : W \to \mathbb{R}$ invariant under time shifts, $g = \theta g$, we show

$$Q_A^{\mathrm{unif}} g = Q_A^{\mathrm{cap}} g.$$

Time shift equivalence:

$$w \sim w' \quad \text{if} \quad w' = \theta^k w, \ \text{some } k.$$

Trajectories modulo time shift:

$$\widetilde{W} := W/\sim, \qquad \pi : W \to \widetilde{W} \quad \text{projection.}$$

$W_A :=$ Trajectories intersecting A; $\widetilde{W}_A := \pi W_A$.

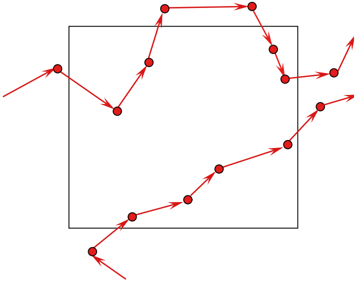
$\exists$ unique σ-finite intensity measure Q^{ri} on $(\widetilde{W}, \widetilde{\mathcal{W}})$ such that

$$\mathbf{1}_{\widetilde{W}_A} Q^{\mathrm{ri}} = \pi \circ Q_A^{\mathrm{cap}} = \pi \circ Q_A^{\mathrm{unif}}.$$

(Sznitman 2010, Cerny-Teixeira 2012).

Gaussian interlacements:

$$\Gamma_\rho^{\mathrm{ri}} := \text{ Poisson process on } \widetilde{W} \text{ with intensity } \rho Q^{\mathrm{ri}},$$
$$\mu_\rho^{\mathrm{ri}} := \text{ Law of } \Gamma_\rho^{\mathrm{ri}}.$$



Like Sznitman 2010 Brownian interlacements

Point density of Gaussian interlacements is ρ.

Construction of a random interlacement at density ρ

Sample a Poisson process χ_0 of parameter ρ.

Take a bounded box Λ

To each point in $\chi_0 \cap \Lambda$ sample a double-infinity Gaussian walk.

Accept the walk with probability 1 over number of visits to Λ.

The accepted walks will be a sample of the random interlacement intersecting Λ.

To sample in $\mathbb{R}^d$, consider a partition $(\Lambda_j)_{j\geq 1}$ of $\mathbb{R}^d$ with Λ_j bounded.

Perform the procedure in each $\Lambda_1, \Lambda_2, \ldots$ successively.

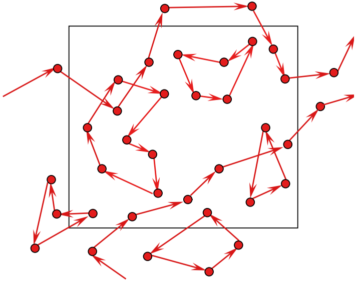
Reject walks with starting point in Λ_j that have points in previous visited boxes.

Gaussian permutation at density $\rho > 0$:

$$(\chi, \sigma)_\rho := \Gamma^{\mathrm{ls}}_{\lambda(\rho \wedge \rho_c)} \cup \Gamma^{\mathrm{ri}}_{(\rho-\rho_c)^+}$$

Superposition of independent realizations of:

- Gaussian loup soup at density $\min\{\rho, \rho_c\}$
- Gaussian interlacement at density $(\rho - \rho_c)^+$.



Conditioned Gaussian loop soup is Canonical permutation

$$\mathcal{H}_{\Lambda,N} := \{\Gamma : \textstyle\sum_{\gamma\in\Gamma}|\gamma|\mathbf{1}_{\{\gamma\}\subset\Lambda} = N\}$$
$$:= \{\Gamma : \text{number of points in cycles of } \Gamma \text{ contained in } \Lambda \text{ is } N\}$$

Proposition 1. *For g depending on the cycles totally included in Λ:*

$$\mu_\lambda^{\mathrm{ls}}(g|\mathcal{H}_{\Lambda,N}) = G_{\Lambda,N}g.$$

where:

$$\mu_\lambda^{\mathrm{ls}} = \text{ loup soup;}$$
$$G_{\Lambda,N}g = \frac{1}{Z_{\Lambda,N}}\sum_{\sigma\in\mathcal{S}_N}\int_{\Lambda^N} g(x,\sigma)e^{-H(x,\sigma)}dx \quad \text{canonical}$$

Grand-canonical measures and loup soup

$$\mu_{\Lambda,\lambda}\, g := \frac{1}{Z_{\Lambda,\lambda}} \sum_{n\geq 0} \frac{\big((\alpha/\pi)^{d/2}\lambda\big)^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \int_{\Lambda^n} g(\underline{x},\sigma) e^{-\alpha H(\underline{x},\sigma)} d\underline{x},$$

where $\underline{x} = (x_1,\dots,x_n)$, $H(\underline{x},\sigma) := \sum_{i=1}^n \|x_i - x_{\sigma(i)}\|^2$.

Proposition 2. *Let $\lambda \leq 1$. The Gaussian loop soup restricted to Λ and the grand-canonical measure are equal,*

$$\mu^{\mathrm{ls}}_{\Lambda,\lambda} = \mu_{\Lambda,\lambda}.$$

Proof. Via Laplace functionals. For $\psi : D_\Lambda \to \mathbb{R}_+$, define

$$g(\Gamma) := \exp\Big(-\textstyle\sum_{\gamma\in\Gamma}\psi(\gamma)\Big).$$

By Campbell's theorem,

$$\mu^{\mathrm{ls}}_{\Lambda,\lambda} g = e^{-Q^{\mathrm{ls}}_\lambda(D_\Lambda)}\, \exp\Big(\sum_{k\geq 1}\frac{1}{k!}a_k\Big),$$

where, using the definition of $Q_{k,\lambda}$ and denoting $\tilde{\lambda} := (\alpha/\pi)^{d/2}\lambda$,

$$\begin{aligned} a_k &:= \tilde{\lambda}^k (k-1)! \int_{\Lambda^k} dx_1 \dots dx_k \, e^{-\alpha H([x_1,\dots,x_k])} \, e^{-\psi([x_1,\dots,x_k])} \\ &= \tilde{\lambda}^k \sum_{\gamma \in \mathcal{C}_k} \int_{\Lambda^k} dx_1 \dots dx_k \, e^{-\alpha H(\underline{x},\gamma)} \, e^{-\psi(\underline{x},\gamma)} \end{aligned}$$

where $\mathcal{C}_k$ is the set of cycles of size k with elements $\{1,\dots,k\}$, $\underline{x} = (x_1,\dots,x_n)$, and $(\underline{x},\gamma) := [x_1, x_{\gamma(1)}, \dots, x_{\gamma^{k-1}(1)}]$.

A combinatorial lemma:

Lemma 3. *Let $(a_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}}$ be such that $\sum_{n\geq 1} \frac{1}{n!}|a_n| < \infty$. Then*

$$\exp\Big(\sum_{n\geq 1} \frac{1}{n!} \, a_n \Big) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{P \in \mathcal{P}_n} \prod_{I \in P} a_{|I|},$$

where $\mathcal{P}_n$ is the set of partitions of $\{1,\dots,n\}$ into non-empty sets.

$$
\begin{aligned}
&\mu^{\mathrm{ls}}_{\Lambda,\lambda} g \\
&= e^{-Q^{\mathrm{ls}}_\lambda(D_\Lambda)} \sum_{n\geq 0} \frac{1}{n!} \sum_{P\in\mathcal{P}_n} \prod_{I\in P} a_{|I|} \\
&= e^{-Q^{\mathrm{ls}}_\lambda(D_\Lambda)} \sum_{n\geq 0} \frac{\tilde{\lambda}^n}{n!} \sum_{P\in\mathcal{P}_n} \prod_{I\in P} \sum_{\gamma\in\mathcal{C}_{|I|}} \int_{\Lambda^{|I|}} dx_1 \dots dx_{|I|} e^{-\alpha H([x_1,\dots,x_{|I|}])}\, e^{-\psi([x_1
\\
&= e^{-Q^{\mathrm{ls}}_\lambda(D_\Lambda)} \sum_{n\geq 0} \frac{\tilde{\lambda}^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \prod_{\gamma\in\sigma} \int_{\Lambda^{|\gamma|}} dx_1 \dots dx_{|\gamma|}\, e^{-\alpha H(\underline{x},\gamma)} e^{-\psi(\underline{x},\gamma)} \\
&= e^{-Q^{\mathrm{ls}}_\lambda(D_\Lambda)} \sum_{n\geq 0} \frac{\tilde{\lambda}^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \int_{\Lambda^n} dx_1 \dots dx_n\, e^{-\alpha H(\underline{x},\sigma)} \prod_{\gamma\in\sigma} e^{-\psi((x_i : i\in\{\gamma\}),\gamma)} \\
&= \mu_{\Lambda,\lambda} g,
\end{aligned}
$$

where $(\underline{x}, \sigma)$ is the spatial permutation that maps x_i to $x_{\sigma(i)}$ and $\{\gamma\}$ is the set of indices that appear in the cycle γ. □

The Gaussian random permutation is Markov and Gibbs

$\Lambda \subset \mathbb{R}^d$ and a spatial permutation $\Gamma = (\zeta, \kappa) \in \mathcal{X}$,

$$
\begin{aligned}
I_\Lambda\zeta &:= \zeta \cap \Lambda, && \text{red points} \\
O_\Lambda\zeta &:= \zeta \cap \Lambda^c, && \text{purple and yellow points} \\
U_\Lambda\zeta &:= \{u \in \zeta \cap \Lambda^c : \kappa(u) \in \Lambda\}, && \text{yellow} \\
V_\Lambda\zeta &:= \{v \in \zeta \cap \Lambda^c : \kappa^{-1}(v) \in \Lambda\}, && \text{yellow}
\end{aligned}
$$

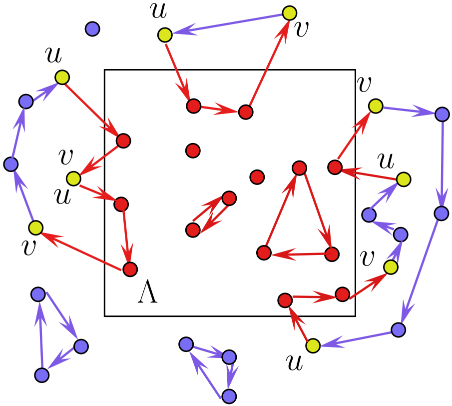
and the maps

$$
\begin{aligned}
I_\Lambda\kappa &: I_\Lambda\zeta \cup U_\Lambda\zeta \to I_\Lambda\zeta \cup V_\Lambda\zeta, \qquad I_\Lambda\kappa(x) = \kappa(x) \quad \text{red arrows,} \\
O_\Lambda\kappa &: O_\Lambda\zeta \setminus U_\Lambda\zeta \to O_\Lambda\zeta \setminus V_\Lambda\zeta, \quad O_\Lambda\kappa(x) = \kappa(x) \quad \text{purple arrows.}
\end{aligned}
$$

Define the inside and outside projections (with respect to Λ) by

$$
I_\Lambda(\zeta, \kappa) := (I_\Lambda\zeta, I_\Lambda\kappa), \qquad O_\Lambda(\zeta, \kappa) := (O_\Lambda\zeta, O_\Lambda\kappa).
$$

Decomposition of a loop soup intersecting Λ



Inside = red

Outside = purple + yellow

Markov property

Proposition 4. *The Gaussian random permutation μ_ρ is Markov:*

$$\mu_\rho\big(dI_\Lambda(\Gamma)\big|\, O_\Lambda(\Gamma) \text{ occurs outside } \Lambda\big) \\ = \mu_\rho\big(dI_\Lambda(\Gamma)\big|\, (U_\Lambda, V_\Lambda\big) \text{ occur outside } \Lambda).$$

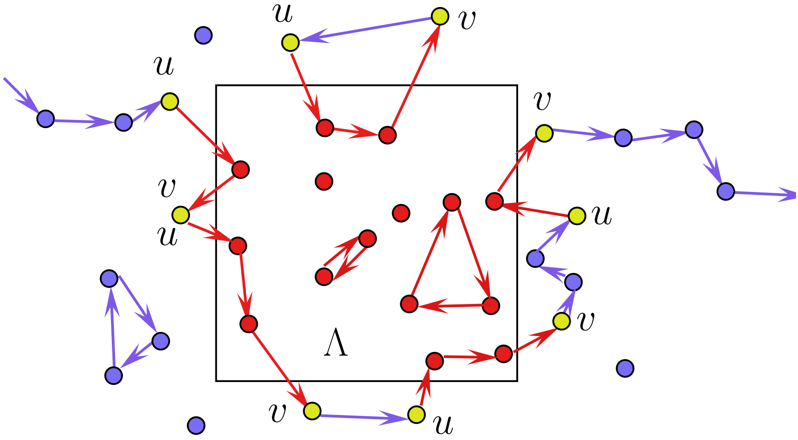
Loop soup:

Conditioning on purple and yellow, the law of red points and arrows depends only on the labeled yellow points.

Conditioned on labeled yellow points, purple points and arrows are independent of red points and arrows.

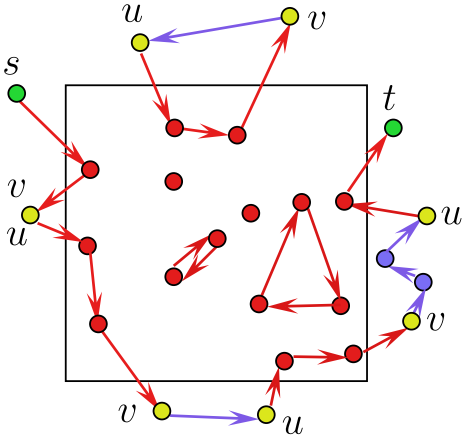
Markov property

For Superposition of loop soup and random interlacements



Markov property

For Superposition of loop soup and random interlacements

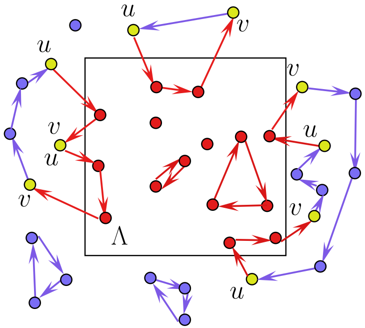


Gaussian random permutation is Gibbs

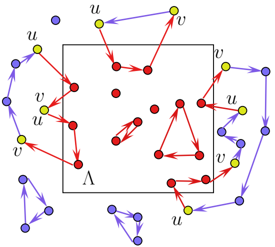
Λ-compatibility between infinite volume random permutations:

$$(\chi, \sigma) \sim_\Lambda (\zeta, \kappa)$$

if they have same yellow points and purple points and arrows.



Gaussian random permutation is Gibbs



Conditioned Hamiltonian: For $(\chi,\sigma) \sim_\Lambda (\zeta,\kappa)$

$$H_\Lambda((\chi,\sigma)|(\zeta,\kappa)) := \sum_{x\in[\chi\cap\Lambda]\cup[\kappa^{-1}(\zeta\cap\Lambda)\setminus\Lambda]} \|x-\sigma(x)\|^2.$$

Fix yellow and purple and sum over red points and arrows.

Specifications $G_{\Lambda,\lambda}(\cdot|(\zeta,\kappa)) :=$ law of red points and arrows.

Gaussian random permutation on $\mathbb{R}^d$ is Gibbs

Theorem 5 (AFY 2019). *For $d \geq 3$ and $\lambda \leq 1$ the loop soup measure*

$\mu_\lambda^{\mathrm{ls}}$ is Gibbs for the specifications $(G_{\Lambda,\lambda} : \Lambda$ compact):

$$\mu_\lambda^{\mathrm{ls}} g = \int d\mu_\lambda^{\mathrm{ls}}(\zeta,\kappa)\, G_{\Lambda,\lambda}(g|(\zeta,\kappa)) \qquad \text{DLR}$$

For all $\rho \geq \rho_c$ the measure

$\mu_1^{\mathrm{ls}} * \mu_{\rho-\rho_c}^{\mathrm{ri}}$ is Gibbs for the specifications $(G_{\Lambda,1} : \Lambda$ compact).

Corollary 6 (AFY 2019). *Point and permutation marginals can be computed explicitely, as follows.*

Point marginal of Gaussian loop soup is permanental

$$\nu_\lambda^{\rm ls} := \text{Point marginal of Gaussian loop soup } \mu_\lambda^{\rm ls}$$

Recall fugacity $\lambda \in (0,1]$ and denote

$$K_{xy} = K_\lambda(x,y) := \sum_{k\geq 1} \lambda^k \Big(\frac{\alpha}{\pi k}\Big)^{d/2} e^{-\frac{\alpha}{k}\|x-y\|^2}, \quad x,y \in \mathbb{R}^d$$

$$= \text{intensity of } \{x,y \in \chi \text{ and } \sigma^k(x) = y \text{ for some } k\}$$

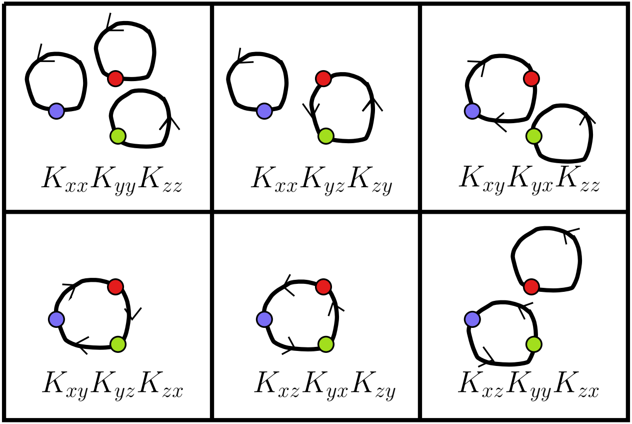
$\nu_\lambda^{\rm ls}$ has correlations

$$\varphi_\lambda^{\rm ls}(x_1,\dots,x_n) = \text{perm}(K_{x_i,x_j})_{i,j=1}^n,$$

Permanental point process.

Point marginal of Gaussian loop soup is permanental

$$\varphi_\lambda^{\mathrm{ls}}(x,y,z) = K_{xx}K_{yy}K_{zz} + K_{xx}K_{yz}K_{zy} + K_{xy}K_{yx}K_{zz} + K_{xy}K_{yz}K_{zx} + K_{xz}K_{yx}K_{zy} + K_{xz}K_{yy}K_{zx},$$



Correlations of point marginal of Gaussian interlacements

$$\nu_\rho^{\mathrm{ri}} := \text{Point marginal of Gaussian interlacements } \mu_\rho^{\mathrm{ri}}$$

Correlations:

$$\varphi_\rho^{\mathrm{ri}}(x_1, \dots, x_n) = \sum_{P \in \mathcal{P}_n} \prod_{I \in P} \sum_{\sigma \in \mathcal{S}_I} V_\rho(x_{\sigma(i_1)}, \dots, x_{\sigma(i_{|I|})}). \tag{2}$$

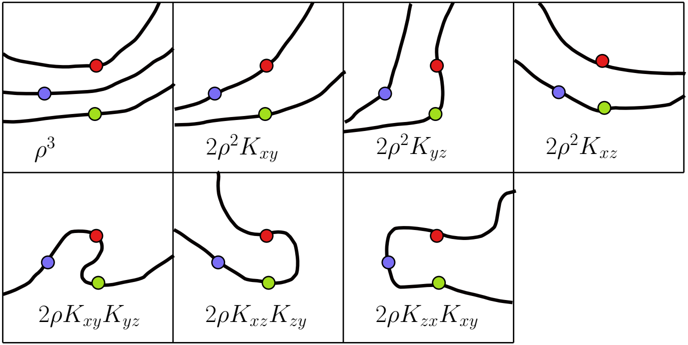
$\mathcal{P}_n :=$ partitions of $\{1, \dots, n\}$ with nonempty sets,
$\mathcal{S}_I :=$ permutations of I,
$(i_1, \dots, i_{|I|})$ arbitrary order of I and

$$V_\rho(x_1, \dots, x_\ell) := \rho K_{x_1,x_2} \dots K_{x_{\ell-1},x_\ell}$$

(Here $\lambda = 1$ and $K_{xy} = K_1(x, y)$)

Correlations of point marginal of Gaussian interlacements

$$\varphi_\rho^{\mathrm{ri}}(x,y,z) = \rho^3 + 2\rho^2 K_{xy} + 2\rho^2 K_{yz} + 2\rho^2 K_{xz}$$
$$+2\rho K_{xy}K_{yz} + 2\rho K_{xz}K_{zy} + 2\rho K_{zx}K_{xy}.$$



Thermodynamic limit of Point marginal

Theorem (Shirai-Takahashi, Tamura-Ito). *Fix density* $\rho > 0$.

$G^{\text{point}}_{\Lambda,|\Lambda|\rho} :=$ *law of point-marginal with* $|\Lambda|\rho$ *points.*

Subcritical $\rho \leq \rho_c$ *or* $d \leq 2$ *Fichtner 1991; Tamura-Ito 2006.*

$$G^{\text{point}}_{\Lambda,|\Lambda|\rho} \Rightarrow \nu^{\text{TI}}_{\rho} \text{ as } \Lambda \nearrow \mathbb{R}^d.$$

Supercritical $\rho > \rho_c$ *and* $d \geq 3$ *Tamura-Ito 2007.*

$$G^{\text{point}}_{\Lambda,|\Lambda|\rho} \Rightarrow \nu^{point}_{\rho} = \nu^{TI}_{\rho_c} * \nu^{\infty}_{\rho-\rho_c}.$$

Theorem (AFY). *Point marginal of Gaussian random permutation coincide with thermodynamic limit above:*

$\nu^{\text{TI}}_{\rho} = \nu^{\text{ls}}_{\lambda(\rho)}$, *point marginal of loop soup at fugacity* $\lambda(\rho)$.

$\nu^{\infty}_{\rho} = \nu^{\text{ri}}_{\rho}$, *point marginal of Gaussian interlacements at* ρ.

Partial "Thermodynamic limit" of permutation marginal

$G^{\text{permut}}_{\Lambda,|\Lambda|\rho} := \sigma$-marginal of $G_{\Lambda,|\Lambda|\rho}$

$$G^{\text{permut}}_{\Lambda,\rho} \Rightarrow \nu^{\text{permut}}_{\rho} \text{ for } \textit{cycle-size distribution}.$$

Macroscopic cycles: cycles with size bigger than $\varepsilon|\Lambda|$.

Subcritical case. $\rho \le \rho_c$ or $d = 1, 2$

The expected fraction of points in macroscopic cycles is zero. BU 2011

Supercritical case. $d \ge 3$ and $\rho > \rho_c$

(a) expected fraction of points in macroscopic cycles is $\frac{\rho-\rho_c}{\rho}$.

(b) Rescaled macroscopic cycles have random lenght:
Benfatto, Cassandro, Merola Presutti 2005.

Poisson-Dirichlet distribution (as uniform permut): Betz-Ueltschi 2011.

Current problems

Thermodynamic limit of canonical measure. That is, the Gaussian random permutation in a box Λ should converge to the infinite volume GRP constructed here.

Extensions: Poisson process on $\mathbb{Z}^d$

Other interactions besides Gaussian.

Quantum case, when the Brownian trajectories from x to y interact. (BCMP 2005 treated the mean field case).

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