Toeplitz determinants, Painlevé equations, and special functions
Part I: an operator approach

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Preliminaries

We will begin with the following two very basic questions. The first is: What can be said about the determinants of finite Toeplitz matrices as the size increases? These are matrices that look like this.

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-n+2} \\ a_2 & a_1 & a_0 & \cdots & a_{-n+3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix}$$

The entries usually correspond to the Fourier coefficients of some function defined on the unit circle S^1, but could be any infinite sequence of complex numbers $\{a_i\}_{i=-\infty}^{\infty}$.

The second question concerns an integral operator on a line segment.

We suppose that $k(x)$ is continuous and we define the operator T on $L^2(a, b)$ which takes the function f to

$$T(f)(x) = \int_a^b k(x - y)\, f(y)\, dy.$$

The function $K(x, y) = k(x - y)$ is called the kernel of T. Now we ask what can be said about the Fredholm determinant

$$D(\lambda) = \det(I - \lambda T)$$

as the size of the interval increases?

It turns out that the way to answer one of these questions will more or less work for both. It is a bit easier to think about the Toeplitz case, so we will begin there.

Since our finite matrix is growing in size it makes sense to ask if we can somehow get information about Toeplitz matrices from the infinite array.

In order to do this, we will view the infinite array as an operator on a Hilbert space.

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

If we imagine that we are multiplying the infinite matrix on the right by a column vector it is quite natural to choose the Hilbert space as an extension of n-dimensional complex space.

We use the Hilbert space of unilateral sequences

$$l^2 = \left\{ \{f_k\}_{k=0}^{\infty} \mid \sum_{k=0}^{\infty} |f_k|^2 < \infty \right\},$$

which we identify with the Hardy space (S^1 is the unit circle)

$$H^2 = \{f \in L^2(S^1) \mid f_k = 0, k < 0\},$$

where

$$f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta} d\theta, \quad f(e^{i\theta}) = \sum_{k=0}^{\infty} f_k e^{ik\theta}.$$

Basic properties of H^2

The space H^2 is a closed subspace of L^2. The inner product of two functions is given by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta.$$

The two-norm of f, $\|f\|_2$, is given by $\langle f, f\rangle^{1/2}$ and is known to be equal to

$$\left(\sum_{k=0}^{\infty}|f_k|^2\right)^{1/2}.$$

Every function in H^2 has an analytic extension into the interior of the unit circle given by

$$f(z) = \sum_{k=0}^{\infty} f_k z^k, \ |z| < 1.$$

We denote the orthogonal projection of L^2 onto H^2 by P.

$$P : L^2 \to H^2$$

$$P : \sum_{k=-\infty}^{\infty} f_k e^{ik\theta} \to \sum_{k=0}^{\infty} f_k e^{ik\theta},$$

that is, P simply takes the Fourier series for f and removes terms with negative index. We denote the orthogonal projection P_n on H^2 by

$$P_n : \sum_{k=0}^{\infty} f_k\, e^{ik\theta} \to \sum_{k=0}^{n-1} f_k\, e^{ik\theta}.$$

Toeplitz operators on H^2

The projection P satisfies $P^2 = P^* = P$. Now let ϕ be a bounded function and define the Toeplitz operator with symbol ϕ by

$$T(\phi) : H^2 \to H^2$$

by

$$T(\phi) f = P(\phi f).$$

The functions $\{e_k(\theta) = e^{ik\theta}\}_{k=0}^{\infty}$ form a Hilbert space basis for H^2. To find the matrix representation of the operator $T(\phi)$ we compute

$$\langle T(\phi)\, e_k, e_j \rangle = \langle P(\phi\, e_k), e_j \rangle = \langle \phi\, e_k, P(e_j) \rangle$$

$$= \langle \phi\, e_k, e_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(e^{i\theta})\, e^{ik\theta}\, e^{-ij\theta} d\theta = \phi_{j-k}.$$

This shows that this operator has exactly the correct matrix representation.

In what follows we will also make use of a Hankel operator defined by

$$H(\phi) = (\phi_{j+k+1}), \qquad 0 \leq j, k < \infty.$$

The matrix display of this operator is

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & \cdots & \cdots \\ a_3 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

with constants along the opposite diagonals.

Lemma

(a) $T(\phi)$ *is a bounded operator*

(b) $aT(\phi) + bT(\psi) = T(a\phi + b\psi)$ *where* $a, b,$ *are complex numbers*

(c) *If* $\phi \equiv 1$ *then* $T(\phi) = I$, *the identity operator.*

(d) $T(\phi\psi) = T(\phi)T(\psi) + H(\phi)H(\tilde{\psi})$

(e) $H(\phi\psi) = T(\phi)H(\psi) + H(\phi)T(\tilde{\psi})$

where $\tilde{\phi}(e^{i\theta}) = \phi(e^{-i\theta})$.

To prove (a) notice that

$$\|T(\phi)f\|_2 = \|P(\phi f)\|_2 \leq \|\phi f\|_2 \leq \|\phi\|_\infty \|f\|_2.$$

The best known result about Toeplitz matrices is the Strong Szegö - Widom Limit Theorem which we will describe and then prove.

let $\mathcal{B}$ stand for the set of all functions ϕ such that the Fourier coefficients satisfy

$$\|\phi\|_{\mathcal{B}} := \sum_{k=-\infty}^{\infty} |\phi_k| + \Big(\sum_{k=-\infty}^{\infty} |k| \cdot |\phi_k|^2 \Big)^{1/2} < \infty.$$

With the norm, and pointwise defined algebraic operations on S^1, the set $\mathcal{B}$ becomes a Banach algebra of continuous functions on the unit circle.

Suppose $\phi \in \mathcal{B}$ and the function ϕ does not vanish on S^1 and has winding number zero. Then

$$D_n(\phi) = \det T_n(\phi) = (\phi_{j-k})_{\,j,k=0,\cdots,n-1}$$

has the asymptotic behavior

$$D_n(\phi) = \det T_n(\phi) \sim G(\phi)^n \, E(\phi) \quad \text{as } n \to \infty.$$

Here are the constants:

$$G(\phi) = e^{(\log \phi)_0}$$

and

$$E(\phi) = \det\left(T(\phi)T(\phi^{-1})\right).$$

What does the constant $\det\left(T(\phi)T(\phi^{-1})\right)$ mean? To understand this, we need to consider trace class operators.

We say that T is trace class if

$$\|T\|_1 = \sum_{n=1}^{\infty} \langle (TT^*)^{1/2} e_n, e_n \rangle < \infty$$

for any orthonormal basis $\{e_n\}$ in our Hilbert space.

It is not always convenient to check whether or not an operator is trace class from the definition.

It is fairly straight forward to check if an operator is Hilbert-Schmidt and it is well known that a product of two Hilbert-Schmidt operators is trace class.

We say an operator is Hilbert-Schmidt if the sum

$$\sum_{i,j} |\langle Se_i, e_j \rangle|^2 < \infty$$

is finite for some choice of orthonormal basis. If it is, then the above sum is independent of the choice of basis and its square root is called the Hilbert-Schmidt norm of the operator, denoted by $||S||_2$.

Properties of trace class operators

(a) Trace class operators form an ideal in the set of all bounded operators and are closed in the topology defined by the trace norm.

(b) Hilbert-Schmidt operators form an ideal in the set of all bounded operators with respect to the Hilbert-Schmidt norm and are closed in the topology defined by the Hilbert-Schmidt norm.

(c) The product of two Hilbert-Schmidt operators is trace class.

(d) If T is trace class, then T is a compact operator.

(e) If T is trace class, then $\sum |\lambda_i| < \infty$ where the λ_is are the eigenvalues of T.

(f) Hence the product $\prod(1 + \lambda_i)$ is always defined and finite and we denote it by $\det(I + T)$.

(g) If T is trace class, then $\det P_n(I + T)P_n \to \det(I + T)$ for orthogonal projections P_n that tend strongly (pointwise) to the identity. (For the first determinant we think of $P_n(I + T)P_n$ as the operator defined on the image of P_n.)

(h) If $A_n \to A, B_n^* \to B^*$ strongly (pointwise in the Hilbert space) and if T is trace class, then $A_n T B_n \to ATB$ in the trace norm.

(i) The functions defined by $\text{trace} T = \sum \lambda_i$ and $\det(I + T)$ are continuous on the set of trace class operators with respect to the trace norm.

(j) If $T_1 T_2$ and $T_2 T_1$ are trace class then
$\text{trace}(T_1 T_2) = \text{trace}(T_2 T_1), \quad \det(I + T_1 T_2) = \det(I + T_2 T_1).$

To apply this to the constant det $\big(T(\phi)T(\phi^{-1})\big)$ we use the identity

$$T(\phi\psi) = T(\phi)T(\psi) + H(\phi)H(\tilde{\psi})$$

to find

$$T(\phi)T(\phi^{-1}) = I - H(\phi)H(\tilde{\phi}^{-1}).$$

Each symbol is in our Banach Algebra. The Hilbert-Schmidt norm of $H(\phi)$ is

$$\Big(\sum_{k=1}^{\infty} |k| \cdot |\phi_k|^2\Big)^{1/2}$$

which is finite by assumption as is the Hilbert-Schmidt norm of $H(\tilde{\phi}^{-1})$. Thus the operator $H(\phi)H(\tilde{\phi}^{-1})$ is trace class and the determinant is defined.

We are ready now to prove the theorem.

It follows from the previous identities

$$T(\phi\psi) = T(\phi)T(\psi) + H(\phi)H(\tilde{\psi})$$

$$H(\phi\psi) = T(\phi)H(\psi) + H(\phi)T(\tilde{\psi})$$

that if ψ_- and ψ_+ have the property that all their Fourier coefficients vanish for $k > 0$ and $k < 0$, respectively, then

$$T(\psi_-\phi\psi_+) = T(\psi_-)T(\phi)T(\psi_+),$$

$$H(\psi_-\phi\tilde{\psi}_+) = T(\psi_-)H(\phi)T(\psi_+).$$

Next we make some simple observations:

1) If $\phi_k = 0$ for $k < 0$ or $k > 0$, then the Toeplitz matrices are triangular and $D_n(\phi) = \det T_n(\phi) = (\phi_0)^n$.

2) $T_n(\phi) = P_n T(\phi) P_n$ i.e. $T_n(\phi)$ is the upper left corner of the Toeplitz operator $T(\phi)$.

The projection P_n has the nice property that if U is an operator whose matrix representation has an upper triangular form. Then

$$P_n U P_n = U P_n.$$

If L is an operator whose matrix representation has a lower triangular form. Then

$$P_n L P_n = P_n L.$$

So if we had an operator of the form LU, then

$$P_n LU P_n = P_n L P_n U P_n$$

and the corresponding determinants would be easy to compute.

What happens for Toeplitz operators is the opposite.

Toeplitz operators generally do not factor as $L\,U$ but rather as $U\,L$.

To see this, we use a Wiener-Hopf factorization. Since the logarithm of ϕ exists, then

$$\phi = \phi_- \, \phi_+$$

where ϕ_+ extends to a function analytic in the inside of the unit circle and ϕ_- extends to a function analytic in the outside of the unit circle and all of these functions and their inverses are in the Banach algebra.

This means that $(\phi_+)_k = 0$ for $k < 0$ and $T(\phi_+)$ is lower triangular and that $(\phi_-)_k = 0$ for $k > 0$ and $T(\phi_-)$ is upper triangular. And one can check that factorization of ϕ also means that

$$T(\phi) = T(\phi_-) \, T(\phi_+).$$

The function ϕ_+ is bounded and in H^2 so the Toeplitz operator is merely multiplication.

We write

$$\begin{aligned} P_n T(\phi) P_n &= P_n T(\phi_-)\, T(\phi_+) P_n \\ &= P_n T(\phi_+) T(\phi_+^{-1}) T(\phi_-)\, T(\phi_+) T(\phi_-) T(\phi_-^{-1}) P_n \\ &= P_n T(\phi_+) P_n T(\phi_+^{-1}) T(\phi_-)\, T(\phi_+) T(\phi_-^{-1}) P_n T(\phi_-) P_n \end{aligned}$$

Taking determinants, we have

$$\det P_n T(\phi_-) P_n = ((\phi_-)_0)^n$$

and

$$\det P_n T(\phi_+) P_n = ((\phi_+)_0)^n,$$

The middle term

$$P_n T(\phi_+^{-1}) T(\phi_-)\, T(\phi_+) T(\phi_-^{-1}) P_n$$

is of the form

$$P_n (I + A) P_n$$

where A is trace class and thus the determinant converges to

$$\det T(\phi_+^{-1}) T(\phi_-)\, T(\phi_+) T(\phi_-^{-1}) = \det T(\phi) T(\phi^{-1})$$

Thus

$$D_n(\phi) \ \sim G(\phi)^n\, E(\phi) \quad \text{as } n \to \infty.$$

The term $E(\phi)$ has a nice concrete description.

If we have a Wiener-Hopf factorization for $\phi = \phi_-\phi_+$, then

$$T(\phi)T(\phi^{-1}) = T(\phi_-)T(\phi_+)T^{-1}(\phi_-)T^{-1}(\phi_+)$$

and this is of the form

$$e^A e^B e^{-A} e^{-B}$$

where

$$A = T(\log(\phi_-)), \quad B = T(\log(\phi_+)).$$

From this we can use a formula for determinants of multiplicative commutators of this form.,

$$\det\left(e^{A}e^{B}e^{-A}e^{-B}\right) = \exp\,\left(\mathrm{trace}(AB - BA)\right)$$

and this then becomes the well-known formula

$$\exp\left(\sum_{k=1}^{\infty} k\,(\log\phi)_k\,(\log\phi)_{-k}\right).$$

Notice this is never zero.

This is a very versatile proof. It can be adapted to the following:

- The same proof works almost word for word for matrix-valued symbols. The only difference is that one needs to assume a two sided factorization and one needs to take care about the order. However, the constant often cannot be simplified and can even vanish.

- The same proof with the addition of a Jacobi identity can be used to prove the an identity called the Borodin-Okounkov-Case-Geronimo (BOCG) identity This says that ($z = e^{i\theta}$)

$$\det T_n(\phi) = G(\phi)^n\, E(\phi) \cdot \det\left(I - H(z^{-n}\phi_-\phi_+^{-1})H(\tilde{\phi}_-^{-1}\tilde{\phi}_+ z^{-n})\right)$$

- It can be adapted to the following case. Suppose F is analytic on a sufficiently large disc so that the operator $F(T(\phi))$ is defined. What are the asymptotics of the determinant of $P_n F(T(\phi)) P_n$? The answer is

$$G(F(\phi))^n \det F(T(\phi)) T((F(\phi))^{-1}).$$

- It can be adapted to perturbations of a Toeplitz operator. For example it can be used to find the asymptotics of the determinants of

$$P_n T(\phi) + H(\phi) P_n.$$

- It can be adapted to the integral operator case

$$T(f)(x) = \int_a^b k(x-y)\, f(y)\; dy,$$

which is what we will now consider.

To define the Fredholm determinant $D(\lambda) = \det(I - \lambda T)$ for an integral operator set

$$K\begin{pmatrix} x_1 & \cdots & x_j \\ y_1 & \cdots & y_j \end{pmatrix} = \det\Big(K(x_p, y_q)\Big), \quad p, q, = 1, \cdots, j.$$

Then $D(\lambda)$ is given by the series

$$1 - \lambda \int K\begin{pmatrix} x_1 \\ x_1 \end{pmatrix} dx_1 + \frac{(\lambda)^2}{2!} \int\int K\begin{pmatrix} x_1 & x_2 \\ x_1 & x_2 \end{pmatrix} dx_1 dx_2 + \cdots$$

To show that this series defines an entire function (at least for a finite interval) we let M be a bound for $|K(x, y)|$. Note that if a matrix A has columns $A_1, \cdots, A_n$ then

$$|\det(A)| \leq ||A_1|| \cdots \leq ||A_n||.$$

This means that each integrand

$$\det\left(K(x_i, x_j)\right)$$

has absolute value at most $M^n n^{n/2}$ and that each term of the series is at most

$$\frac{|\lambda|^n (b-a)^n M^n n^{n/2}}{n!}$$

and thus the series converges for all λ.

Thus $D(\lambda)$ is an entire function and it can be shown λ is a zero of $D(\lambda)$ then $\frac{1}{\lambda}$ is an eigenvalue of the operator T.

The set of non-zero eigenvalues is discrete and either constitutes a finite set or the eigenvalues have a limit point of zero.

So a natural question is how does this agree with our other definition of determinant? How do we show the integral operator is trace class?

To make things a bit more precise, the truncated Wiener-Hopf operator $W_\alpha(\sigma)$ is defined by

$$f(x) \to g(x) = f(x) + \int_0^\alpha k(x-y)\, f(y)\, dy,$$

where k is given by

$$k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\sigma(\xi) - 1) e^{-ix\xi} d\xi.$$

The symbol σ is defined on the real line and it is assumed that $\sigma - 1$ is in $L^1(R)$. If $\alpha = \infty$, then $W_\alpha(\sigma)$ is the usual Wiener-Hopf operator and will be denoted by $W(\sigma)$.

This last condition assures us that $W_\alpha(\sigma) - I$ is trace class. To see this, think of ξ as fixed and

$$e^{-i(x-y)\xi}(\sigma(\xi) - 1) = e^{-ix\xi}e^{iy\xi}(\sigma(\xi) - 1)$$

as the kernel of a rank one operator.

The kernel in question is a L^1 limit of sums of such operators and since trace class operators are closed in the trace norm, the truncated Wiener-Hopf operator is trace class.

So for T corresponding

$$f(x) \to \int_0^\alpha k(x-y)\, f(y)\, dy$$

then

$$\tilde{D}(\lambda) = \det(I - \lambda T)$$

is well defined.

One can show that if the integral operator is trace class then $D(\lambda) = \tilde{D}(\lambda)$ by using a limiting argument.

The standard Wiener-Hopf operator

$$f(x) \to g(x) = f(x) + \int_0^\infty k(x-y)\, f(y)\, dy,$$

is, using the properties of convolution, ($\widehat{\ }$ $^\vee$ mean Fourier and inverse Fourier transform)

$$f \to P(\sigma \hat{f}\,)^\vee$$

Thus the Wiener-Hopf operator is the same as the Toeplitz operator with the Fourier transform replacing the discrete transform, and where P is the projection onto $L^2(0,\infty)$.

In an analogous way, the truncated operator is the same as

$$P_\alpha W(\sigma) P_\alpha$$

where P_α is the projection onto $L^2(0,\alpha)$. We require that not only $\sigma - 1$ be in L^1, but that k is as well and that

$$\int_{-\infty}^{\infty} (1+|x|)|k(x)|^2 dx < \infty.$$

Then if σ is bounded away from zero, has index zero, then

$$\det(W_\alpha(\sigma)) \sim G(\sigma)^\alpha \; E(\sigma) = G(\sigma)^\alpha \; \det W(\sigma^{-1}) W(\sigma),$$

where $G(\sigma) = \exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \log\sigma\,(x)\,dx\right)$.

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