A. Its.

$\varphi(z)$, $z \in C : |z| = 1$

$T_n[\varphi] = \{\varphi_{j-k}\}_{j,k=0 \ldots n-1}$ truncated Toeplitz matrices

$D_n = \det T_n$

Multiple integral representation:

$$D_n = \frac{1}{(2\pi i)^n n!} \int_C \cdots \int_C \prod_{j<k} |z_j - z_k|^2 \prod_{j=1}^{n} \frac{\varphi(z_j)}{z_j} dz_j$$

Also, $D_n = \prod_{k=0}^{n-1} h_k$, where h_k is related to OPUC:

Given the symbol φ, we can define monic orthogonal polynomials $\{P_n(z)\}_{n=0}^{\infty}$, $P_n(z) = z^n + \cdots$ such that

$$\int_C z^{-k} P_n(z) \varphi(z) \frac{dz}{2\pi i z} = 0, \quad k = 0, 1, \ldots n-1.$$

or $$\int_C z^{-k} P_n(z) \varphi(z) \frac{dz}{2\pi i z} = \boxed{h_n} \delta_{n,k}, \quad k = 0 \ldots n.$$

We also define $Q_n(z) = -\frac{z^n}{h_n} \widehat{P}_n(z^{-1})$, where

$$\int_C \widehat{P}_n(z^{-1}) z^k \varphi(z) \frac{dz}{2\pi i z} = \hat{h}_n \delta_{n,k}, \quad k = 0 \ldots n.$$

Additionally, $\int_C P_n(z) \widehat{P}_k(z^{-1}) \varphi(z) \frac{dz}{2\pi i z} = h_n \delta_{n,k}$

Classical situation: $\varphi(z) = \overline{\varphi(z)}$, then $\widehat{P}(z^{-1}) = \overline{P(z)}$

and $\widehat{P}(z|\varphi) = \overline{P(\bar{z}|\bar{\varphi})}$

Obs. $\ln D_n(z) = -\int_0^1 \left[\int_C \left(P_n'(z|\gamma) Q_{n-1}(z|\gamma) \right.\right.$

$\left.\left. - P_n(z|\gamma) Q_{n-1}'(z|\gamma) \right) z^{-n} \frac{\varphi(z)-1}{2\pi i} \, dz \right] d\gamma$

where $P_n(z|\gamma) : \varphi_\gamma = \gamma\varphi + 1 - \gamma$, and

$Q_n(z) = \frac{-1}{h_n} z^n \hat{P}_n(z^{-1})$.

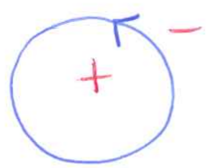
<u>Riemann-Hilbert representation of OPUC</u>

(Baik–Deift–Johansson).

Define $Y(z) = \begin{pmatrix} P_n(z) & \int_C \frac{P_n(s) s^{-n} \varphi(s)}{s-z} \frac{ds}{2\pi i} \\ Q_{n-1}(z) & \int_C \frac{Q_{n-1}(s) s^{-n} \varphi(s)}{s-z} \frac{ds}{2\pi i} \end{pmatrix}$,

it satisfies

I. $Y(z) \in \mathcal{H}(\mathbb{C} \setminus C)$



II. $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n}\varphi(z) \\ 0 & 1 \end{pmatrix}$

III. $Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

Obs. $Y(z) = \begin{pmatrix} z^n & \mathcal{O}(z^{-n-1}) \\ \mathcal{O}(z^{n-1}) & z^{-n} + \mathcal{O}(z^{-n-1}) \end{pmatrix}$

(1,2) entry: $\int_C \frac{P_n(s) s^{-n} \varphi(s)}{s-z} \frac{ds}{2\pi i}$

$= -\sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \int_C P_n(s) s^{k-n+1} \varphi(s) \frac{ds}{2\pi i s}$

But $\int_C P_n(z) z^{-k} \varphi(z) \frac{dz}{2\pi i z} = 0$, so orthogonality gives $Y_{1,2}(z) = \mathcal{O}(z^{-n-1})$ as $z \to \infty$.

Obs. 1) $\det Y(z) = 1$

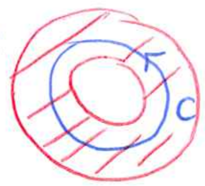
2) Suppose that $\tilde{Y}(z)$ satisfies I – III, then take $\tilde{Y}(z) Y^{-1}(z)$, this is identity, so I – III determines $Y(z)$ uniquely.

Taking I – III as definition of Y, we can try to obtain large n asymptotics, and then $P_n(z) = Y_{11}(z)$ and $h_n = Y_{12}(0)$.

We rewrite:

$$\ln D_n(t) = -\int_0^1 \left[\int_C \left(Y_{11}'(z) Y_{21}(z) - Y_{11}(z) Y_{21}'(z) \right) z^{-n} \frac{\varphi(z)-1}{2\pi i} dz \right] dt$$

Assume that $\varphi(z)$ is analytic in an annulus:



$\varphi(z) = e^{V(z)}$, where

$V(z) = \sum_{k=-\infty}^{\infty} V_k z^k$ (Convergent Laurent series)

First transformation: $Y(z) \longmapsto T(z) = Y(z) \begin{cases} z^{n\sigma_3}, & |z|>1 \\ I, & |z|<1 \end{cases}$

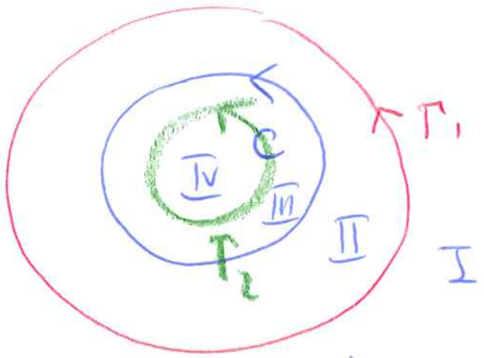
so for $T(z)$ we have the jump:

$$T_+(z) = T_-(z) \begin{pmatrix} z^n & \varphi(z) \\ 0 & z^{-n} \end{pmatrix}$$ Oscillatory matrix!

and the asymptotics $T(\infty) = I$.

Now,

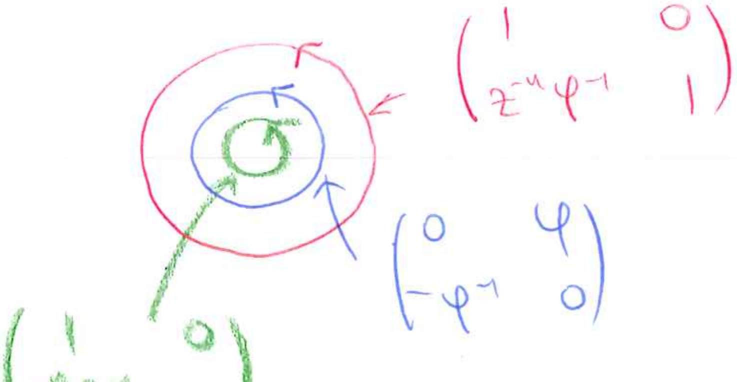
$$\begin{pmatrix} z^n & \varphi \\ 0 & z^{-n} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ z^{-n}\varphi^{-1} & 1 \end{pmatrix}}_{\text{Deform to outer contour}} \begin{pmatrix} 0 & \varphi \\ -\varphi^{-1} & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ z^{n}\varphi^{-1} & 1 \end{pmatrix}}_{\text{Deform to inner contour}}$$



Second transformation:

$$T \longrightarrow S(z) = T(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ z^{-n}\varphi^{-1} & 1 \end{pmatrix}, & z \in \mathrm{II} \\ \begin{pmatrix} 1 & 0 \\ -z^{n}\varphi^{-1} & 1 \end{pmatrix}, & z \in \mathrm{III} \\ I, & z \in \mathrm{I} \cup \mathrm{IV} \end{cases}$$

So S satisfies the RHP on three contours:



plus identity at infinity. The jumps on $\Gamma_1 \cup \Gamma_2$ are exponentially close to identity, so we expect that as $n \to \infty$, $S(z) \sim S^{(\infty)}$, where $S^{(\infty)}$ satisfies a RHP with a jump on C only:

$$S^{(\infty)}_+ = S^{(\infty)}_- \begin{pmatrix} 0 & \varphi \\ -\varphi^{-1} & 0 \end{pmatrix}, \quad z \in C$$

Szegő function:

$$D(z) = \exp\left\{\frac{1}{2\pi i}\int_C \frac{\ln\varphi(s)}{s-z}\,ds\right\},$$ which solves the following scalar problem: $D_+ = D_-\varphi$ on C, so

$$S^{(\infty)}(z) = D^{\sigma_3}(z)\begin{cases} I, & |z|>1 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & |z|<1 \end{cases}$$

Tracing back the transformations, we obtain for the first column

$$\left[Y(z)\right]_1 \cong D_+^{\sigma_3}(z)\begin{pmatrix} \varphi^{-1}z^n \\ -1 \end{pmatrix} + O(\rho^{-n}), \quad |\rho|>1.$$

Obs. $\varphi = \varphi_\gamma$, deformation of the original symbol

$$\varphi_\gamma \equiv \gamma\varphi(z) + 1 - \gamma$$

Now, $Y_{11} = D_+\varphi_\gamma^{-1}z^n$

$Y_{21} = -D_+^{-1}$, and therefore

$$Y_{11}'Y_{21} - Y_{11}Y_{21}' = -2D_+'D_+^{-1}\varphi_\gamma^{-1}z^n + \varphi_\gamma^{-2}\varphi_\gamma'z^n - n\varphi_\gamma^{-1}z^{n-1},$$

and then

$\ln D_n \cong I_1 + I_2 + I_3$, where

$$I_1 = \int_0^1\int_C D_+'D_+^{-1}\varphi_\gamma^{-1}\,\frac{\varphi_\gamma-1}{2\pi i}\,dz\,d\gamma$$

$$I_2 = -\int_0^1\int_C \varphi_\gamma^{-2}\varphi_\gamma'\,\frac{\varphi_\gamma-1}{2\pi i}\,dz\,d\gamma$$

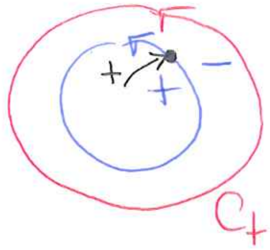
$$I_3 = n\int_0^1\int_C \varphi_\gamma^{-1}\,\frac{\varphi_\gamma-1}{2\pi i}\,\frac{dz}{z}\,d\gamma$$

Obs $\varphi_\gamma - 1 = \frac{d\varphi_\gamma}{d\gamma}$, so

$$I_3 = n\int_0^1 \int_C \frac{d}{d\gamma}\ln\varphi_\gamma \, \frac{dz}{2\pi i z}\, d\gamma$$

$$= n\int_C \ln\varphi \, \frac{dz}{2\pi i z} = n(\ln\varphi)_0 .$$

We can check that $I_2 = 0$, integrating with respect to z for each fixed γ (winding 0 for any γ)



Now, $$D_+(z) = \exp\left(\frac{1}{2\pi i}\int_C \frac{\ln\varphi_\gamma(s)}{s - z_+}\, ds\right)$$

$$= \exp\left(\frac{1}{2\pi i}\int_{C_+} \frac{\ln\varphi_\gamma(s)}{s-z}\, ds\right)$$

by analiticity of φ_γ.

So $D_+' D_+^{-1} = \frac{1}{2\pi i}\int \frac{\ln\varphi_\gamma(s)}{(s-z)^2}\, ds$, and then

$$I_1 = 2\int_0^1\int_C\int_{C_+} \frac{\ln\varphi_\gamma(s)}{(s-z)^2} \frac{d}{d\gamma}\ln\varphi_\gamma \frac{ds\,dz}{(2\pi i)^2}$$

$$= \int_0^1\int_C\int_{C_+} \frac{\ln\varphi_\gamma(s)\frac{d}{dz}\ln\varphi_\gamma(z) + \ln\varphi_\gamma(z)\frac{d}{d\gamma}\ln\varphi_\gamma(s)}{(s-z)^2} \frac{ds\,dz}{(2\pi i)^2}$$

$+$(residue when switching C and C_+)
$\quad \| $
$\quad 0$ (exercise)

$$= \int_C\int_{C_+} \frac{\ln\varphi(s)\ln\varphi(z)}{(z-s)^2} \frac{ds\,dz}{(2\pi i)^2}$$

$$= \int_C \ln \varphi_{\pm}(z) \frac{d}{dz} \int_{C_+} \frac{\ln \varphi_{\pm}(s)}{s-z} \frac{ds\, dz}{(2\pi i)^2}$$

$$= \int_C \ln \varphi_{\pm}(z) \Bigg(\underbrace{\sum_{k=1}^{\infty} k z^{k-1} \int_{C_+} \ln \varphi(s)\, s^{-k-1} \frac{ds}{2\pi i}}_{(\ln \varphi(z))_k} \Bigg) \frac{dz}{2\pi i}$$

$$= \sum_{k=1}^{\infty} k (\ln \varphi)_k \int_C \ln \varphi(z) z^{k-1} \frac{dz}{2\pi i}$$

$$= \sum_{k=1}^{\infty} k (\ln \varphi)_k (\ln \varphi)_{-k}.$$

$\boxed{/\!/}$

<u>Fisher-Hartwig symbols</u>

$$\varphi(z) = e^{V(z)} \underbrace{|z-1|^{2\alpha}}_{\text{root}} \underbrace{z^{\beta} e^{-\beta\pi i}}_{\text{jump}}, \quad \begin{array}{l} 2\alpha > -1 \\ 0 < \arg z < 2\pi \end{array}$$

In this case

$$\ln D_n = \underbrace{n V_0 + \sum_{k=1}^{\infty} k V_k V_{-k}}_{\substack{\text{regular part} \\ \text{(SSLT)}}} - (\alpha - \beta) \sum_{k=1}^{\infty} V_k - (\alpha+\beta) \sum_{k=1}^{\infty} V_{-k}$$

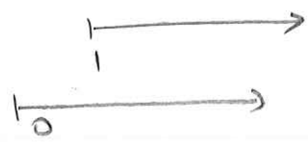
$$+ \underbrace{(\alpha^2 - \beta^2) \ln n}_{\substack{\text{power-type} \\ \text{behaviour}}} + \ln \frac{G(1+\alpha-\beta)\, G(1+\alpha+\beta)}{G(1+2\alpha)} + o(1),$$

where $G(x)$ is the Barnes G-function, $G(x+1) = \Gamma(x) G(x)$

<u>Obs</u>. We can write

$$\varphi(z) = e^{V(z)} (z-1)^{2\alpha} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)}, \text{ since}$$

$$|z-1|^{2\alpha} = (z-1)^{2\alpha} z^{-\alpha} e^{-\alpha\pi i} \qquad \begin{array}{l} (z-1)^{2\alpha} \\ z^{-\alpha} \end{array}$$

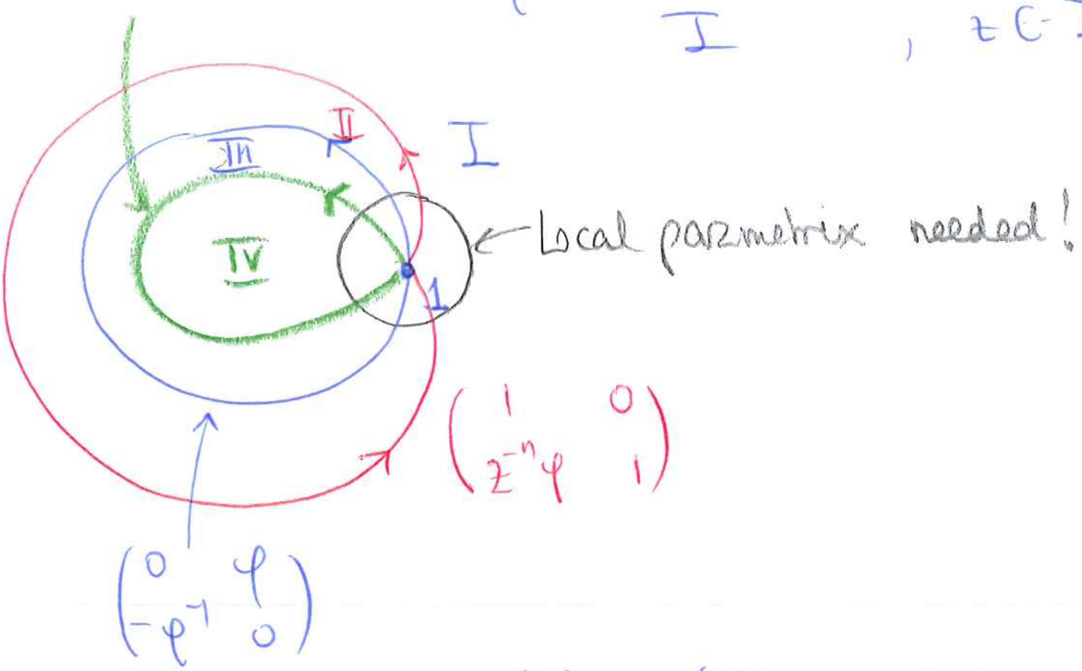


Check: $|z-1|^{2\alpha} = (z-1)^\alpha (\bar{z}-1)^\alpha$
$$= (z-1)^\alpha \left(\frac{1}{z}-1\right)^\alpha$$

$$Y(z) \longrightarrow T(z) = Y(z) \begin{cases} z^{-n\sigma_3}, & |z|>1 \\ I, & |z|<1 \end{cases}$$

with $T_+ = T_- \begin{pmatrix} z^n & \varphi \\ 0 & z^{-n} \end{pmatrix} = T_- \begin{pmatrix} 1 & 0 \\ z^{-n}\varphi^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & \varphi \\ -\varphi^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^{n}\varphi^{-1} & 1 \end{pmatrix}$

then $T \longrightarrow S = T \begin{cases} \begin{pmatrix} 1 & 0 \\ z^{-n}\varphi & 1 \end{pmatrix}, & z \in \mathrm{II} \\ \begin{pmatrix} 1 & 0 \\ -z^{n}\varphi & 1 \end{pmatrix}, & z \in \underline{\mathrm{III}} \\ I, & z \in \mathrm{I} \cup \underline{\mathrm{IV}} \end{cases}$



Global parametrix: $S^{(\infty)} = P^{(\infty)}(z)$, which satisfies

$$P_+^{(\infty)} = P_-^{(\infty)} \begin{pmatrix} 0 & \varphi \\ -\varphi^{-1} & 0 \end{pmatrix}, \quad z \in C, \quad P^{(\infty)}(\infty) = I,$$

and again

$$P^{(\infty)}(z) = D^{\sigma_3}(z) \begin{cases} I, & |z|>1 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & |z|<1. \end{cases}$$

If we write

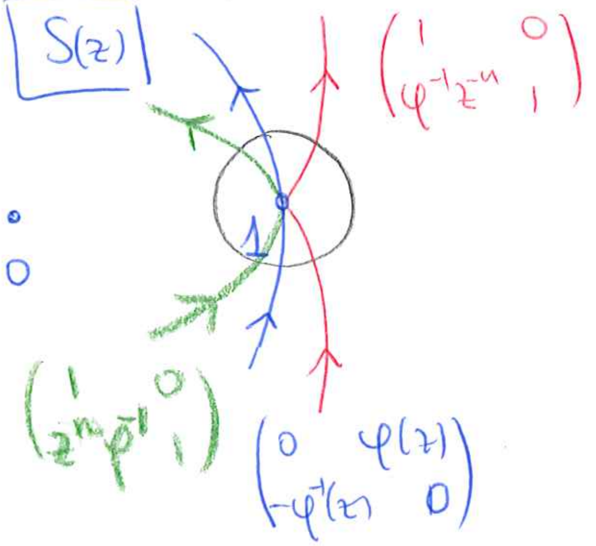
$$V(z) = V_+ + V_- = \sum_{k=0}^{\infty} z^k V_k + \sum_{k=-\infty}^{-1} z^k V_k \,, \quad \text{then}$$

$$\mathcal{D}(z) = \begin{cases} e^{V_+} (z-1)^{\alpha+\beta} e^{-i\pi(\alpha+\beta)} , & |z|<1 \\ e^{-V_-} \left(\frac{z-1}{z}\right)^{-\alpha+\beta} , & |z|>1 \end{cases}$$

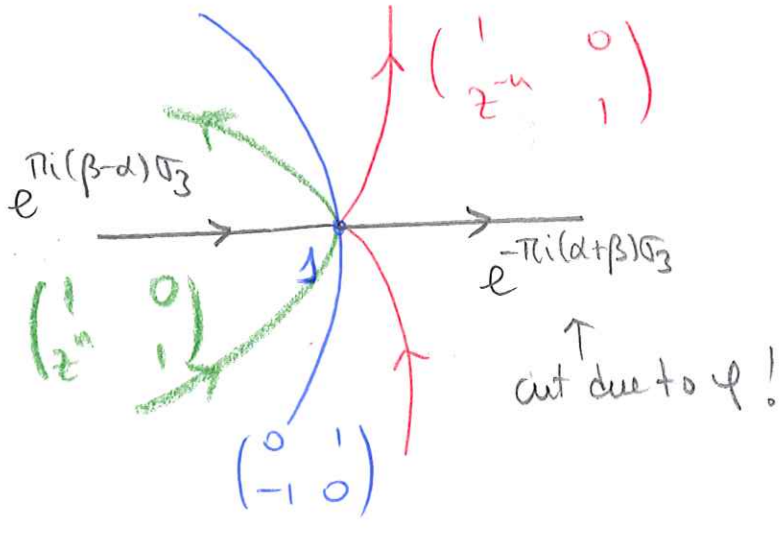
as explicit factorisation ($\mathcal{D}_+ = \mathcal{D}_- \varphi$).

Local parametrix?

$\boxed{S(z)}$



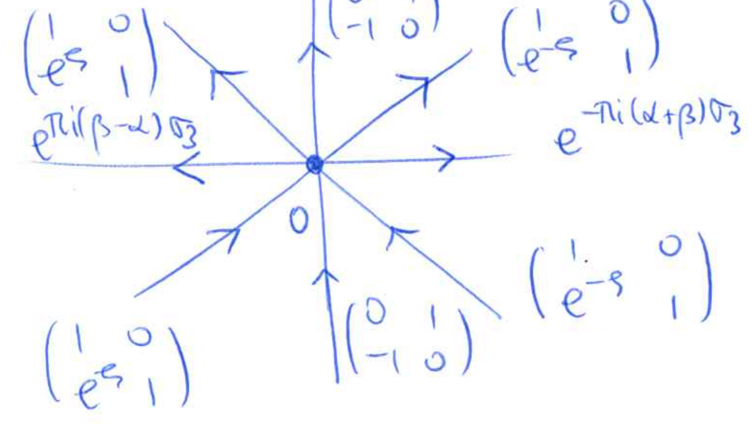
Define $\tilde{S}(z) = S(z)\,\varphi^{\sigma_3/2}$, then



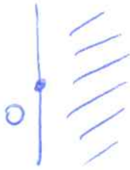
Local variable:

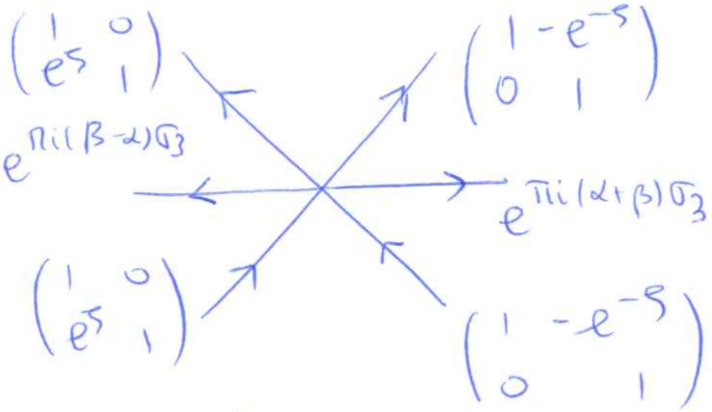
$$\zeta = n \cdot \ln z = n(z-1) + \cdots \,, \quad z \to 1$$

$\mathring{\Phi}(\zeta)$:

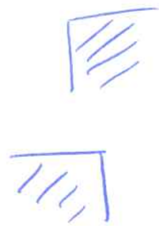


Then $\overset{\circ}{\Phi} \to \overset{1}{\Phi}(\zeta) = \overset{\circ}{\Phi}(\zeta)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

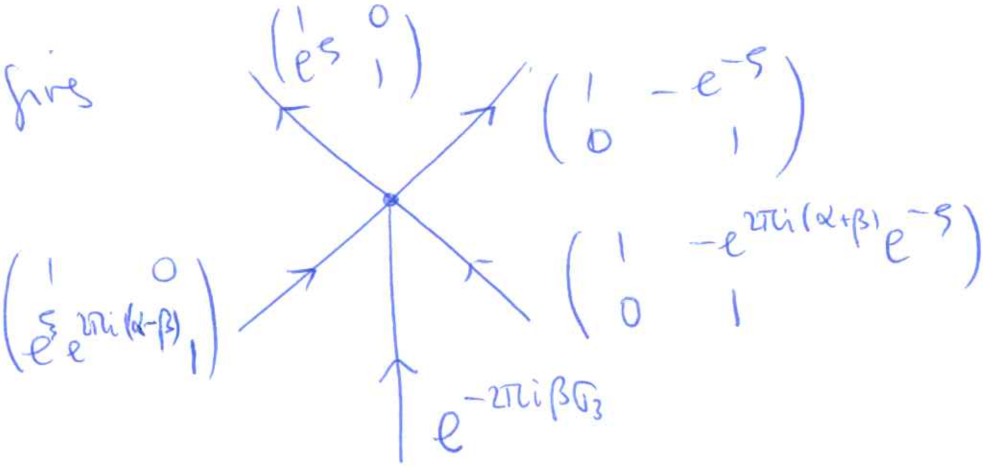




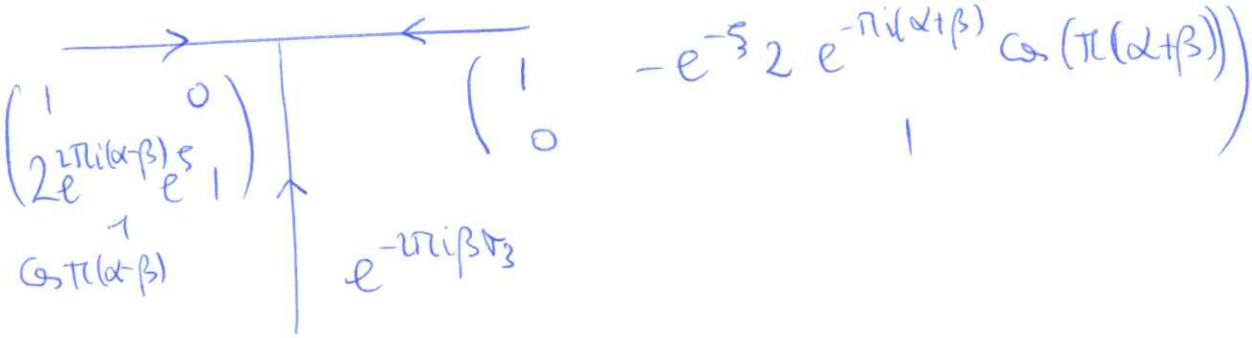
Now $\overset{1}{\Phi}(\zeta) \to \overset{2}{\Phi}(\zeta) = \overset{1}{\Phi}(\zeta) \cdot \begin{cases} e^{\pi i(\alpha+\beta)\sigma_3} \\ e^{-\pi i(\beta-\alpha)\sigma_3} \end{cases}$



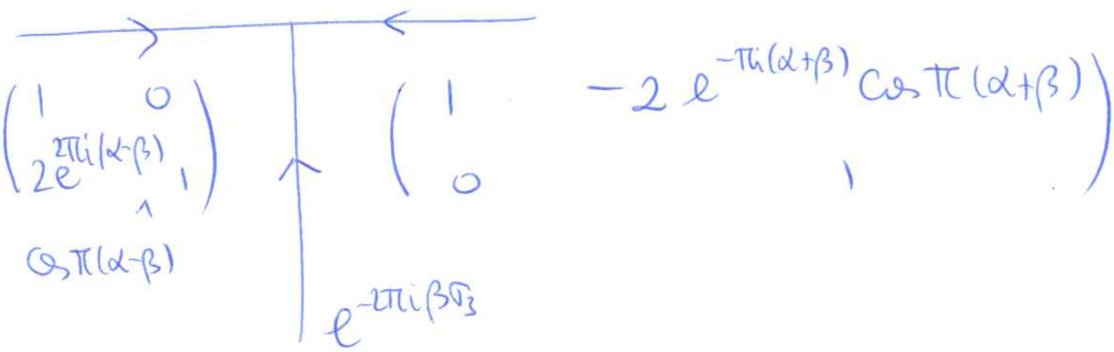
gives



Next, $\overset{2}{\Phi}(\zeta) \to \overset{3}{\Phi}(\zeta) \to \overset{3}{\Psi}(\zeta) = \overset{3}{\Phi}(\zeta)\, e^{-\frac{\zeta}{2}\sigma_3}$



$\overset{3}{\Phi}(\zeta)$



$\overset{3}{\Psi}(\zeta)$

Also, $\overset{3}{\Psi}(\xi) = \left(I + O(1/\xi)\right) e^{-\frac{\xi}{2}\sigma_3} \xi^{-\beta\sigma_3}$, $\xi \to \infty$

and $\overset{3}{\Psi}(\xi) = \hat{\Psi}(\xi)\, \xi^{\alpha\sigma_3}\, C_j$ constant matrices depending on different regions

Notation: $\Psi(\xi) \equiv \overset{3}{\Psi}(\xi)$, and consider

$\frac{d\Psi}{d\xi} \cdot \Psi^{-1}$ is holomorphic in $\mathbb{C} \setminus \{0\}$ (no jumps)

$\boxed{\frac{d\Psi}{d\xi} \cdot \Psi^{-1} = -\frac{1}{2}\sigma_3 + \frac{1}{\xi} A_0}$, where $A_0 = \begin{pmatrix} -\beta & a \\ b & \beta \end{pmatrix}$, and eigs $\pm\alpha$

if $\Psi(\xi) = \left(I + \frac{m}{\xi} + \ldots\right) e^{-\frac{\xi}{2}\sigma_3} \xi^{-\beta\sigma_3}$, then

$a = 2(m)_{12}$, $b = -2(m)_{21}$.

So $\frac{d\Psi}{d\xi} = \left(-\frac{1}{2}\sigma_3 + \frac{1}{\xi} A_0\right)\Psi$, linear ODE with rational coeffs. with irregular sing at ∞ and regular sing. at 0.

Monodromy data: $\mathcal{M} = \{S_1, S_2, C\}$, with Stokes matrices $S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}$, $S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}$, with the relation $s_1 s_2 = 2e^{-2\pi i\beta}(\cos 2\pi\alpha - \cos 2\pi\beta)$.

Obs $\dim \mathcal{M} = \dim \mathcal{A}$.

$(a,b) \leftrightarrow (s_1, s_2)$ (Direct/inverse monodromy problem)

First component:

$$\rho \Psi_{11}'' + \Psi_{11}' + \left(\frac{1}{2} - \frac{\rho}{4} - \frac{\alpha^2}{\rho} - \beta\right)\Psi_{11} = 0$$

We can write $\Psi_{11} = \rho^\alpha w(\rho)$, which gives

$$\rho w'' + (2\alpha+1) w' + \left(\frac{1}{2} - \beta - \frac{\rho}{4}\right) w = 0$$

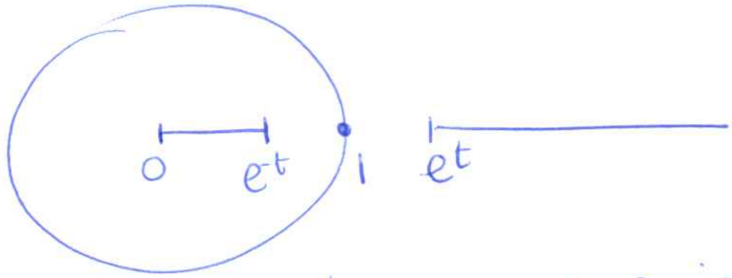
Last transformation: $w(\rho) = e^{-\rho/2} v(\rho)$, then

$$\boxed{\rho v'' + (2\alpha+1-\rho) v' - (\alpha+\beta) v = 0}$$

Confluent hypergeometric function, $v = \Psi(\alpha+\beta, 1+2\alpha; \rho)$.

Transition asymptotics

$$\varphi(z) = e^{V(z)} (z - e^{-t})^{\alpha+\beta} (z - e^{t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-\pi i(\alpha+\beta)}$$



$0 \leq t < t_0$.

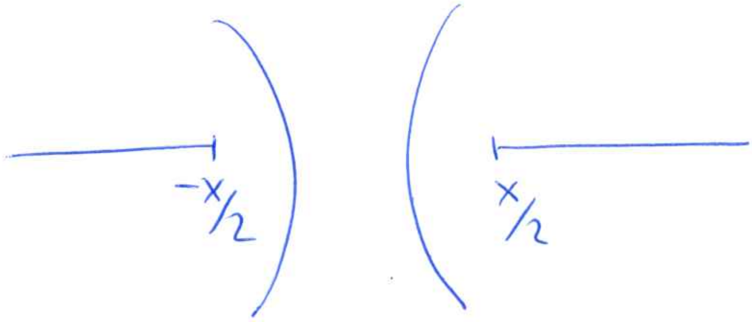
Monodromy theory and Painlevé equations

$$\frac{d\Psi}{d\rho} = \left[-\frac{\sigma_3}{2} + \frac{A_0}{\rho}\right]\Psi$$ → Confluent hypergeometric

We add one more singularity

$$\frac{d\Psi}{d\rho} = \left[-\frac{\sigma_3}{2} + \frac{A_0}{\rho} + \frac{1}{\boxed{\rho - x}} A_x\right]\Psi$$ → Painlevé V

Now $\dim \mathcal{M} = \dim \mathcal{A} - 1$, so there is a one-parameter family of equations with the same monodromy data.



We can say that P_{V} is a non-linear confluent hypergeometric function.

$$\frac{d\Psi}{d\xi} = [\xi A_1 + A_0]\Psi \longrightarrow \text{Parabolic cylinder function}$$

$$\frac{d\Psi}{d\xi} = [\xi^2 \boxed{A_2} + \xi A_1 + A_0]\Psi \longrightarrow P_{II} \quad \text{(one parameter)}$$

A_2 nilpotent $\longrightarrow P_{I}$.

But also $$\frac{d\Psi}{d\xi} = \left[\xi A_1 + A_0 + \frac{A_{-1}}{\xi}\right]\Psi \longrightarrow P_{IV}.$$

(two parameters)

Bessel: $$\frac{d\Psi}{d\xi} = \left[A_1 + \frac{A_0}{\xi}\right]\Psi$$ and then

$$\frac{d\Psi}{d\xi} = \left[A_1 + \frac{A_0}{\xi} + \frac{A_2}{\xi^2}\right]\Psi \longrightarrow P_{III}.$$

Hypergeometric $$\frac{d\Psi}{d\xi} = \left(\frac{A_1}{\xi} + \frac{A_2}{\xi-1}\right)\Psi$$ and then

$$\frac{d\Psi}{d\xi} = \left(\frac{A_1}{\xi} + \frac{A_2}{\xi-1} + \frac{A_3}{\xi-x}\right)\Psi \longrightarrow P_{VI}.$$