Localisation of RW in dimensions $d \geq 3$

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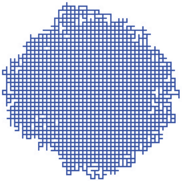

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Statement of problem

Bolthausen's question

For $N \in \mathbb{N}$, consider

$$d\mathbb{Q} = \frac{\exp(-|R_N|)}{Z_N} d\mathbb{P}$$

where $R_N = \{X_1, \ldots, X_N\}$ is range.
Q: What can be said about typical realisation of $\mathbb{Q}$?

$Z_N =$ partition function estimated by Donsker–Varadhan (1979):

$$Z_N = \exp(-(1 + o(1))\chi_d N^{d/(d+2)})$$

for some explicit χ_d, $d \geq 2$.

Bolthausen's conjecture

Theorem (Bolthausen 1994)

Let $d = 2$. Then RW localises on $B(x, \rho_2 N^{1/4})$ for some random $x \in \mathbb{Z}^d$ and explicit $\rho_2 > 0$.

Conjecture for $d \geq 3$ (1994):

Localisation in $B(x, \rho_d N^{1/(d+2)})$ for some $\rho_d > 0$.

Part of his analysis is directly written for general $d \geq 2$.

This talk

Solution to Bolthausen's conjecture (joint with Raphaël Cerf, 2018).

Related works

• Independent solution by Ding, Fukishima, Sun, and Xu. See Ryoki's course.

• Sznitman did continuous version of this problem in the 90s, using his enlargement of obstacles method.

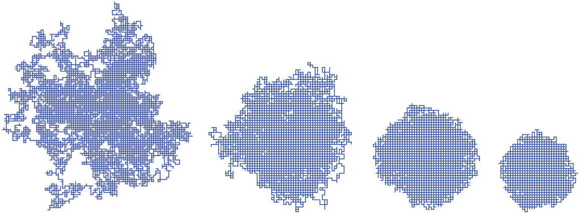
• Work by [DFSX] uses (discrete version of) Sznitman's method for upper bound. Plus separate argument for filling a ball.

• Unlike [DSFX] we did not get estimates on the size of the boundary.

Motivation

- One motivation for us is work by B.-Yadin (2015) which considers

$$d\mathbb{Q} = \frac{\exp(-\beta|\partial R_N|)}{Z_N} d\mathbb{P}.$$



- Results show localisation (but no shape theorem) at scale $n = N^{1/(d+1)}$.
- Bolthausen's question is much easier...
- Note: random medium representation not available here.

Ideas and heuristics

Why is scale $L = N^{1/(d+2)}$ the correct scale ?

$$\mathbb{P}(R_N \subset B(0,L)) \approx \exp(-N/L^2).$$

Contribution is $\exp(-|R_N|) \approx \exp(-L^d)$.
Hence $L^d = N/L^2$.

More precisely, if $U \subset \mathbb{R}^d$, let $\lambda_U =$ principal EV of $-\Delta$ with Dirichlet boundary conditions in U.

$$\mathbb{P}(R_N \subset LU) \approx \exp(-\frac{\lambda_U}{2} L^d).$$

Contributes $\exp(-|U|L^d)$.

Faber–Krahn inequality

Consequently Z_N should be obtained by minimising

$$\inf_{U \subset \mathbb{R}^d} \{ \frac{\lambda_U}{2} + |U| \}$$

Faber–Krahn inequality: U is a Euclidean ball. Then problem equivalent to

$$\inf_{r>0} \{ \frac{\lambda}{2r^2} + \omega_d r^d \}$$

with $\lambda = \lambda_{B(0,1)}$ and $\omega_d = |B(0,1)|$. The value is χ_d from Donsker–Varadhan asymptotics. Radius $r = \rho_d$.

Quantitative Faber–Krahn

• Bolthausen's starting point is quantitative Faber–Krahn (:= qFK).

• To use Large Deviation theory, must be on compact set $\implies$ qFK is needed on a (continuous) torus of large but fixed ($O(1)$) size.

• Unfortunately, qFK on the torus only known in $d = 2$. (Even quantitative isoperimetric not known.)

• But known in $\mathbb{R}^d$ by work of Brasco, De Phillipis, Volkachev (2015).

Consequence of quantitative Faber–Krahn

Let ϕ_x = 1st eigenfunction in $B(x, \rho_d)$, normalised so $\|\phi_x\|_2 = 1$.

Lemma

If $g : \mathbb{R}^d \to \mathbb{R}^+$ is C^∞ and such that $||g||_2 = 1$, $g \geq 0$, and if

$$\varepsilon \; = \; \inf_{x \in \mathbb{R}^d} \; ||g - \phi_x||_2 \; > \; 0$$

is small enough, then

$$\left|\{\, g > 0 \,\}\right| + \frac{1}{2d} \int_{\mathbb{R}^d} \left|\nabla g\right|^2 dx \;\geq\; \chi_d + \varepsilon^{48} \,.$$

LDP in $\mathbb{R}^d$

For $t \in \mathbb{N}$ and $D \subset \mathbb{R}^d$, let

$$\tau(D,t) = \inf\{k \geq 1 : L(D,k) \geq t\}$$

and

$$L_t^D(x) = \sum_{k=0}^{\tau(D,t)-1} 1_{\{X_k = x\}}$$

Theorem (B.-Cerf 2018)

Let C be closed convex in $\ell^1(D)$. Then for all $t \geq 1$,

$$\inf_{x \in \bar{D}} \mathbb{P}^x \left(\frac{1}{t} L_t^D \in C; \tau(D,t) < \infty \right) \leq \exp\left(-t \inf_{h \in C} \frac{1}{2} \mathcal{E}^D(\sqrt{h}, \sqrt{h}) \right)$$

where $\mathcal{E}^D$ is Dirichlet form

$$\mathcal{E}^D(f,f) = \frac{1}{2d} \sum_{y,z \in D : |y-z|=1} (f(y) - f(z))^2$$

Two parts of proof

Part I: ℓ^1 shape theorem

Local times are close to ϕ_x^2 in ℓ^1 sense for some $x \in \mathbb{R}^d$.

Part II: ball is filled

If local times are close to ϕ_x^2 then all points in $B(x, L\rho_d - L^{1-\kappa})$ are visited.

By Bolthausen (1994) this implies containment and so conjecture.

Coarse–graining, I

• Improved lower bound on partition function already proved by Bolthausen:

$$Z_N \geq \exp(-\chi_d L^d - cL^{d-1})$$

for some $c \in \mathbb{R}$.

• So to show $\mathcal{A}$ is unlikely, it suffices

$$\mathbb{E}(\exp(-|R_N|)1_{\mathcal{A}}) \leq \exp(-\chi_d L^d - \varepsilon L^{d-\varepsilon}).$$

• To single out the minimiser we would like to use LDP and "sum" over all possible functions.

Coarse-graining, II

- Too much entropy, of course.
- Instead use a coarse–grained version of local time profile.

Key tool for this is the observation that: whp,
(i) $|R_N| \leq CL^d$
(ii): if $f_N = \sqrt{L_N}$, $\mathcal{E}(f_N, f_N) \leq CL^d \log L$ by LDP.

Poincaré-Sobolev inequality: this implies

$$\|f_N\|_{2^*} \leq C\sqrt{L^d \log L}$$

where $2^* = 2d/(d-2) > 2$.
Controls number of blocks of size L where f_N large (high density) so reduces entropy.
Poincaré-Wirtinger controls L^2 distance to coarse-grained profile.

Putting LDP + Coarse–Graining + qFK together

We obtain shape theorem in ℓ^1:

$$\text{set } \ell_N = \frac{L^d}{N} L_N(\lfloor Lx \rfloor) = \text{ rescaled local time} \in \ell^1(\mathbb{R}^d)$$

Proposition

Let

$$\mathcal{L}_n = \{ f \in \ell^1(\mathbb{R}^d) : \|f\|_1 = 1, \inf_{x \in \mathbb{R}^d} \|\phi_x^2 - f\|_1 \geq L^{-1/800} \}$$

Then

$$\mathbb{E}(e^{-|R_N|} 1_{\ell_N \in \mathcal{L}_N}) \leq \exp(-\chi_d L^d - L^{d-1/17}).$$

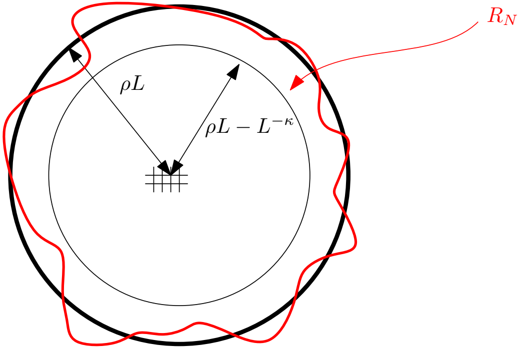
Filling the ball

Suppose ℓ^1 shape theorem holds. Show "all" of ball visited.

Proposition

For κ small enough,

$$\mathbb{E}(e^{-|R_N|}1_{\{\|\ell_N-\phi_0^2\|_1\leq L^{-s}\}}1_{\{B(0,\rho L-L^{-\kappa})\text{not full}\}}) \leq Z_N e^{-L^{-\kappa}}$$



Ideas

Pick $x \in B(0, \rho L - L^{1-\kappa})$ and $m = L^{1-2\kappa}$. Show $B(x, m)$ visited.

- By shape thm, walk spends much time in $B(x, m)$.

- Hence many bridges of duration m^2:
[staying in $B(x, m)$, from bulk to bulk. (bulk $= B(x, m/2)$)].

- We want to sum over all $\mathcal{X} \subset B(x, m)$, probability to avoid $\mathcal{X}$.

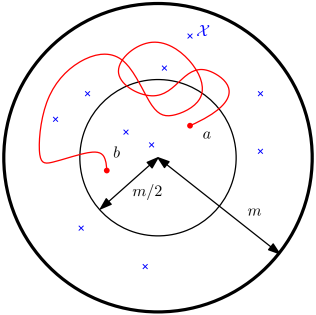
Key: can condition on everything that happens outside $B(x, m)$.
All bridges are independent!

Avoidance probability

- Need lower bound for

$$\mathbb{P}^{a\to b;m^2}(\text{bridge } X \text{ hits } \mathcal{X})$$

uniformly in a, b and geometry of $\mathcal{X}$: depending only on $k = |\mathcal{X}|$.



We will get $\geq \frac{k}{(\log m)m^d}$ uniformly.

Isoperimetric ideas

- Intuitively: worst case when $\mathcal{X}$ clumped as ball (of radius $k^{1/d}$).
- Ignore boundary effects for now: assume $\mathcal{X} \subset B(x, m/2)$.
Change bridge into SRW.
- "First" moment method:

$$\mathbb{P}(\text{visit } \mathcal{X}) = \frac{\mathbb{E}(L_{m^2}(\mathcal{X}))}{\mathbb{E}(L_{m^2}(\mathcal{X}) | L_{m^2}(\mathcal{X}) > 0)} \geq \frac{km^2/m^d}{\max_{z \in \mathcal{X}} \mathbb{E}_z(L_\infty(\mathcal{X}))}$$

- $\mathbb{E}_z(L_\infty(\mathcal{X})) = \sum_{x\in\mathcal{X}} G(z,x)$, $G =$ Green function
- But $G(z,x)$ is essentially monotone in distance.
- So worst indeed when $\mathcal{X}$ is ball!

Hence

$$\mathbb{P}(\text{visit } \mathcal{X}) \geq \frac{km^{2-d}}{k^{2/d}} = k^{1-2/d} m^{2-d}.$$

This would be enough...

BUT this ignores boundary effects: walk conditioned to stay in $B(x,m)$.

Isoperimetry for conditioned walks

• What if $\mathcal{X}$ is clumped close to $\partial B(x, m)$?

• Idea: decompose into dyadic annuli at distance $1, \ldots, 2^j, \ldots$ from $\partial B(x, m)$.

• Suppose $\mathcal{X} \subset A_j$ at distance $r = 2^j$. Then Gambler's ruin:

$$\mathbb{E}^{a \to b; m^2}(L_{m^2}(\mathcal{X})) \asymp k m^{2-d} (r/m)^2$$

• On the other hand, if $z \in \mathcal{X}$,

$$\mathbb{E}_z(L(\mathcal{X})) \le \mathbb{E}_z(L(A_j)) \le r^2$$

Hence

$$\mathbb{P}^{a \to b; m^2}(\text{visit } \mathcal{X}) \ge k m^{-d}, \ \text{indep. of } r$$

• If $\mathcal{X}$ not fully contained in an annulus, take the annulus with the biggest number of points. Then $k \to k / \log m$!
END OF PROOF!

THANK YOU!