Operator limits of random matrices

I. Stochastic Airy

Brian Rider

Temple University

A random matrix

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Now with $\lambda_1, \lambda_2, \ldots, \lambda_n$ that spectrum, the typical eigenvalue distributes itself according to:

$$\frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k}(\lambda) \to \frac{1}{2\pi} \sqrt{4 - \lambda^2}\, d\lambda.$$

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We'll be interested in *local* fluctuations.

For example, it is clear that in the bulk an individual eigenvalue should experience $O(1/n)$ fluctuations. But I'll not talk about the bulk at all...

The Tracy-Widom law(s)

With slightly stronger assumptions on the matrix entries one has $\lambda_{max} \to 2$ and $\lambda_{min} \to -2$ with probability one.

A *local fluctuation* at the edge would be to ask weather there is exponent γ such that for some random variable ζ, one has

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In the mid-90's Craig Tracy and Harold Widom showed, in the complex Gaussian case ("GUE"):

$$\lim_{n\to\infty} P\Big(n^{2/3}(\lambda_{\max} - 2) \leq t\Big) = \exp\left(-\int_t^{\infty}(s-t)u^2(s)ds\right),$$

where u solves $u''(t) = tu(t) + 2u^3(t)$ (Painlevé II) with $u(t) \sim Ai(t)$ at $+\infty$.

Determinantal structure

The essential fact in the business is that GUE is "exactly solvable".

In particular, the joint density of eigenvalues of GUE is proportional to:

$$\prod_{k=1}^{n} e^{-\frac{1}{2} n \lambda_k^2} \times \prod_{j<k} |\lambda_j - \lambda_k|^2 \propto \det\Big(K_n(\lambda_i, \lambda_j)\Big)_{1 \le i,j \le n}$$

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In fact, all finite dimensional correlations have the same structure:

$$\int_{\mathbb{R}^{n-k}} \det\Big(K_n(\lambda_i, \lambda_j)\Big)_{1\le i,j\le n} d\lambda_{k+1}\cdots d\lambda_n = C_{n,k} \det\Big(K_n(\lambda_i, \lambda_j)\Big)_{1\le i,j\le k}.$$

(GUE is your favorite *determinantal process*).

Gaps

Any such determinantal process possesses a closed "gap formula". In particular, for a point process on $\mathbb{R}$ with correlations

$$P_n(\lambda_1, \ldots, \lambda_k) \propto \det\Big(K_n(\lambda_i, \lambda_j)\Big)_{1 \leq i,j, \leq k}$$

with K_n nonnegative, symmetric, trace class, it holds: for any $B \subset \mathbb{R}$

$$P\Big(\text{no points in } B\Big) = \det_{L^2(B)}(I - K_n).$$

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This is a Fredholm determinant on the right. In particular,

$$\det_{L^2(B)}(I - K_n) \\ := 1 - \int_B K_n(\lambda, \lambda) + \frac{1}{2}\int_B \int_B \det \begin{pmatrix} K_n(\lambda, \lambda) & K_n(\mu, \lambda) \\ K_n(\lambda, \mu) & K_n(\mu, \mu) \end{pmatrix} d\lambda d\mu - \cdots$$

In the case of $n < \infty$ points (like we have here) this truncates. That is to say you can treat the right hand side as a definition.

Airy kernel and process

A first form of the (soft-edge) Tracy-Widom law is then

$$F_2(t) := \lim_{n\to\infty} P\Big(n^{2/3}(\lambda_{\max} - 2) \le t\Big) = \det_{L^2[t,\infty)}(I - K_{Airy}).$$

Here

$$K_{Airy}(x, y) = \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x - y},$$

with Ai the Airy function from before.

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Painlevé formulas for the largest (and next largest...) point distributions come after.

Outside the complex case

If we go back to the start and replace the complex Gaussian entries with real or quaternion Gaussians, the eigenvalue density is changed as in:

$$\prod_{j<k} |\lambda_j - \lambda_k|^2 \text{ is replaced } \prod_{j<k} |\lambda_j - \lambda_k|^1 \text{ or } \prod_{j<k} |\lambda_j - \lambda_k|^4.$$

Speak of the $\beta = 1, 2,$ or 4 ensembles (or G{O,U,S}E).

When $\beta = 1$, 4, the eigenvalue processes are Pfaffian (not determinantal), but still exist closed formulas for the correlation functions in terms of OPs.

And there exist limit laws F_1 and F_4 for λ_{max} in terms of Painlevé II:

$$F_1(t) = \exp\left(-\frac{1}{2}\int_t^\infty u(s)ds\right) F_2^{1/2}(t),$$

$$F_4(t) = \cosh\left(\frac{1}{2}\int_{\sqrt{2}t}^\infty u(s)ds\right) F_2^{1/2}(\sqrt{2}t),$$

for the record

General beta ensembles

For any $\beta > 0$, introduce the law $P_{n,\beta}$ on n real points with density:

$$\propto \prod_{k=1}^{n} e^{-\frac{\beta}{4} n\lambda_k^2} \times \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$
$$= \exp\left[-\beta\left(n\sum_{k=1}^{n}\frac{\lambda_k^2}{4} - \sum_{j<k}\log|\lambda_j - \lambda_k|\right)\right].$$

For $\beta = 1, 2, 4$ these are the eigenvalue densities for G{O,U,S}E.

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Is there a one-parameter family of Tracy-Widom laws?

Stochastic Airy Operator

Theorem (Ramírez, R., Virág)

For $x \mapsto b(x)$ a standard Brownian motion, and any $\beta > 0$ define

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x).$$

Let $\Lambda_0 \le \Lambda_1 \le \cdots$ denote the eigenvalues of $\mathcal{H}_\beta$ acting on $L^2[0,\infty)$ with Dirichlet conditions at the origin. Then, with $\lambda_1 > \lambda_2 > \cdots$ the ordered points under $P_{n,\beta}$ it holds that

$$\left\{ n^{2/3}(2 - \lambda_\ell) \right\}_{\ell=1,k} \Rightarrow \left\{ \Lambda_\ell \right\}_{\ell=0,k-1}$$

for any fixed k as $n \to \infty$.

As $b'(x)$ is a random distribution (Brownian motion is almost everywhere non-differentiable), some work is required to make sense of $\mathcal{H}_\beta$

General beta Tracy-Widom

The limiting largest point of the Hermite β-ensemble then converges to the (*negative)* ground state eigenvalue of $\mathcal{H}_\beta$. In particular,

$$-TW_\beta = \inf_{f \in \mathcal{L}} \left\{ \int_0^\infty \left[(f'(x))^2 + x f^2(x) \right] dx + \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x) db(x) \right\}$$

for

$$\mathcal{L} = \left\{ f : f(0) = 0, \ \int_0^\infty f^2(x) dx = 1, \int_0^\infty \left[(f'(x))^2 + x f^2(x) \right] dx < \infty \right\}.$$

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Form is densely defined, and tempting to get a lower bound via

$$\left| \int_0^\infty f^2 db \right| = 2 \left| \int_0^\infty f'(x) f(x) b(x) dx \right| \le c \int_0^\infty (f')^2(x) dx + c' \int_0^\infty b^2(x) f^2(x) dx,$$

but the *law of the iterated log* shows you have to be a bit more clever (even for large beta).

Where does this come from?

For all $\beta > 0$ there is a simple tridiagonal matrix model for P_β.

Theorem (Dumitriu-Edelman)

Let $g_1, g_2, \ldots, g_n$ be independent $N(0,2)$ and $\chi_{\beta n}, \chi_{\beta(n-1)}, \ldots, \chi_\beta$ be independent "chi" variables of the indicated parameter. Then the joint distribution of eigenvalues of the random Jacobi matrix

$$H_{n,\beta} = \frac{1}{\sqrt{n\beta}} \begin{bmatrix} g_1 & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & g_2 & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & g_{n-1} & \chi_\beta \\ & & & \chi_\beta & g_n \end{bmatrix}$$

is given by $P_{n,\beta}$.

(A χ_r has density $\propto x^{r-1} e^{-x^2/2}$, otherwise referred to as a certain Γ variable),

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With $M = M_n = [m_{ij}]_{1 \le i,j \le n}$, $m_{ij} = \overline{m_{ji}}$ write

$$M = \begin{bmatrix} m_{ii} & \mathbf{m}^\dagger \\ \mathbf{m} & M_{n-1} \end{bmatrix}$$

and build a $(n-1) \times (n-1)$ unitary $U = [\mathbf{u}_1 \dots \mathbf{u}_{n-1}]$ with $\mathbf{m}^\dagger \mathbf{u}_1 = \|\mathbf{m}\|$. Then

$$\begin{bmatrix} 1 & \mathbf{0}^\dagger \\ \mathbf{0} & U^\dagger \end{bmatrix} M \begin{bmatrix} 1 & \mathbf{0}^\dagger \\ \mathbf{0} & U \end{bmatrix} = \begin{bmatrix} m_{ii} & (\|\mathbf{m}\|, 0 \ \cdots \ 0)^\dagger \\ (\|\mathbf{m}\|, 0 \ \cdots \ 0) & U^\dagger M_{n-1} U \end{bmatrix},$$

repeat.

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repeat.

Exercise: Convince yourself that when you carry out the above for GOE or GUE you get the advertised $\beta = 1$ or $\beta = 2$ tridiagonal. *Note:* (i) Gaussian vectors are rotation invariant, (ii) the squared norm of a d-dim Gaussian vector is a χ^2_d.

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With $T(A, B) = \text{tridiag}(B, A, B)$ for $B = (B_1, \ldots, B_{n-1}) \in \mathbb{R}^+_{n-1}$ and $A = (A_1, \ldots, A_n) \in \mathbb{R}_n$ their result reads:

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Distribute (A,B) according to the density

$$\propto e^{-n\frac{\beta}{4}(\sum_{i=1}^n a_i^2 + 2\sum_{i=1}^{n-1} b_i^2)} \prod_{i=1}^{n-1} b_i^{\beta(n-i)} = e^{-n\frac{\beta}{4}\,\mathrm{tr}\left(T^2(a,b)\right)} \prod_{i=1}^{n-1} b_i^{\beta(n-i)}$$

then the eigenvalues of $T(A,B)$ have density

$$\propto \prod_{k=1}^{n} e^{-\frac{\beta}{4}n\lambda_k^2} \times \prod_{j<k} |\lambda_j - \lambda_k|^\beta.$$

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The map needed is to go from tridiagonal (a, b)-coordinates to eigenvalue and eigenvector (really *norming constant*) (λ, q)-coordinates.

Stochastic Airy heuristics

Edelman-Sutton had conjectured the Stochastic Airy limit via the natural continuum limit of the tridiagonals. That is, they suggested that

$$n^{2/3}(2I - H_{n,\beta}) \rightsquigarrow -\frac{d^2}{dx^2} + x + \tfrac{2}{\sqrt{\beta}} b'(x)$$

as operators. (Scaling $H_{n,\beta}$ itself like λ_{max} in Tracy-Widom.)

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The only thing really moving in $H_{n,\beta}$ is those off diagonal χs.

Excerise: Make precise the statement that, for fixed k and $n \to \infty$, $\frac{1}{\sqrt{\beta n}} \chi_{\beta(n-k)} \simeq 1 - \frac{k}{2n} + \mathfrak{g}$ for $\mathfrak{g}$ a Gaussian.

This give the leading order $n^{2/3}(2I - H_{n,\beta}) = n^{2/3}\text{tridiag}(-1, 2, -1) + \cdots$ which has the clear interpretation as $-\frac{d^2}{dx^2}$, discretized on scale $(\Delta x) = n^{-1/3}$.

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Excerise: Convince yourself that the natural continuum interpretation of $n^{2/3}(\text{tridiag}(1, 0, 1) - H_{n,\beta})$ as $n \to \infty$ is $\otimes(x + \frac{2}{\sqrt{\beta}} b'(x))$.

The Riccati substitution

Consider $\tau = -\frac{d^2}{dx^2} + q(x)$ for a nice (deterministic, smooth) potential q and its Dirichlet eigenvalue problem on $[0, L < \infty]$

$$\tau\psi(x) = \lambda\psi(x), \quad \psi(0) = \psi(L) = 0.$$

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Sturm's Oscillation theorem tells you: Consider the corresponding solution $\psi = \psi(x, \lambda)$ for fixed λ to the initial value problem with $\psi(0, \lambda) = 0$ and $\psi'(0, \lambda) = 1$. Then it holds that

$$\#\Big\{\text{eigenvalues } \le \lambda\Big\} = \#\Big\{\text{zeros of } x \mapsto \psi(x, \lambda) \text{ in } [0, L]\Big\}.$$

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The Riccati substitution takes the equation satisfied by $p(x) = \frac{\psi'(x,\lambda)}{\psi(x,\lambda)}$:

$$p'(x) = q(x) - \lambda - p^2(x).$$

This starts at $p(0) = +\infty$, hits $-\infty$ when ψ hits zero, immediately "reappearing" at $+\infty$.

The Riccati diffusion

What this means for $q(x) = x + \frac{2}{\sqrt{\beta}} b'(x)$:

Theorem

Consider the solution $p_t = p_t^\lambda$ *to the Itô equation*

$$dp_t = \frac{2}{\sqrt{\beta}} db_t + (\lambda + t - p_t^2) dt,$$

started from $+\infty$ *at time zero, and restarted there after any explosion to* $-\infty$*. Then*

$$P(TW_\beta \leq \lambda) = P_{(+\infty,0)}(p^\lambda \textit{ never explodes}),$$

with the distribution of the k^{th} *largest point being given by the probability of at most* k *explosions.*

Note: Can absorb the spectral parameter λ into a starting time, or, replace the probabilities on the right with $P_{(+\infty,\lambda)}$ for $p = p^0$.

Exercise: Show that $p_t - \frac{2}{\sqrt{\beta}} b_t$ solves an ODE with random coefficients - convince yourself that the process really can be started from ∞

Application: Tracy-Widom(β) tails

Combining the defining variational principle

$$-TW_\beta = \inf_{f \in \mathcal{L}} \int_0^\infty \left[(f_x')^2 + x f_x^2\right] dx + \frac{2}{\sqrt{\beta}} \int_0^\infty f_x^2 db_x$$

with the Riccati diffusion description

$$P(TW_\beta \leq \lambda) = P_{(+\infty,\lambda)}(p \text{ never explodes}), \quad dp_t = \tfrac{2}{\sqrt{\beta}} db_t + (t - p_t^2) dt$$

we can prove:

Theorem (Ramírez, R., Virág)

For all $\beta > 0$ it holds

$$P(TW_\beta > a) = e^{-\frac{2}{3}\beta a^{\frac{3}{2}}(1+o(1))}$$

and

$$P(TW_\beta < -a) = e^{-\frac{\beta}{24} a^3 (1+o(1))}$$

as $a \to \infty$.

Proof of left-tail upper bound

Using that $-TW_\beta$ is the ground state eigenvalue of $\mathcal{H}_\beta$ one has

$$P(TW_\beta < -a) = P(\Lambda_0(\mathcal{H}_\beta) > a) \le P\left(\frac{\int (f_x'^2 + x f_x^2)dx] + \frac{2}{\sqrt{\beta}}\int f_x^2 db_x}{\int f_x^2 dx} > a\right)$$

for any nice function $f \not\equiv 0$ vanishing at the origin.

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$$P(TW_\beta < -a) = P(\Lambda_0(\mathcal{H}_\beta) > a) \leq P\left(\frac{\int (f_x'^2 + x f_x^2) dx] + \frac{2}{\sqrt{\beta}} \int f_x^2 db_x}{\int f_x^2 dx} > a \right)$$

for any nice function $f \not\equiv 0$ vanishing at the origin.

Exercise: For deterministic f it holds $\int f_x^2 db_x \sim \sqrt{\int f_x^4} \times \mathfrak{g}$ for $\mathfrak{g} \sim N(0,1)$.

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Choose

$$f(x) = (x\sqrt{a}) \wedge \sqrt{(a-x)^+} \wedge (a-x)^+$$

and collect:

$$a \int f_x^2 dx \sim \frac{a^3}{2}, \quad \int x f_x^2 dx \sim \frac{a^3}{6}, \quad \int f_x^4 dx \sim \frac{a^3}{3},$$

while $\int f'(x)^2 dx = O(a)$ to finish.

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We look at the event that the diffusion $dp_t = \frac{2}{\sqrt{\beta}} db_t + (t - p_t^2)dt$, started from position $+\infty$ at time $-a$ never explodes (hits $-\infty$).

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With that

$$
\begin{aligned}
P(TW_\beta < -a) &= P_{(\infty,-a)}(p \text{ never explodes }) \\
&\geq P_{(1,-a)}(p \text{ never explodes}) \\
&\geq P_{(1,-a)}(p_t \in [0,2] \text{ for all } t \in [-a,0]) P_{0,0}(p \text{ never explodes})
\end{aligned}
$$

What we've bought: The second factor has no dependence on $a \to \infty$.

Left-tail lower bound con't

Cameron-Martin-Girsanov: Let P denote the measure induced on continuous paths by the solution of $x_t = \sqrt{\sigma} b_t + \int_{\cdot}^{t} f(x_s) ds$. Over finite time windows this will be absolutely continuous to Brownian motion measure with

$$\frac{dP}{dBM}\bigg|_{\mathcal{F}[S,T]} = e^{\frac{1}{\sigma}\int_S^T f(b_t)db_t - \frac{1}{2\sigma}\int_S^T f^2(b_t)dt}$$

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Applied to p_t for which $f(p_t) = (t - p_t^2)$ over the widow $t \in [-a, 0]$:

$$\begin{aligned} P(TW_\beta < -a) \geq & \; c_\beta P_{(1,-a)}\Big(p_t \in [0,2] \text{ for all } t \in [-a,0]\Big) \\ = & \; c_\beta E_{(1,-a)}\left[1_A \, e^{\frac{\beta}{4}\int_{-a}^{0}(t-b_t^2)db_t - \frac{\beta}{8}\int_{-a}^{0}(t-b_t)^2 dt}\right] \end{aligned}$$

with $A = \{b_t \in [0,2], t \in [-a,0]\}$.

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with $A = \{b_t \in [0,2], t \in [-a,0]\}$.

Exercise: Granted Itô's rule $f(b_t) - f(b_0) = \int_0^t f'(b_t)db_t + \frac{1}{2}\int_0^t f''(b_t)dt$ finish the job.

Operator limits of random matrices

II. Stochastic Airy: proofs and extensions

Brian Rider

Temple University

Task for the hour

(1) Show that Stochastic Airy

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x)$$

(on $\mathbb{R}_+$ with Dirichlet boundaries) can be made sensible.

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(2) Show the β-Hermite matrix $H_{n,\beta}$, with

$$\frac{g_1}{\sqrt{n\beta}}, \frac{g_2}{\sqrt{n\beta}}, \dots \text{ on diagonal}$$

and

$$\frac{\chi_{\beta(n-1)}}{\sqrt{n\beta}}, \frac{\chi_{\beta(n-2)}}{\sqrt{n\beta}}, \dots \text{ on the off-diagonals}$$

satisfies

$$n^{2/3}(2I - H_{n,\beta}) \to \mathcal{H}_\beta \text{ in some operator sense.}$$

(3) Payoffs for other beta ensembles.

Return to the quadratic form

Advertised that $-TW_\beta$ can be defined as the infimum of

$$\langle f, \mathcal{H}_\beta f\rangle = \int_0^\infty [(f')^2(x) + xf^2(x)]dx + \frac{2}{\sqrt{\beta}}\int_0^\infty f^2(x)db_x$$

over f satisfying $f(0) = 0$, $\int_0^\infty f^2(x) = 1$, $\int_0^\infty [(f')^2(x) + xf^2(x)]dx < \infty$ (i.e., $f \in \mathcal{L}$).

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Decompose

$$b_x = \bar{b}_x + (b_x - \bar{b}_x), \quad \bar{b}_x = \int_x^{x+1} b_y dy$$

and then

$$\langle f, b' f\rangle = \int_0^\infty f^2(x)\bar{b}'_x dx + 2\int_0^\infty f'(x)f(x)(\bar{b}_x - b_x)dx.$$

and least for smooth compactly supported f.

Key inequality

For any $c > 0$ there is an almost surely finite $C(c, b)$ with

$$\left| \int_0^\infty f^2(x) db_x \right| \leq c \int_0^\infty [(f')^2(x) + x f^2(x)] dx + C(c, b) \int_0^\infty f^2(x) dx.$$

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Recall from above, first for "nice" test functions,

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Exercise: There is an $C(b) < \infty$ (almost surely) so that

$$\sup_{x>0} \sup_{0<y\leq 1} \frac{|b_{x+y} - b_x|}{\sqrt{\log(1+x)}} \leq C(b).$$

It follows that $|\bar{b}'_x|$ and $|\bar{b}_x - b_x|$ are similarly bounded. (Just uses that b_x has independent homogeneous increments, and a bound on $P_0(\sup_{x<1} |b_x| > c)$).

Existence of the groundstate

Let's introduce the natural norm on $\mathcal{L}$:

$$\|f\|_*^2 = \int_0^\infty [(f')^2(x) + (1+x)f^2(x)]dx.$$

Then what we have shown can be summarized as: there are constants c (deterministc) and C, C' (random) such that for all $f \in \mathcal{L}$

$$c\|f\|_*^2 - C\|f\|_2^2 \leq \langle f, \mathcal{H}_\beta f\rangle \leq C'\|f\|_*^2.$$

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- The (a.s.) uniform bound on $\|f_n\|_*$ produces a subsequence $f_{n'} \to f_0$ occuring: weakly in H^1, uniformly on compacts, and in L^2.
- From here can conclude $\langle f_0, \mathcal{H}_\beta f_0 \rangle = \tilde{\Lambda}_0$. (And $\tilde{\Lambda}_0 = \Lambda_0 = -TW_\beta$.)

Higher eigenvalues and more

We can now define $\Lambda_1 < \Lambda_2 < \cdots$ by Rayleigh-Ritz, for example

$$\tilde{\Lambda}_1 := \inf_{f \in \mathcal{L}, f \perp f_0} \langle f, \mathcal{H}_\beta f \rangle.$$

The same type of argument will show a pair $(\tilde{\Lambda}_1, f_1)$ exists. Then can check it is an eigenvalue/eigenvector (and announce the former $= \Lambda_1$).

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Exercise: For any $\epsilon > 0$ there is a random C so that

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in the sense of operators (quadratic forms).

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Exercise: Granted the classical asymptotics $\lambda_k(\mathcal{A}) = (\frac{3}{2}\pi k)^{2/3} + o(1)$, show that

$$k^{-2/3}\Lambda_k \to (\frac{3}{2}\pi)^{2/3}$$

with probability one.

Convergence proof: setup

Bring back the matrix model $H_{n,\beta}$ with $\frac{1}{\sqrt{\beta n}} g_k$ and $\frac{1}{\sqrt{\beta n}} \chi_{\beta(n-k)}$ on the diagonals/offdiagonals. No controversy to declare:

$$TW_\beta(n) := \min_{\|v\|=1} \langle v, \hat{H}_{n,\beta} v \rangle, \quad \hat{H}_{n,\beta} = n^{2/3}(2I - H_{n,\beta}).$$

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Now write:

$$\langle v, \hat{H}_{n,\beta} v \rangle = n^{2/3} \sum_{k=0}^{n} (v_{k+1} - v_k)^2 + \sum_{k=0}^{n} \eta_{n,k} v_k v_{k+1}$$
$$+ \tfrac{2}{\sqrt{\beta}} \sum_{k=0}^{n} y_{n,k}^{(1)} v_k^2 + y_{n,k}^{(2)} v_k v_{k+1}$$

in which $v_0 = v_{n+1} = 0$ and

$$\eta_{n,k} = \tfrac{2}{\sqrt{\beta}} n^{1/6} (\sqrt{\beta n} - E\chi_{\beta(n-k)}), \quad y_{n,k}^{(1)} = -\frac{1}{2} n^{1/6} g_k$$

and $y_{n,k}^{(2)}$ a centered/scaled $\chi_{\beta(n-k)}$.

Convergence proof: improved heuristics

Want to show

$$\min_{\|v\|=1} \langle v, \hat{H}_{n,\beta} v \rangle \to \inf_{f \in \mathcal{L}} \langle f, \mathcal{H}_\beta f \rangle.$$

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Embed the discrete minimization problem in L^2: any $v \in \mathbb{R}^n$ is identified with a piecewise constant $f_v(x) = v(\lceil n^{1/3} x \rceil)$ for $x \in [0, \lceil n^{2/3} \rceil]$, $f_v = 0$ otherwise.

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With this point of view better to consider

$$\langle v, \hat{H}_{n,\beta} v \rangle = n^{1/3} \sum_{k=0}^{n} (v_{k+1} - v_k)^2 + n^{-1/3} \sum_{k=0}^{n} \eta_{n,k} v_k v_{k+1}$$

$$+ \tfrac{2}{\sqrt{\beta}} n^{-1/3} \sum_{k=0}^{n} y_{n,k}^{(1)} v_k^2 + y_{n,k}^{(2)} v_k v_{k+1}$$

A calculation shows:

$$n^{-1/3} \sum_{k=1}^{\lceil n^{1/3} x \rceil} \eta_{n,k} \to \frac{x^2}{2}, \quad \tfrac{2}{\sqrt{\beta}} n^{-1/3} \sum_{k=1}^{\lceil n^{1/3} x \rceil} (y_{n,k}^{(1)} + y_{n,k}^{(2)}) \Rightarrow \tfrac{2}{\sqrt{\beta}} b_x.$$

Convergence proof: An actual estimate

Need to show the discrete quadratic form is bounded below, as $n \to \infty$.

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Very much as in the proof that Stochastic Airy is well defined: show the noise part of the form can be controlled by deterministic part: e.g., for any $c > 0$,

$$\left| n^{-1/3} \sum_{k=0}^{n} y_{n,k}^{(1)} v_k^2 \right| \leq c \|v\|_{n,*} + C_n \sum_{k=1}^{n} v_k^2 n^{-1/3}$$

where

$$C_n = C_n(y^{(1)}, c) \text{ is a } \textit{tight} \text{ random sequence}$$

and

$$\|v\|_{n,*}^2 = \sum_{k=0}^{n} n^{1/3} (v_{k+1} - v_k)^2 + \sum_{k=0}^{n} k n^{-2/3} v_k^2$$

is the analog of our $\| \cdot \|_*^2$ norm from before.

And similarly for the $y^{(2)}$ noise term.

Convergence proof: Now what?

The bound just described gives (also similar to the continuum):

$$c\|v\|_{n,*}^2 - C_n\|v\|_{\ell_2}^2 \le \langle v, \hat{H}_{n,\beta} v\rangle \le C_n'\|v\|_{n,*}^2$$

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• Can select a subsequence of eigenvalue and (normalized) eigenvectors $(\lambda_0(n'), v_{n'})$ such that you have the convergence

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• Gives at least

$$\lambda_* = \langle f_*, \mathcal{H}_\beta f_*\rangle \ge \Lambda_0 = -TW_\beta$$

for any such limit point... (and that limit point is an eigenvalue of $\mathcal{H}_\beta$...)

Other ensembles: Wishart matrices

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The appropriate general beta version is to take the density on n positive points with joint density: for $\beta > 0$ and $\kappa > n-1$

$$P^{\beta}_{n,\kappa}(\lambda_1, \dots, \lambda_n) \propto \prod_{j<k} |\lambda_j - \lambda_k|^{\beta} \times \prod_{k=1}^{n} \lambda_k^{\frac{\beta}{2}(\kappa-n+1)-1} e^{-\frac{\beta}{2} n \lambda_k}.$$

(When $\beta = 1, 2, 4$ and $\kappa = m \in \mathbb{Z}$ this realizes the $MM^\dagger$ real, complex, or quaternion Gaussian Wishart ensemble.)

β-Laguerre

There is again a tridiagonal matrix model, due to Dumitriu-Edelman.

Let $B = B_{n,\beta,\kappa}$ be the random upper bidiagonal

$$B = \frac{1}{\sqrt{\beta n}} \begin{bmatrix} \chi_{\beta\kappa} & \chi_{\beta(n-1)} & & & \\ & \chi_{\beta(\kappa-1)} & \chi_{\beta(n-2)} & & \\ & & \ddots & & \\ & & & \chi_{\beta(\kappa-n+2)} & \chi_{\beta} \\ & & & & \chi_{\beta(\kappa-n+1)} \end{bmatrix},$$

with all variables independent.

Then the eigenvalues of $W = BB^{\dagger}$ have joint density $P^{\beta}_{n,\kappa}$.

β-Laguerre

There is again a tridiagonal matrix model, due to Dumitriu-Edelman.

Let $B = B_{n,\beta,\kappa}$ be the random upper bidiagonal

$$B = \frac{1}{\sqrt{\beta n}} \begin{bmatrix} \chi_{\beta\kappa} & \chi_{\beta(n-1)} & & & \\ & \chi_{\beta(\kappa-1)} & \chi_{\beta(n-2)} & & \\ & & \ddots & & \\ & & & \chi_{\beta(\kappa-n+2)} & \chi_{\beta} \\ & & & & \chi_{\beta(\kappa-n+1)} \end{bmatrix},$$

with all variables independent.

Then the eigenvalues of $W = BB^{\dagger}$ have joint density $P^{\beta}_{n,\kappa}$.

Exercise: For M an $m \times n$ matrix of independent real/complex Gaussians show there are U and V unitary with $UMV =$ the advertised B.

Tracy-Widom(β) for β-Laguerre

Previous procedure gives:

Theorem (Ramírez, R, Virág)

Let $\lambda_1 \geq \lambda_2 \ldots$ denote the ordered β-Laguerre eigenvalues and set

$$\mu_{n,\kappa} = (\sqrt{n} + \sqrt{\kappa})^2, \quad \text{and} \quad \sigma_{n,\kappa} = \frac{(\sqrt{n\kappa})^{1/3}}{(\sqrt{n} + \sqrt{\kappa})^{4/3}}.$$

Then for any k, as $n \to \infty$ with arbitrary $\kappa = \kappa_n > n - 1$ we have

$$\Big(\sigma_{n,\kappa}(\mu_{n,\kappa} - \lambda_\ell)\Big)_{\ell=1,\ldots,k} \Rightarrow \Big(\Lambda_0, \Lambda_1, \ldots, \Lambda_{k-1}\Big),$$

the ordered eigenvalues for Stochastic Airy.

An application: Spikes

Johnstone raised the question: What happens to Tracy-Widom for non-null Wishart ensembles?

Or, what is λ_{max} for $M\Sigma M^{\dagger}$ for "general" $\Sigma \neq I$? Even in the "spiked" case: $\Sigma = \Sigma_r \oplus I_{n-r}$, for $\Sigma_r = \mathrm{diag}(c_1, \ldots, c_r)$.

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If $c = \mathfrak{c} - wn^{-1/3}$: $P\Big(\sigma_n(\lambda_{\max} - \mu_n) \leq t\Big) \to F(t,w) = F_2(t)f(t,w)$ where f can again be described in terms of Painlevé II.

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That $\beta = 2$ is absolutely critical to the analysis.

Spiked beta ensemble

Can still tri-diagonalize. Get the same product of random bidiagonal B matrices, but with a multiplicative shift by $\sqrt{c}$ in the $(1,1)$ entry. (*Exercise*?)

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Theorem (Bloemendal-Virág)

At criticality, the appropriately scaled $B_c B_c^\dagger$ *with* $c = \mathfrak{c} - wn^{-1/3}$*, converges in the now familiar operator sense to*

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x),$$

but subject now to $f'(0) = wf(0)$ *at the origin.*

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So have a "general beta spiked" Tracy-Widom law $TW_{\beta,w}$, with $TW_\beta = TW_{\beta,\infty}$

PDE for $TW_{\beta,w}$ distributions

Can again use the Riccati trick.

The Robin boundary condition means that any $x \mapsto \psi(x,\lambda)$ satisfying $\mathcal{H}_\beta \psi = \lambda\psi$ is subject to $(\psi(0,\lambda), \psi'(0,\lambda)) = (1, w)$, or $p(0,\lambda) = \frac{\psi'(0,\lambda)}{\psi(0,\lambda)} = w$.

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The upshot is:

$$P(TW_{\beta,w} \leq \lambda) = P_{\lambda,w}(p \text{ never explodes}),$$

with p our friend from before: $dp_t = \frac{2}{\sqrt{\beta}} db_t + (t - p_t^2)dt$, now begun at place w at time λ.

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Now view $F(\lambda, w) = F_\beta(\lambda, w) = P(TW_{\beta,w} \le \lambda)$ as a hitting distribution for the "space-time" Markov process (p_t, t). By general theory any such function is killed by the *generator*:

$$\frac{\partial F}{\partial \lambda} + \frac{2}{\beta}\frac{\partial^2 F}{\partial^2 w} + (\lambda - w^2)\frac{\partial F}{\partial w} = 0.$$

This PDE has been used by Rumanov to find the first Painlevé formulas for TW_β outside of $\beta = 1, 2, 4$ - for $\beta = 6$!

Operator limits of random matrices

III. Hard edge

Brian Rider

Temple University

Back to complex Wishart

Have $n \times m$ matrices M of independent complex Gaussians and form the appropriately scaled $\frac{1}{n} M M^\dagger$.

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$$\frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k}(\lambda) \to \sqrt{(\lambda - \ell)(r - \lambda)} \, \frac{d\lambda}{2\pi\lambda}$$

where $\ell = (1 - \sqrt{\gamma})^2$ and $r = (1 + \sqrt{\gamma})^2$

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When $\gamma = 1$, then $\ell = 0$ and eigenvalues feel the "hard edge" of the origin.

In fact, if $m = n + a$ as $n \uparrow \infty$ there is a one-parameter family of limit laws for λ_{min} (also due Tracy-Widom).

Hard edge kernel/process

Using the determinantal structure: with $m - n \equiv a$ as $n \to \infty$ it holds,

$$\mathbb{P}\Big(n^2 \lambda_{min} \geq t\Big) \to \det{}_{L^2[0,t)}(I - K_{Bessel})$$

where

$$K_{Bessel}(x,y) = \frac{J_a(\sqrt{x})\sqrt{y}J_a'(\sqrt{y}) - J_a(\sqrt{y})\sqrt{x}J_a'(\sqrt{x})}{x-y},$$

and J_a is the Bessel function of first kind. (The Fredholm determinant itself can be expressed in terms of Painlevé V).

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As $a \to \infty$ recover the soft edge:

$$a^{4/3} K_{Bessel}(a^2 - a^{4/3}\lambda, a^2 - a^{4/3}\mu) \to K_{Airy}(\lambda, \mu),$$

with similar statement at the point distribution (Painlevé) level.

General beta

Tuned for the hard edge (and in a slightly different form then before), define:

$$B = \frac{1}{\sqrt{n\beta}} \begin{bmatrix} \chi_{(n+a)\beta} & -\chi_{(n-1)\beta} & & & \\ & \chi_{(n+a-1)\beta} & -\chi_{(n-2)\beta} & & \\ & & \ddots & \ddots & \\ & & & \chi_{(a+2)\beta} & -\chi_{\beta} \\ & & & & \chi_{(a+1)\beta} \end{bmatrix}$$

here $a > -1$, $\beta > 0$ and all entries are independent.

Then, the eigenvalues of $W = BB^{\dagger}$ have joint density

$$P_{\beta,a} \propto \prod_{k=1}^{n} \lambda_k^{\frac{\beta}{2}(a+1)-1} e^{-\frac{\beta}{2} n \lambda_k} \cdot \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}.$$

Hard edge operator

Here's a version of the result:

Theorem (Ramírez, R.)

For all $\beta > 0$, $a > -1$ *and* $x \mapsto b_x$ *a standard Brownian motion define*

$$\tau = \tau_{\beta,a} = -e^x \left(\frac{d^2}{dx^2} - \left(a + \tfrac{2}{\sqrt{\beta}} b'_x\right) \frac{d}{dx} \right).$$

Acting on functions supported on $\mathbb{R}_+$ *which vanish at the origin* τ *has eigenvalues* $0 < \Lambda_0(\beta, a) < \Lambda_1(\beta, a) < \cdots$. *Also, as* $n \to \infty$ *and for any fixed* k

$$\{n^2 \lambda_i\}_{i=1,\dots,k} \Rightarrow \{\Lambda_0(\beta, a), \dots, \Lambda_{k-1}(\beta, a)\}$$

for $\{\lambda_i\}_{i=1,2,\dots}$ *the ordered points of* $P_{\beta,a}$.

Other formulations of $\tau_{\beta,a}$

While it is suggestive to write τ as the (negative of the) generator for a "Brownian motion with white noise drift", perhaps better to note

$$-\tau = \frac{1}{m(x)} \frac{d}{dx} \frac{1}{s(x)} \frac{d}{dx}$$

with $m(x) = e^{-(a+1)x - \frac{2}{\sqrt{\beta}} b(x)}$ and $s(x) = e^{ax + \frac{2}{\sqrt{\beta}} b(x)}$.

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Then the eigenvalue problem $-\tau f = \lambda f$ can be written as a system:

$$f'(x) = s(x)g(x), \quad g'(x) = \lambda m(x) f(x), \quad (f(0), g(0)) = (0, 1)$$

and $g(x) = s(x)^{-1} f'(x)$ can be solved for in C^1.

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In other words, τ is really a "classical" Sturm-Louiville operator.

The integral operator $\tau_{\beta,a}^{-1}$

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Better still:

$$(\tau^{-1} f)(x) = \int_0^\infty \left(\int_0^{x \wedge y} s(z) dz \right) f(y) m(y) dy$$

is (a.s.) non-negative and compact in $L^2[\mathbb{R}_+, m(dx)]$.

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Further $\tau^{-1} = \kappa^\dagger \kappa$ with

$$(\kappa f)(x) = e^{x/2} \int_x^\infty e^{\frac{a+1}{2}(x-y) + \frac{1}{\sqrt{\beta}}(b_y - b_x)} f(y) dy.$$

This kernel satisfies $\int_0^\infty \int_0^\infty |\kappa(x,y)|^2 dm(x) dm(y) < \infty$ (so τ^{-1} is product Hilbert-Schmidt).

Noting the matrix model $W = BB^\dagger$ has the same structure, we actually pin down the integral operator limit of $(nB)^{-1}$.

Embedding

$A = a_{ij} \in \mathbb{R}^{n\times n}$ can be embedded into $L^2[0,1]$ without changing the spectrum: for $x_i = i/n$ for $i = 0, 1, \ldots, n$ and $f \in L^2[0,1]$,

$$(Af)(x) := \sum_{j=1}^{n} a_{ij} n \int_{x_{j-1}}^{x_j} f(x)dx, \quad \text{when } x_{i-1} \leq x < x_i.$$

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By the inversion formula for bidiagonal matrices we can view $(nB)^{-1}$ as an $(L^2[0,1] \mapsto L^2[0,1])$ integral operator with the discrete upper-triangular kernel

$$k_n(x,y) = \frac{\sqrt{\beta n}}{\chi_{\beta(n+a-i)}} \prod_{k=i+1}^{j} \frac{\chi_{\beta(n-k)}}{\chi_{\beta(n+a-k)}} \mathbf{1}_{\Gamma_{ij}},$$

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Exercise: Convince yourself of all that!

Pointwise limit of the kernel

A bit more streamlined:

$$k_n(x,y) \simeq \frac{\sqrt{\beta n}}{\chi_{\beta([n(1-x)]+a)}} \exp\left[\sum_{k=[nx]}^{[ny]} \log \frac{\widetilde{\chi}_{\beta(n-k)}}{\chi_{\beta(n+a-k)}}\right] \mathbf{1}_{x<y}.$$

The most complicated bit here is a sum of independent variables.

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Exercise: For fixed $x \in [0,1)$

$$\frac{\sqrt{\beta n}}{\chi_{\beta(n+a-\lfloor nx \rfloor)}} \Rightarrow \frac{1}{\sqrt{1-x}}, \qquad \sum_{k=1}^{[nx]} \log \frac{\chi_{\beta(n-k)}}{\chi_{\beta(n+a-k)}} \Rightarrow N\left((1-x)^{a/2}, \frac{1}{\beta}\log\frac{1}{(1-x)}\right)$$

in law.

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The process version of this produces $k_n(x,y) \to k_{\beta,a}(x,y)$ where

$$k_{\beta,a}(x,y) := (1-x)^{-\frac{1+a}{2}} \exp\left[\int_x^y \frac{db_z}{\sqrt{\beta(1-z)}}\right] (1-y)^{a/2}\, \mathbf{1}_{x<y}.$$

and b_z is a Brownian motion.

Putting it together

The inverse of the full matrix model should then converge to the integral operator with kernel $(k^T k)(x,y) =$

$$(1-x)^{a/2} e^{-\int_0^x \frac{db_t}{\sqrt{\beta(1-t)}}} \left(\int_0^{x \wedge y} \frac{e^{2\int_0^z \frac{db_t}{\sqrt{\beta(1-t)}}}}{(1-z)^{a+1}} dz \right) (1-y)^{a/2} e^{-\int_0^y \frac{db_t}{\sqrt{\beta(1-t)}}}$$

on $L^2[0,1]$.

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on $L^2[0,1]$.

Get the advertised limit by a change of variable:

$$(k^T k)(1-e^{-x}, 1-e^{-y}) e^{-x/2} e^{-y/2} = \left(\int_0^{x \wedge y} s(z) dz \right) [m(x)m(y)]^{\frac{1}{2}},$$

on $L^2[0,\infty)$.

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on $L^2[0,1]$.

Get the advertised limit by a change of variable:

$$(k^T k)(1-e^{-x}, 1-e^{-y}) e^{-x/2} e^{-y/2} = \left(\int_0^{x \wedge y} s(z) dz \right) [m(x)m(y)]^{\frac{1}{2}},$$

on $L^2[0,\infty)$.

You'll recall: $\tau^{-1}(x,y) = \left(\int_0^{x \wedge y} s(z)dz\right) m(y)$ on $L^2[[0,\infty), m]$ and that

$$s(x) = e^{ax + \frac{2}{\sqrt{\beta}} b_x}, \quad m(x) = e^{-(a+1)x - \frac{2}{\sqrt{\beta}} b_x}.$$

What actually gets proved

Let $K_{\beta,a}$ be the integral operator on $L^2[0,1]$ with kernel

$$k_{\beta,a}(x,y) = (1-x)^{-\frac{1+a}{2}} e^{\int_x^y \frac{db_z}{\sqrt{\beta(1-z)}}} (1-y)^{a/2} \mathbf{1}_{x<y}$$

and K_n the integral operator derived form the embedded bidiagonal random matrix $(nB)^{-1}$, with kernel

$$k_n(x,y) \simeq \frac{\sqrt{\beta n}}{\chi_{\beta([n(1-x)]+a)}} \exp\left[\sum_{k=[nx]}^{[ny]} \log \frac{\widetilde{\chi}_{\beta(n-k)}}{\chi_{\beta(n+a-k)}}\right] \mathbf{1}_{x<y}$$

also acting in $L^2[0,1]$.

Theorem (Ramírez, R.)

For any sequence of the operators K_n, there is a subsequence $n' \to \infty$ and suitable probability space on which

$$P\left(\lim_{n'\to\infty} \int_0^1 \int_0^1 \Big| k_{n'}(x,y) - k_{\beta,a}(x,y) \Big|^2 dxdy = 0\right) = 1.$$

Fun Fact

Return to the β-Laguerre density:

$$c_{n,\beta} \prod_{i \neq j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{n} w(\lambda_i), \quad w(\lambda) = \lambda^{\frac{\beta}{2}(a+1)-1} e^{-\frac{\beta}{2} n\lambda}.$$

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When $\frac{\beta}{2}(a+1) = 1$ (e.g, $\beta = 2$ and $a = 0$) immediate that

$$P(\lambda_{min} > t) = c_{n,\beta} \int_t^\infty \cdots \int_t^\infty \prod_{i \neq j} |\lambda_i - \lambda_j|^\beta e^{-\beta \frac{n}{2} \sum_{k=1}^n \lambda_k} d\lambda_1 \dots d\lambda_n = e^{-\beta \frac{n^2}{2} t},$$

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This means for example that:

$$\inf_{f \not\equiv 0, f(0)=0} \frac{\int_0^\infty (f_x')^2 e^{\frac{2}{\sqrt{\beta}} b_x - \frac{2}{\beta} x} dx}{\int_0^\infty (f_x)^2 e^{\frac{2}{\sqrt{\beta}} b_x - \frac{2}{\beta} x} dx} \sim \exp(\beta/2),$$

but I have no direct proof of this.

Riccati at the hard edge

Write out $\psi(t) = \lambda \tau^{-1} \psi(t)$:

$$\psi(t) = \lambda \int_0^\infty \left(\int_0^{t \wedge s} e^{au + \frac{2}{\sqrt{\beta}} b_u} \, du \right) \psi(s) e^{-(a+1)s - \frac{2}{\sqrt{\beta}} b_s} \, ds.$$

Read off that $\psi(0) = 0$.

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Taking one derivative throughout, followed by an Itô differential gives the system:

$$\begin{aligned} d\psi_t' &= \tfrac{2}{\sqrt{\beta}} \psi_t' db_t + \left((a + \tfrac{2}{\beta}) \psi_t' - \lambda e^{-t} \psi_t \right) dt, \\ d\psi_t &= \psi_t' dt, \end{aligned}$$

And $q = \frac{\psi'}{\psi}$ solves:

$$dq_t = \tfrac{2}{\sqrt{\beta}} q_t db_t + ((a + \tfrac{2}{\beta}) q_t - q_t^2 - \lambda e^{-t}) dt.$$

Passages of this process (started at $+\infty$) will count eigenvalues of τ.

Riccati at the hard edge - more precise

Theorem (Ramírez, R.)

Take the law induced by q defined by

$$dq_t = \frac{2}{\sqrt{\beta}} q_t db_t + ((a + \tfrac{2}{\beta})q_t - q_t^2 - \lambda e^{-t})dt.$$

started at $+\infty$, and restarted at $+\infty$ after any passage to $= -\infty$. Then,

$$P(\Lambda_0(\tau_{\beta,a}) > \lambda) = P_{(+\infty,0)}(q \textit{ never hits } 0),$$
$$P(\Lambda_k(\tau_{\beta,a}) < \lambda) = P_{(+\infty,0)}(p \textit{ hits } 0 \textit{ at least } k+1 \textit{ times}).$$

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An easier observation: When $a \geq 0$ the process q will hit $-\infty$ with probability one once it hits zero.

General hard to soft transition

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In fact it holds that:

Theorem (Ramírez, R.)

For all $\beta > 0$,

$$\frac{a^2 - \Lambda_0(\tau_{\beta,2a})}{a^{4/3}} \Rightarrow TW_\beta$$

as $a \to \infty$.

Have a proof via Riccati - haven't succeeded in showing this directly through the operators.

Proof for the transition (sketch)

On one hand, $P(TW_\beta \leq \lambda)$ is the probability that

$$dp_t = \tfrac{2}{\sqrt{\beta}} db_t + (t + \lambda - p_t^2) dt$$

never hits $-\infty$ (started from $+\infty$). And for $a \gg 1$, $P(\Lambda_0(\tau_{\beta,2a}) > \mu)$ is the probability that

$$dq_t = \tfrac{2}{\sqrt{\beta}} q_t db_t + ((2a + \tfrac{2}{\beta}) q_t - q_t^2 - \mu e^{-t}) dt$$

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never hits $-\infty$ (started from $+\infty$).

Should be enough to show there is the convergence

$$\Big(t \mapsto q_t^{2a,\mu}, \quad \mu = a^2 - a^{4/3}\lambda\Big) \Rightarrow \Big(t \mapsto p_t^\lambda\Big),$$

as measures on paths (started from $+\infty$).

Proof for the transition (sketch con't)

Given $q = q^{2a, a^2 - a^{4/3}\lambda}$ satisfies:

$$dq_t = \frac{2}{\sqrt{\beta}} q_t db_t + \left((2a + \tfrac{2}{\beta}) q_t - q_t^2 - (a^2 - a^{4/3}\lambda) e^{-t} \right) dt$$

make the change of variables

$$\eta(t) = a^{-2/3} q(a^{-2/3} t) - a^{1/3},$$

noting $\eta_0 = +\infty$ and η_t hits $-\infty$ if and only if q does.

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noting $\eta_0 = +\infty$ and η_t hits $-\infty$ if and only if q does. Then:

$$\begin{aligned} d\eta_t &= \tfrac{2}{\sqrt{\beta}} \Big[1 + a^{-1/3} \eta_t \Big] db_t \\ &\quad + \Big[\lambda e^{-a^{-2/3} t} + a^{2/3} (1 - e^{-a^{-2/3} t}) - \eta_t^2 + \tfrac{2}{\beta} (a^{-1/3} + a^{-2/3} \eta_t) \Big] dt \\ &\sim \tfrac{2}{\sqrt{\beta}} db_t + [\lambda + t - \eta_t^2] dt, \end{aligned}$$

for bounded sets of time and space. And this is just the equation for the TW_β Riccati diffusion.

Operator limits of random matrices

IV. Universality and exotic limits

Brian Rider

Temple University

Universality

Back in the *measure on matrices* worldview, the natural form of universality would be to ask whether replacing say

$$GUE : \quad e^{-n\frac{1}{2}\mathrm{tr} M^2} dM$$

(where $dM =$ Lebesgue measure on the space of $n \times n$ Hermitian matrices) with

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alters *local* statistics.

Importantly, these ensembles maintain the same analytic structure at the eigenvalue density level:

$$\propto e^{-n\sum_{k=1}^{n} V(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^2 = \det\Big(K_n^V(\lambda_i, \lambda_j)\Big)$$

with K_n^V the projection kernel onto the span of the first n OPs for weight $e^{-nV(\lambda)}$ on $\mathbb{R}$.

RHPs and $\beta = 1, 2, 4$

Sticking with $\beta = 2$ for a moment, the universality of any local statistic is passed onto the universality of the appropriately scaled K_n^V.

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This in turn is passed onto asymptotics for the family of OPs with nonclassical weight(s) $e^{-nV(\lambda)}$, and the mighty hammer that is the RHP method has basically settled the matter: universality holds at *regular* points of the non-universal equilibrium measure:

$$\begin{aligned}\mu_V &= \lim_{n\uparrow\infty} \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i^V} \\ &= \operatorname{argmin}_\mu \left(\int V(\lambda)\mu(d\lambda) - 2\int\int \log|\lambda-\gamma|\mu(d\lambda)\mu(d\gamma) \right).\end{aligned}$$

With more tears, $\beta = 1$ and $\beta = 4$ can be pushed through.

Both general V and β

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Again denote

$$T(A, B) = \mathrm{tridiag}(B, A, B),$$

for $(A_1, \ldots, A_n)$ real coordinates and $(B_1, \ldots, B_{n-1})$ all positive.

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Then if you draw (A, B) according to the law with density

$$\propto \exp\Big(-n\beta \operatorname{tr} V(T(a, b))\Big) \prod_{k=1}^{n-1} b_k^{\beta(n-k)-1},$$

the random Jacobi matrix $T(A, B)$ has joint eigenvalue density

$$P_{\beta,V} \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\beta n \sum_{k=1}^n V(\lambda_k)}.$$

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Note if $V(\lambda) = \frac{1}{4}\lambda^2$ get the β-Hermite ensemble of Dumitriu-Edelman. The proof is the same.

A metatheorem (for Stochastic Airy Universality)

The idea is that if there is a centering ($\mathcal{E}$) scaling rate ($\gamma_n \to \infty$) after which top the eigenvalues of $T_n = T_n(A, B)$ approach those of the Stochastic Airy operator, the game is the following.

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Write

$$\gamma_n(\mathcal{E} I - T_n(A, B)) = m_n^2 \,\mathrm{tridiag}(-1, 2, -1) + \mathrm{tridiag}(\tilde{A}, \tilde{B}, \tilde{A}),$$

and, interpreting the $\tilde{A}s$ and $\tilde{B}s$ as combining to a potential on discretization scale m_n^{-1}: show that,

$$\sum_{k=1}^{[tm_n]} (\tilde{A}_k + 2\tilde{B}_k) \Rightarrow \frac{t^2}{2} + \frac{2}{\sqrt{\beta}} b(t)$$

for b a Brownian motion.

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for b a Brownian motion. Along with *sufficient compactness* should mean

$$\lambda_{max}\Big(\gamma_n(\mathcal{E} I - T_n(A, B)\Big) \Rightarrow TW_\beta.$$

What compactness?

Re-notate on/off diagonals of already centered/scaled tridiagonal matrix as in

$$2m_n^2 + m_n(X_{n,k} - X_{n,k-1}), \quad -m_n^2 + m_n(Y_{n,k} - Y_{n,k-1}).$$

With $X_n(t) = X_{n,[m_n t]}$, etc., in addition to $X_n(t) + 2Y_n(t) \Rightarrow \frac{1}{2}t^2 + \frac{2}{\sqrt{\beta}}b(t)$ require...

There are decompositions:

$$X_{n,k} = \frac{1}{m_n}\sum_{\ell=1}^{k} \eta_{n,\ell}^X + w_{n,k}^X, \quad Y_{n,k} = \frac{1}{m_n}\sum_{\ell=1}^{k} \eta_{n,\ell}^Y + w_{n,k}^Y,$$

such that

$$t/C_n - C_n \le \eta_n^X(t) + \eta_n^Y(t) \le C_n t + C_n,$$

and

$$|w_n^X(t) - w_n^X(s)|^2 + |w_n^Y(t) - w_n^Y(s))|^2 \le C_n(1 + t/\phi(t)).$$

for all n and $t, s \in [0, n/m_n]$ with $|t-s| \le 1$ with tight C_n and some $\phi(t) \to \infty$.

Universality of Stochastic Airy

Theorem (Krishnapur, R., Virág)

Let V be a strictly convex polynomial. There exists a coupling of the random matrices T_n realizing $P_{\beta,V}$ on the same probability space and constants γ and $\mathcal{E}$ depending only on V so that: almost surely,

$$\gamma n^{2/3}(\mathcal{E} I - T_n) \to -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x)$$

The indicated convergence is such: for every k, the bottom kth eigenvalue converges and the corresponding eigenvector converges in norm.

Note similar to before we view $\mathcal{E} I - T_n$ acting on $\mathbb{R}^n \subset L^2(\mathbb{R}_+)$ with coordinate vectors

$$e_j = (\vartheta n)^{1/6} \mathbf{1}_{[j-1,j](\vartheta n)^{-1/3}},$$

with ϑ yet another constant depending on V.

Full disclosure - there are better universality results

Around the same time two separate groups proved stronger forms of soft-edge universality:

- *Bekerman-Figalli-Guionnet* by transportation of measure.
- *Bourgade-Erdös-Yau* by their relaxation of Dyson Brownian motion approach.

Both groups require only some number of derivatives of V, along with μ_V having one band of support and being regular.

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Aside: Convexity of V is the only simple geometric condition that produces one band plus regularity of μ_V.

Both these alternate methods are "by comparison". The philosophical advantage of the operator approach is that it (re)identifies the limit.

The set-up

One has a Gibbsian type law on tridiagonal (A, B): $e^{-n\beta H(a,b)} dadb$ for Hamiltonian

$$H = \mathrm{tr}(V(T_n(a,b))) - \sum_{k=1}^{n-1} (1 - \frac{k}{n} - \frac{1}{n\beta}) \log(b_k).$$

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Convexity of V yields convexity of H.

Polynomial V gives a Markov field property: the (A, B) variables in past/future are independent given a block of length $d = \frac{1}{2} deg(V) - 1$.

The bad:

Want to use convexity of H to show the variables fluctuate in a small window about the minimizer (a^*, b^*). You're not actually going to compute the minimizer...

Rough idea

Proceed in blocks. Consider a stretch of coordinates (A_k, B_k) with $k \in \mathcal{I}$ and $|\mathcal{I}| = n^{\alpha}$ with α "small".

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Induced law reads

$$dP_{\mathcal{I}} = \frac{1}{Z} \int e^{-n\beta H_q(a,b)} dQ(q).$$

Convexity/concentration gives

$$H_q(a, b) \simeq \bar{H}_q + \nabla \bar{H}_q \cdot (a - a_q, b - b_q) + \frac{1}{2}(a - a_q, b - b_q)^\dagger (\nabla^2 \bar{H}_q)(a - a_q, b - b_q)$$

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Will yield that $P_{\mathcal{I}}$ is a *mixture* of Gaussians - in total variation norm.

Doesn't look very universal - now have the problem of estimating/computing these conditional minimizers (a_q, b_q).

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In the resulting function, set all a_k and b_k to the same quantity. Produces a Hamiltonian in two variables:

$$H^{(t)} = H^{(t)}(a,b) = W(a,b) - (1-t)\log b.$$

Now define $(\hat{a}_t, \hat{b}_t)$, the minimizers of this expression, as your "local minimizers".

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Remark: Let C is the symmetric circulant matrix derived from the $tridiag(b, a, b)$ matrix. Then

$$W(a, b) = \frac{1}{\dim C} \mathrm{tr} V(C),$$

assuming $\dim C > \deg V$.

Local minimizers and equilibrium measures

This "local potential" W may also be written as in

$$W(a, b) = [1] V(a + b(z + 1/z))$$

where $[1]$ denotes the coefficient of the constant term in the Laurent series in z. See this by counting random walk paths.

Local minimizers and equilibrium measures

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Using the integral formula for the Laurent coefficient, the equations for $(\hat{a}_t, \hat{b}_t)$ are equivalent to

$$\int_{L_t}^{R_t} \frac{sV_t'(s)\,ds}{\sqrt{(s - L_t)(R_t - s)}} = 2\pi, \quad \int_{L_t}^{R_t} \frac{V_t'(s)\,ds}{\sqrt{(s - L_t)(R_t - s)}} = 0,$$

where

$$V_t = \frac{1}{1-t} V, \quad L_t = \hat{a}_t - 2\hat{b}_t, \quad R_t = \hat{a}_t + 2\hat{b}_t.$$

This identifies (L_t, R_t) as the left and right endpoints of support for the equilibrium measure associated with the family of potentials V_t.

Beyond regularity - higher order Tracy-Widom

It is possible to cook up Vs where the limiting eigenvalue density vanishes faster that square root at its right-most edge of support $\mathcal{E}$:

$$\psi_V(t) \sim (\mathcal{E} - t)^{\frac{4k+1}{2}}, \quad \text{for } k = 1, 2, \ldots.$$

Claeys-Its-Krasovsky (2010) showed at $\beta = 2$ that

$$P\left(n^{\frac{2}{4k+3}}(\lambda_{max} - \mathcal{E}) \leq t\right) \to \text{Painlevé Stuff}.$$

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Conjecture

Let T_n be a tridiagonal ensemble realizing the k^{th} order degeneracy. Then $H_{n,k} = \gamma n^{\frac{2}{4k+3}}(\mathcal{E}I - T_n)$, with a constant $\gamma = \gamma_V$ converges to the operator

$$\mathcal{H}_{\beta,k} = -\frac{d^2}{dx^2} + x^{\frac{1}{2k+1}} + \frac{2}{\sqrt{\beta}} x^{-\frac{k}{2k+1}} b'(x).$$

The problem: cannot produce this sort of behavior with convex potentials.

A concrete example

Take the quartic:

$$V(x) = \frac{1}{20}x^4 - \frac{4}{15}x^3 + \frac{1}{5}x^2 + \frac{8}{5}x,$$

then

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The density on tri-diagonal matrix coordinates reads: $e^{-n\beta H}$ for

$$\begin{aligned} H(a,b) = & \frac{1}{10}\sum(b_k^4 + 2b_k^2 b_{k+1}^2) - \sum(1 - \frac{k+1/\beta}{n})\log b_k \\ & + \frac{1}{20}\sum\left(a_k^4 - \frac{16}{3}a_k^3 + 4a_k^2 + 32a_k\right) \\ & + \frac{1}{5}\sum b_k^2\Big(2 + a_k a_{k+1} + a_k^2 + a_{k+1}^2 - 4(a_k + a_{k+1})\Big). \end{aligned}$$

Just need to prove a CLT for the running sum of the (a,b)-coordinates!

The hard and soft edges meet

Claeys-Kuijlaars introduces a Wishart-like model that mimics a vanishing inside the bulk. With $dM =$ Lebesgue on

$$Z^{-1}(\det M)^a e^{-n \operatorname{tr} V(M)} dM, \quad V(x) = \frac{1}{2c}(x-2)^2.$$

The parameter c can be tuned so that the equilibrium measure:

Has a hard edge at the origin for $c > 1$.

Is supported away from the origin for $c < 1$

Has an exact square-root vanishing right at the origin when $c = 1$

There's even a double scaling limit around $c = 1 + sn^{-2/3}$. At the level of correlations Claeys-Kuiljaars show that:

$$\lim_{n \to \infty} n^{-2/3} K_{n,s,a}(xn^{-2/3}, yn^{-2/3}) = K(x, y; a, s).$$

Another Painlevé object, but get back the Bessel and Airy kernels by taking limits $s \to \pm\infty$ afterwards.

General β hard-meets-soft case

Want a tridiagonal matrix with eigenvalue density:

$$P_{\beta,as} = \frac{1}{Z} \exp\left(-\frac{\beta}{4c} n \sum_{j=1}^{n} (\lambda_j - 2)^2 \right) \prod_{j=1}^{n} \lambda_j^{\frac{\beta}{2}(a+1)-1} \prod_{j<k} |\lambda_j - \lambda_k|^{\beta},$$

for $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^n$ and $c = c(s,n) = 1 - sn^{-2/3}$.

Draw $(X, Y) \in (\mathbb{R}_+^n, \mathbb{R}_+^{n-1})$ according to density $e^{-\beta cnH}$ for

$$H(x,y) = \frac{1}{4} \sum_{k=1}^{n} (x_k^2 + y_k^2) + \frac{1}{2} \sum_{k=1}^{n} x_k (y_k + y_{k-1}) - \sum_{k=1}^{n} (x_k + y_k)$$
$$- \sum_{k=1}^{n} \frac{c}{2} \left(1 - \frac{k}{n} + \frac{a + 1 - 2\beta^{-1}}{n} \right) \log x_k - \sum_{k=1}^{n} \frac{c}{2} \left(1 - \frac{k}{n} - \frac{2\beta^{-1}}{n} \right) \log y_k.$$

The matrix model is $W_n = B_n(X,Y) B_n^{\dagger}(X,Y)$ where now B has $\sqrt{X_k}$'s on diagonal and $-\sqrt{Y_k}$ on off-diagonal.

A (forthcoming) theorem

Easier to state in the special case $\frac{\beta}{2}(a+1) = 1$.

Theorem (Ramírez, R.)

Let $\lambda_1 < \lambda_2 < \cdots$ be the ordered points under the law $P_{\beta,a,s}$. Then $\{n^{2/3}\lambda_k\}$ converge in the sense of finite dimensional distributions to those of the random Schrödinger operator

$$-\frac{d^2}{dx^2} + Z^2(x) + Z'(x)$$

(with Dirichlet conditions on the positive half line). Here $Z(x) = Z(x;\beta,s)$ is defined at follows. Let $x \mapsto z(x)$ be the diffusion

$$dz_x = \frac{2}{\sqrt{\beta}}db_x + (s + x - z_x^2)dx, \quad z(0) = 0.$$

Then $Z(x)$ is z conditioned never to explode.