Intermediate Dimensions, Capacities and Projections

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Overview

• The talk concerns sets in $\mathbb{R}^n$ with differing Hausdorff and box-counting dimensions.

• Hausdorff and box-counting dimensions can be regarded as particular cases of a spectrum of 'intermediate' dimensions $\dim_\theta F$ $(0 \leq \theta \leq 1)$ with

$$\dim_0 F = \dim_{\mathrm{H}} F \quad \text{and} \quad \dim_1 F = \dim_{\mathrm{B}} F$$

• Intermediate dimensions give an idea of the range of sizes of covering sets needed to get good estimates for Hausdorff dimension.

• Potential theoretic methods enable us to study geometric properties of these dimensions such as the effect of orthogonal projection.

Hausdorff and box dimension - alternative definitions

Recall that Hausdorff dimension may be defined without introducing Hausdorff measures: for $E \subset \mathbb{R}^n$

$$\dim_{\mathrm{H}} E = \inf \Big\{ s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists a cover } \{U_i\} \text{ of } E \text{ such that } \sum |U_i|^s \leq \epsilon \Big\}.$$

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The lower/upper box-counting dimensions of a non-empty compact $E \subset \mathbb{R}^n$ are

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{r\to 0} \frac{\log N_r(E)}{-\log r}, \quad \overline{\dim}_{\mathrm{B}} E = \lim_{r\to 0} \frac{\log N_r(E)}{-\log r}$$

where $N_r(E)$ is the least number of sets of diameter r covering E.

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where $N_r(E)$ is the least number of sets of diameter r covering E. Equivalently $\underline{\dim}_{\mathrm{B}}$ may be defined

$$\underline{\dim}_{\mathrm{B}} E = \inf \Big\{ s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists a cover } \{U_i\} \text{ of } E \text{ such that } |U_i| = |U_j| \text{ for all } i, j \text{ and } \sum |U_i|^s \leq \epsilon \Big\}.$$

Intermediate dimensions

Let $E \subset \mathbb{R}^n$ be non-empty and bounded. For $0 \leq \theta \leq 1$ define the lower θ-intermediate dimension of E by

$$\underline{\dim}_{\theta} E = \inf \big\{ s \geq 0 : \text{ for all } \epsilon > 0 \text{ there exist arbitrarily small } \delta > 0 \text{ s.t.} \\ \text{and } \{U_i\} \text{ covering } E \text{ s.t. } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ and } \textstyle\sum |U_i|^s \leq \epsilon \big\}.$$

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Similarly, define the upper θ-intermediate dimension of E by

$$\overline{\dim}_\theta E = \inf\big\{s \geq 0 : \text{ for all } \epsilon > 0 \text{ and all sufficiently small } \delta > 0 \text{ there is a cover } \{U_i\} \text{ of } E \text{ s.t. } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ and } \sum |U_i|^s \leq \epsilon\big\}.$$

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Then

$$\underline{\dim}_0 E = \overline{\dim}_0 E = \dim_{\mathrm{H}} E, \quad \underline{\dim}_1 E = \underline{\dim}_{\mathrm{B}} E \text{ and } \overline{\dim}_1 E = \overline{\dim}_{\mathrm{B}} E.$$

Moreover, for bounded E and $\theta \in [0, 1]$,

$$\dim_{\mathrm{H}} E \le \underline{\dim}_{\theta} E \le \overline{\dim}_{\theta} E \le \overline{\dim}_{\mathrm{B}} E \text{ and } \underline{\dim}_{\theta} E \le \underline{\dim}_{\mathrm{B}} E.$$

SImple properties

- $\overline{\dim}_\theta$ is finitely stable, that is
$$\overline{\dim}_\theta(E_1 \cup E_2) = \max\{\overline{\dim}_\theta E_1, \overline{\dim}_\theta E_2\}.$$

- For $\theta \in (0, 1]$, both $\underline{\dim}_\theta E$ and $\overline{\dim}_\theta E$ are unchanged on replacing E by its closure.

- For $E, F \subseteq \mathbb{R}^n$ be non-empty and bounded and $\theta \in [0, 1]$,

$$\underline{\dim}_\theta E + \underline{\dim}_\theta F \leq \underline{\dim}_\theta(E \times F) \leq \overline{\dim}_\theta(E \times F) \leq \overline{\dim}_\theta E + \overline{\dim}_{\mathrm{B}} F.$$

- For $\theta \in [0, 1]$, $\underline{\dim}_\theta$ and $\overline{\dim}_\theta$ are bi-Lipschitz invariant.

Continuity and monotonicity

Proposition Let $E \subset \mathbb{R}^n$ and let $0 \leq \theta < \phi \leq 1$. Then

$$\underline{\dim}_\theta E \;\leq\; \underline{\dim}_\phi E \;\leq\; \underline{\dim}_\theta E + \Big(1 - \frac{\theta}{\phi}\Big)(n - \underline{\dim}_\theta E),$$

similarly for upper dimensions.

In particular, $\theta \mapsto \underline{\dim}_\theta E$ and $\theta \mapsto \overline{\dim}_\theta E$ are continuous for $\theta \in (0,1]$ and (not necessarily strictly) increasing.

Intermediate dimensions and Assouad dimension

The Assouad dimension of $E \subseteq \mathbb{R}^n$ is defined by

$$\dim_{\mathrm{A}} E = \inf \Big\{ s \geq 0 \ : \ \text{there exists } C > 0 \text{ such that for all } x \in E,$$
$$\text{and for all } 0 < r < R,\ N_r(E \cap B(x,R)) \leq C \left(\frac{R}{r}\right)^s \Big\}$$

where $N_r(A)$ denotes the smallest number of sets of diameter at most r required to cover a set A. In general $\underline{\dim}_{\mathrm{B}} E \leq \overline{\dim}_{\mathrm{B}} E \leq \dim_{\mathrm{A}} E \leq n$,

Proposition For non-empty bounded $E \subseteq \mathbb{R}^n$ and $\theta \in (0,1]$,

$$\underline{\dim}_{\,\theta} E \geq \dim_{\mathrm{A}} E - \frac{\dim_{\mathrm{A}} E - \underline{\dim}_{\mathrm{B}} E}{\theta},$$

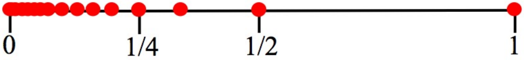
with a similar conclusion using $\overline{\dim}_{\,\theta}$ and $\overline{\dim}_{\mathrm{B}}$.

Example

For $p > 0$ let

$$E_p = \left\{0, \frac{1}{1^p}, \frac{1}{2^p}, \frac{1}{3^p}, \dots\right\}.$$

E_1:



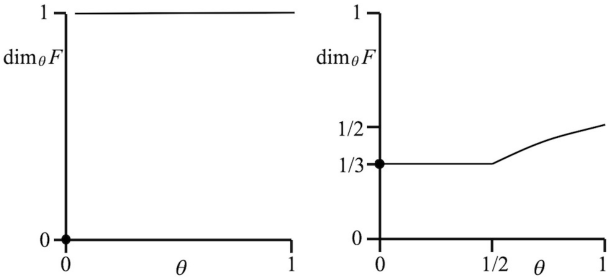
Since E_p is countable, $\dim_{\mathrm{H}} E_p = 0$.

It is well-known that $\dim_{\mathrm{B}} E_p = 1/(p+1)$.

For $p > 0$ and $0 \le \theta \le 1$,

$$\underline{\dim}_{\,\theta} E_p = \overline{\dim}_{\theta} E_p = \frac{\theta}{p+\theta}.$$

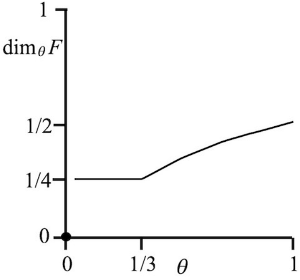
Examples



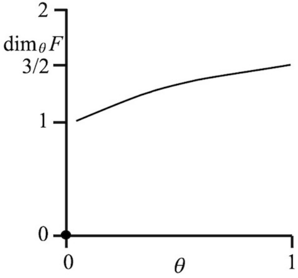
$E_{\log} = \{0, 1/\log 2, 1/\log 3, \ldots\}$

$E_1 \cup E$ where
$\dim_{\mathrm{H}} E = \dim_{\mathrm{B}} E = 1/3$

Examples

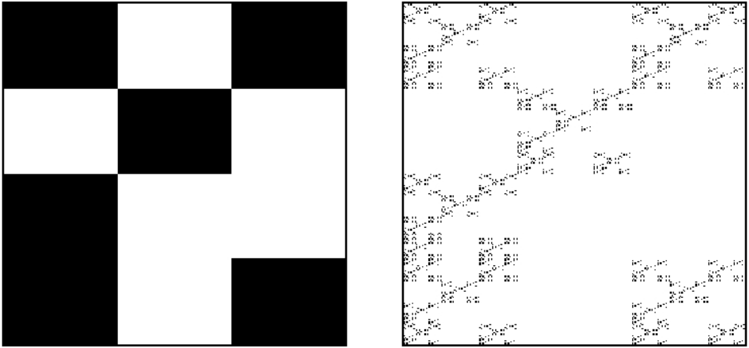


$E_1 \cup E$ where
$\underline{\dim}_B E = \dim_A E = 1/4$



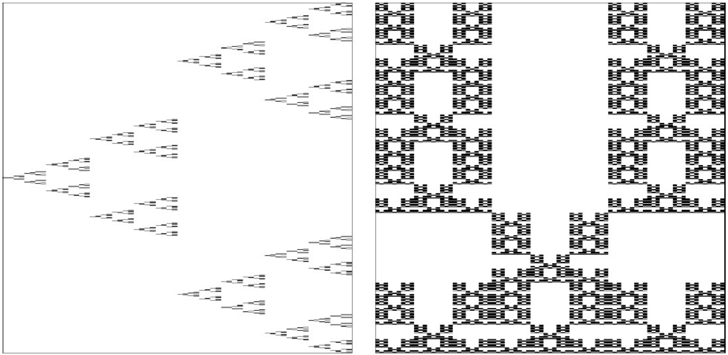
$E_1 \times E_{\log}$

Bedford-McMullen carpets



3 × 4 Bedford-McMullen self-affine carpet

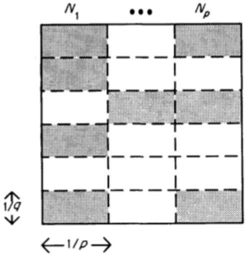
Bedford-McMullen carpets



2 $\times$ 3 and 3 $\times$ 5 Bedford-McMullen self-affine carpets

Bedford-McMullen carpets

$p \times q$ carpet, $p < q$ (Bedford 1984, McMullen 1984)



$$\dim_H E = \frac{1}{\log p} \log \left(\sum_{j=1}^{p} N_j^{\log p / \log q} \right)$$

$$\dim_B E = \frac{\log N}{\log p} + \frac{\log \frac{1}{N} \sum_{j=1}^{p} N_j}{\log q}$$

N_j rectangles selected in jth column, N non-empty columns.

Bedford-McMullen carpets

Proposition Let E be the Bedford-McMullen carpet as above. Then for $0 < \theta < \frac{1}{4}(\log p / \log q)^2$,

$$\overline{\dim}_\theta E \;\leq\; \dim_{\mathrm{H}} E + \left(\frac{2\log(\log p/\log q)\log(\max_j N_j)}{\log q}\right)\frac{1}{-\log\theta}. \tag{1}$$

In particular, $\underline{\dim}_\theta E$ and $\overline{\dim}_\theta E$ are continuous at $\theta = 0$ and so are continuous on $[0, 1]$.

Proof Put a natural Bernoulli measure μ on E and show that for all $x \in E$, $\mu(S(x, p^{-k})) \geq (p^{-k})^{d+\epsilon}$ for some $K \leq k \leq K/\theta$ for all large K, where $S(x, p^{-k})$ is an 'approximate square' of centre x and side p^{-k}.

Bedford-McMullen carpets

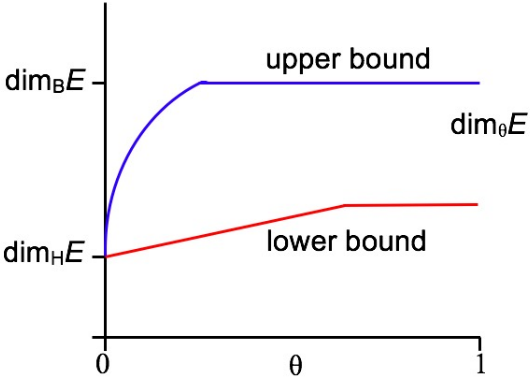
Proposition Let E be the Bedford-McMullen carpet as above. Then for $0 \leq \theta \leq \log p / \log q$,

$$\underline{\dim}_\theta E \geq \dim_{\mathrm{H}} E + \theta \frac{\log \sum_{j=1}^{p} N_j - H(\mu)}{\log p}. \qquad (2)$$

where $H(\mu) < \log \sum_{j=1}^{p} N_j$ is the entropy of the Bernoulli measure on E.

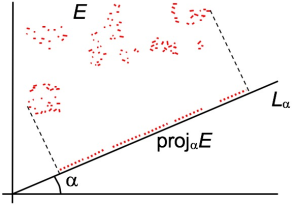
Proof For each K, construct a measure ν_K on E and show that for some $E_0 \subset E$ with $\nu_K(E_0) \geq \frac{1}{2}, \quad \nu_K(S(x, p^{-k})) \leq (p^{-k})^{d'-\epsilon}$ for all $x \in E_0$ and $K \leq k \leq K/\theta$.

Bedford-McMullen carpets



Lower bound for $\underline{\dim}_\theta E$, upper bound for $\overline{\dim}_\theta E$

Marstrand's projection theorems



Theorem (Marstrand 1954, Mattila 1975) Let $E \subset \mathbb{R}^n$ be Borel. For all $\alpha \in G(n, m)$

$$\dim_{\mathrm{H}} \text{proj}_\alpha E \le \min\{\dim_{\mathrm{H}} E, m\} \equiv \dim_{\mathrm{H}}^m E$$

with equality for almost all $\alpha \in G(n, m)$,

[proj_α is orthogonal projection onto the m-dimensional subspace α]

Think of $\dim_{\mathrm{H}}^m E$ as 'the dimension of E when viewed from an m-dimensional viewpoint' or the m-dimensional Hausdorff dimension profile of E.

Capacities and Hausdorff dimension of projections

That $\dim_{\rm H}\text{proj}_\alpha E \leq \min\{\dim_{\rm H} E, m\}$ for all α follows since projection is a Lipschitz map which cannot increase dimension.

The lower bound may be derived from the capacity characterisation of Hausdorff dimension. Let $\mathcal{M}(E)$ be the set of probability measures on E. With the capacity $C^s(E)$ of $E \subset \mathbb{R}^n$ given by

$$\frac{1}{C^s(E)} = \inf_{\mu\in\mathcal{M}(E)} \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s},$$

then $\quad \dim_{\rm H} E = \sup\left\{s : C^s(E) > 0\right\}.$

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$$\text{then} \quad \dim_{\rm H} E = \sup\left\{s : C^s(E) > 0\right\}.$$

Let μ_α be the projection of μ onto line in direction α. If $0 < s < 1$

$$\begin{aligned}\int_0^\pi \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\mu_\alpha(t)d\mu_\alpha(u)}{|t-u|^s}\right] d\alpha &= \int_0^\pi \left[\int_E\int_E \frac{d\mu(x)d\mu(y)}{|x\cdot\alpha - y\cdot\alpha|^s}\right] d\alpha \\ &\leq c\int_E\int_E \frac{d\mu(x)d\mu(y)}{|x-y|^s} < \infty\end{aligned}$$

Box-counting dimension

Recall that the box-counting dimensions of a non-empty and compact $E \subset \mathbb{R}^n$ are

$$\underline{\dim}_{\rm B} E \;=\; \liminf_{r\to 0} \frac{\log N_r(E)}{-\log r} \quad \text{and} \quad \overline{\dim}_{\rm B} E \;=\; \limsup_{r\to 0} \frac{\log N_r(E)}{-\log r}$$

where $N_r(E)$ is the least number of sets of diameter r covering E.
Is there a Marstrand-type theorem for box-dimensions of projections? For $E \subset \mathbb{R}^n$, for a.a. $\alpha \in G(n,m)$,

$$\frac{\underline{\dim}_{\rm B} E}{1 + (\frac{1}{m} - \frac{1}{n})\underline{\dim}_{\rm B} E} \leq \underline{\dim}_{\rm B}\mathrm{proj}_\alpha E \leq \min\{\underline{\dim}_{\rm B} E, m\} \;;$$

Examples show that these bounds are best possible.

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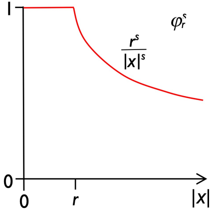
$$\frac{\underline{\dim}_{\rm B} E}{1 + (\frac{1}{m} - \frac{1}{n})\underline{\dim}_{\rm B} E} \leq \underline{\dim}_{\rm B} \mathrm{proj}_\alpha E \leq \min\{\underline{\dim}_{\rm B} E, m\} \; ;$$

Examples show that these bounds are best possible.

Even so, $\underline{\dim}_{\rm B} \mathrm{proj}_\alpha E$ and $\overline{\dim}_{\rm B} \mathrm{proj}_\alpha E$ must be constant for almost all α; for a messy argument and indirect value see (F & Howroyd, 1996, 2001). Using capacities things become much simpler.

Define kernels $\phi_r^s(x)$ for $s > 0$, $x \in \mathbb{R}^n$ by

$$\phi_r^s(x) = \begin{cases} 1 & 0 \le |x| < r \\ \left(\frac{r}{|x|}\right)^s & r \le |x| \end{cases}.$$



The reason for using this kernel is that (for $n = 2, m = 1$)

$$\phi_r^1(x-y) = \min\left\{1, \left(\frac{r}{|x-y|}\right)^s\right\} \asymp \mathcal{L}\{\alpha : |\text{proj}_\alpha(x-y)| \le r\} \ (x, y \in \mathbb{R}^2).$$

The capacity $C_r^s(E)$ of a compact $E \subset \mathbb{R}^n$ w.r.t. ϕ_r^s is

$$\frac{1}{C_r^s(E)} = \inf_{\mu \in \mathcal{M}(E)} \int\int \phi_r^s(x-y)d\mu(x)d\mu(y),$$

where $\mathcal{M}(E)$ are the probability measures on E. The infimum is attained by some equilibrium measure $\mu \in \mathcal{M}(E)$.

Box-counting dimensions of projections

Then for $E \subset \mathbb{R}^n$, with $N_r(E)$ the least number of sets of diameter r that can cover E,

$$c_1 C_r^s(E) \ \leq \ N_r(E) \ \leq \ \begin{cases} c_2 \log(1/r) \ C_r^s(E) & \text{if } s = n \\ c_2 \ C_r^s(E) & \text{if } s > n \end{cases} \qquad (1),$$

$(c_1, c_2$ independent of $r)$.

In particular for $E \subset \mathbb{R}^n$

$$\liminf_{r\to 0} \frac{\log C_r^n(E)}{-\log r} = \liminf_{r\to 0} \frac{\log N_r(E)}{-\log r} = \underline{\dim}_{\mathrm{B}} E.$$

Similarly for $\overline{\dim}_{\mathrm{B}}$ taking lim sup.

Note: Inequalities (1) fail if $0 < s < n$.

Box-counting dimensions of projections

Theorem Let $E \subset \mathbb{R}^n$ be non-empty compact.
Then

$$\overline{\dim}_B \operatorname{proj}_\alpha E \;\leq\; \limsup_{r\to 0} \frac{\log C_r^m(E)}{-\log r} \;\equiv \overline{\dim}_B^m E$$

with equality for almost all $\alpha \in G(n,m)$,

Similarly for $\underline{\dim}_B$ taking $\liminf$.

We call

$$\overline{\dim}_B^s E := \limsup_{r\to 0} \frac{\log C_r^s(E)}{-\log r} \quad (E \subset \mathbb{R}^n),$$

using capacity with respect to the kernels $\phi_r^s(x) = \min\left\{1, \left(\frac{r}{|x|}\right)^s\right\}$, the (upper)$s$-box-dimension profile of E, which should be thought of as the 'box-dimension of E when regarded from an s-dimensional viewpoint'.

Box-counting dimensions of projections

Lower bound proof (n=2, m=1): Let $F \subset \mathbb{R}$ be compact, ν a probability measure on F, and $\mathcal{I}_r(F)$ the intervals $[ir,(i+1)r),(i \in \mathbb{Z})$ that intersect F.

$$1 = \Big(\sum_{I \in \mathcal{I}_r(F)} \nu(I)\Big)^2 \leq N_r(F) \sum_{I \in \mathcal{I}_r(F)} \nu(I)^2 \leq$$

$$N_r(F) \sum_{I \in \mathcal{I}_r(F)} (\nu \times \nu)\{(w,z) \in I \times I\} \leq N_r(F)(\nu \times \nu)\{(w,z) : |w-z| \leq r\}. \quad (1)$$

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$$N_r(F)\sum_{I\in\mathcal{I}_r(F)}(\nu\times\nu)\{(w,z)\in I\times I\} \leq N_r(F)(\nu\times\nu)\{(w,z):|w-z|\leq r\}. \quad (1)$$

Let μ be an equilibrium measure for ϕ_r^1 on $E \subset \mathbb{R}^2$, and let μ_α be the projection of μ onto the line in direction α.

$$\int(\mu_\alpha\times\mu_\alpha)\{(w,z):|w-z|\leq r\}d\alpha = \int(\mu\times\mu)\{(x,y):|\mathrm{proj}_\alpha x-\mathrm{proj}_\alpha y|\leq r\}d\alpha$$

$$= \iint\mathcal{L}\{\alpha:|\mathrm{proj}_\alpha(x-y)|\leq r\}d\mu(x)d\mu(y) \leq c\iint\phi_r^1(x-y)d\mu(x)d\mu(y) = \frac{c}{C_r^1(E)}.$$

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Hence taking $\nu=\mu_\alpha$ and $F=\mathrm{proj}_\alpha E$ in (1) and integrating w.r.t. α:

$$\int\frac{d\alpha}{N_r(\mathrm{proj}_\alpha E)} \leq \frac{c}{C_r^1(E)}.$$

Box-counting dimensions of projections

As above

$$\int \frac{d\alpha}{N_r(\mathrm{proj}_\alpha E)} \leq \frac{c}{C_r^1(E)}.$$

If $\sum_k 2^{sk} C^1_{2^{-k}}(E)^{-1} < \infty$ then there are $M_\alpha < \infty$ for a.a. α such that

$$\frac{2^{sk}}{N_{2^{-k}}(\mathrm{proj}_\alpha E)} \leq M_\alpha \quad (\text{for all } k \in \mathbb{N}),$$

so , $N_{2^{-k}}(\mathrm{proj}_\alpha E) \geq 2^{sk}\frac{1}{M_\alpha}$.

Hence if $\overline{\dim}^1_{\mathrm{B}}(E) > s$ then $\overline{\dim}_{\mathrm{B}}(\mathrm{proj}_\alpha E) \geq s$ for almost all α.

Box-counting dimensions of projections

Upper bound proof (n=2, m=1): Recall that for $F \subset \mathbb{R}$,

$$c_1 C_r^1(F) \le N_r(F) \le c_2 \log(1/r) C_r^1(F).$$

With μ the equilibrium measure on $E \subset \mathbb{R}^2$, for all $x \in E$,

$$\frac{1}{C_r^1(E)} \le \int \phi_r^1(x-y)d\mu(y) \le \int \phi_r^1(\mathrm{proj}_\alpha x - \mathrm{proj}_\alpha y)d\mu(y)$$

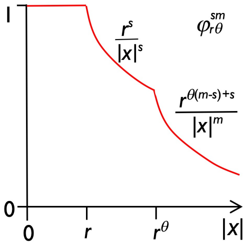
$$= \int \phi_r^1(z-w)d\mu_\alpha(w)$$

for all $z \in \mathrm{proj}_\alpha E$.
This is enough to imply that

$$N_r(\mathrm{proj}_\alpha E) \le c_2 \log(1/r) C_r^1(E).$$

Now define kernels $\phi_{r,\theta}^{s,m}$ for $0 \le s \le m$, $r > 0$ for $x \in \mathbb{R}^n$ by

$$\phi_{r,\theta}^{s,m}(x) = \begin{cases} 1 & 0 \le |x| < r \\ \left(\frac{r}{|x|}\right)^s & r \le |x| < r^\theta \\ \frac{r^{\theta(m-s)+s}}{|x|^m} & r^\theta \le |x| \end{cases}$$



Again the capacity $C_{r,\theta}^{s,m}(E)$ of $E \subset \mathbb{R}^n$ is given by

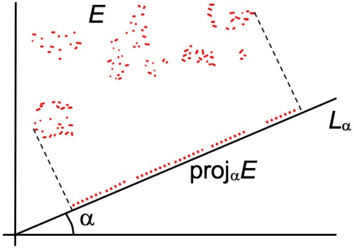
$$\frac{1}{C_{r,\theta}^{s,m}(E)} = \inf_{\mu \in \mathcal{M}(E)} \int\int \phi_{r,\theta}^{s,m}(x-y)d\mu(x)d\mu(y).$$

For $E \subset \mathbb{R}^n$ define for $1 \le m \le n$,

$$\underline{\dim}_\theta^m E = \left\{ \text{the unique } s \in [0,n] \text{ such that } \liminf_{r\to 0} \frac{\log C_{r,\theta}^{s,m}(E)}{-\log r} = s \right\},$$

Similarly for $\overline{\dim}_\theta^m E$. Then for $E \subset \mathbb{R}^n$

$$\underline{\dim}_\theta E = \underline{\dim}_\theta^n E \quad \text{and} \quad \overline{\dim}_\theta E = \overline{\dim}_\theta^n E.$$



Theorem Let $E \subset \mathbb{R}^2$ be a non-empty bounded Borel set and $\theta \in [0,1]$. Then

$$\underline{\dim}_\theta \mathrm{proj}_\alpha E \le \underline{\dim}^1_\theta F \text{ with equality for almost all } \alpha \in [0,\pi),$$

$$\overline{\dim}_\theta \mathrm{proj}_\alpha E \le \overline{\dim}^1_\theta F \text{ with equality for almost all } \alpha \in [0,\pi),$$

Similarly for projections in higher dimensions.

Thank you!