Universality in tiling models : a Master kernel

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Joint work with Mark Adler and Kurt Johansson:

Mark Adler, Kurt Johansson and P. van Moerbeke, *Tilings of non-convex Polygons, skew-Young Tableaux and determinantal Processes*, Comm Math Phys, **364**, 287-342 (2018) (arXiv: 1609.06995).

Mark Adler, Kurt Johansson and P. van Moerbeke, *Lozenge tilings of hexagons with cuts and asymptotic fluctuations: a new universality class*, Math Phys Anal Geom **21** 1-53. (2018)(arXiv:1706.01055).

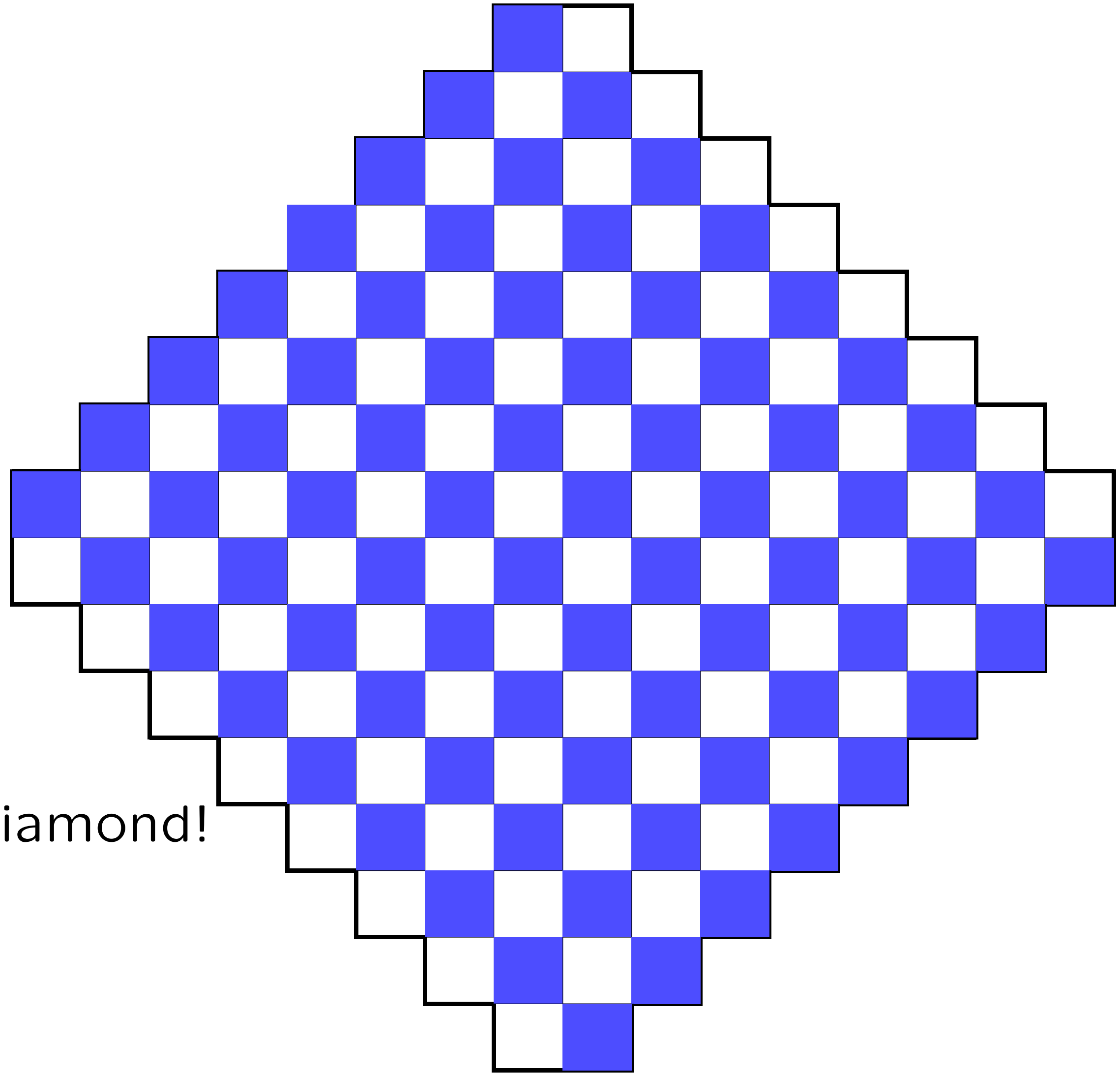
Mark Adler and P. van Moerbeke, *Probability distributions related to tilings of non-convex Polygons*, Journal of Math. Phys **59**, 091418 (2018) (Special volume in memory of Ludvig Faddeev)

Mark Adler, Kurt Johansson & P. van Moerbeke: *The discrete tacnode process, a universal and a master kernel.* (2019)

1. **Domino tilings of rectangular Aztec diamonds**

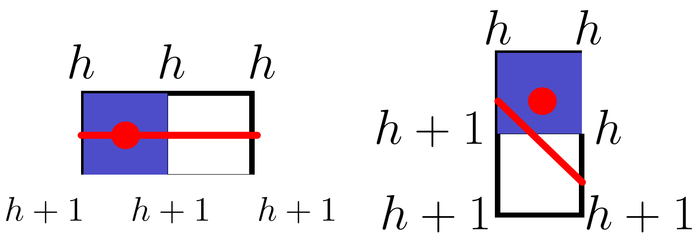
2. **Lozenge tilings of non-convex polygons !**

1. Domino tilings of Aztec diamonds



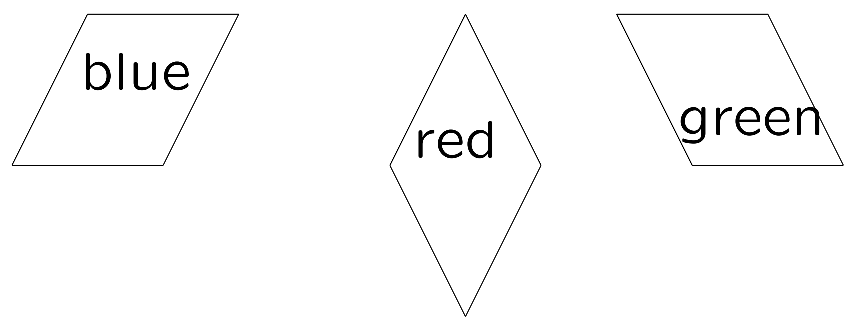
Usual Aztec diamond!

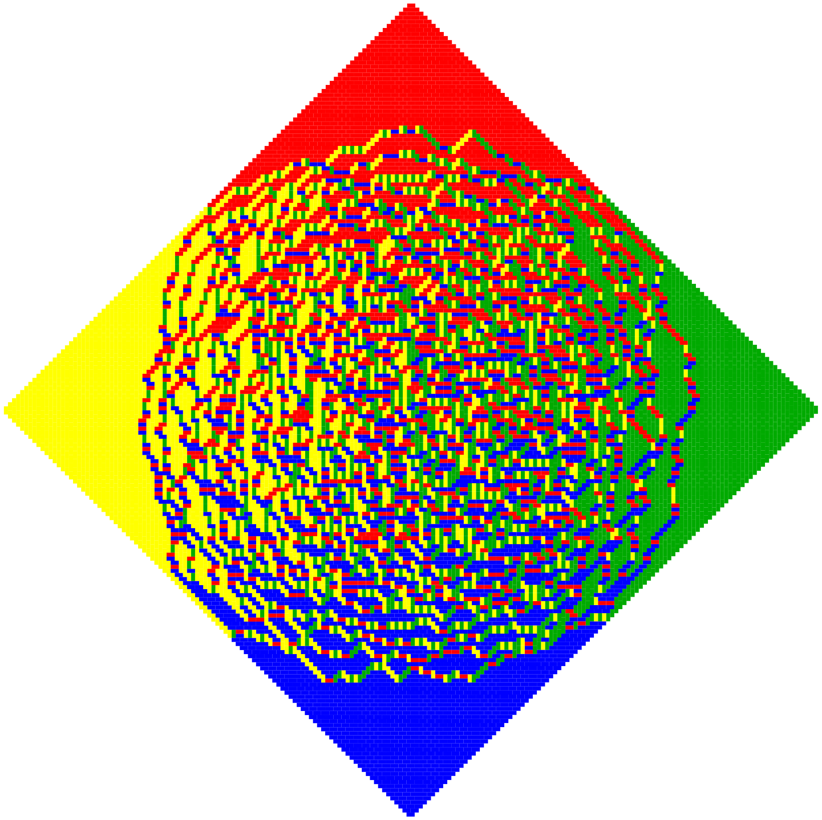
- Domino tilings by 4 kind of domino's, with a height function



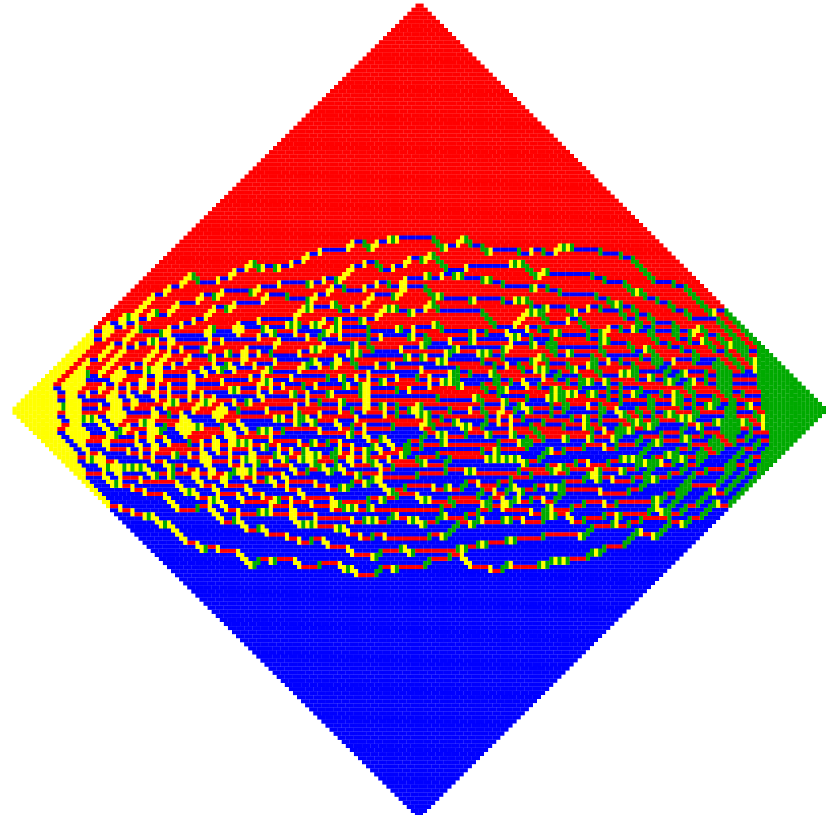
Blue Green Red Yellow

- Lozenge tilings of hexagonal regions





$a = 1$
Arctic circle (n=100)

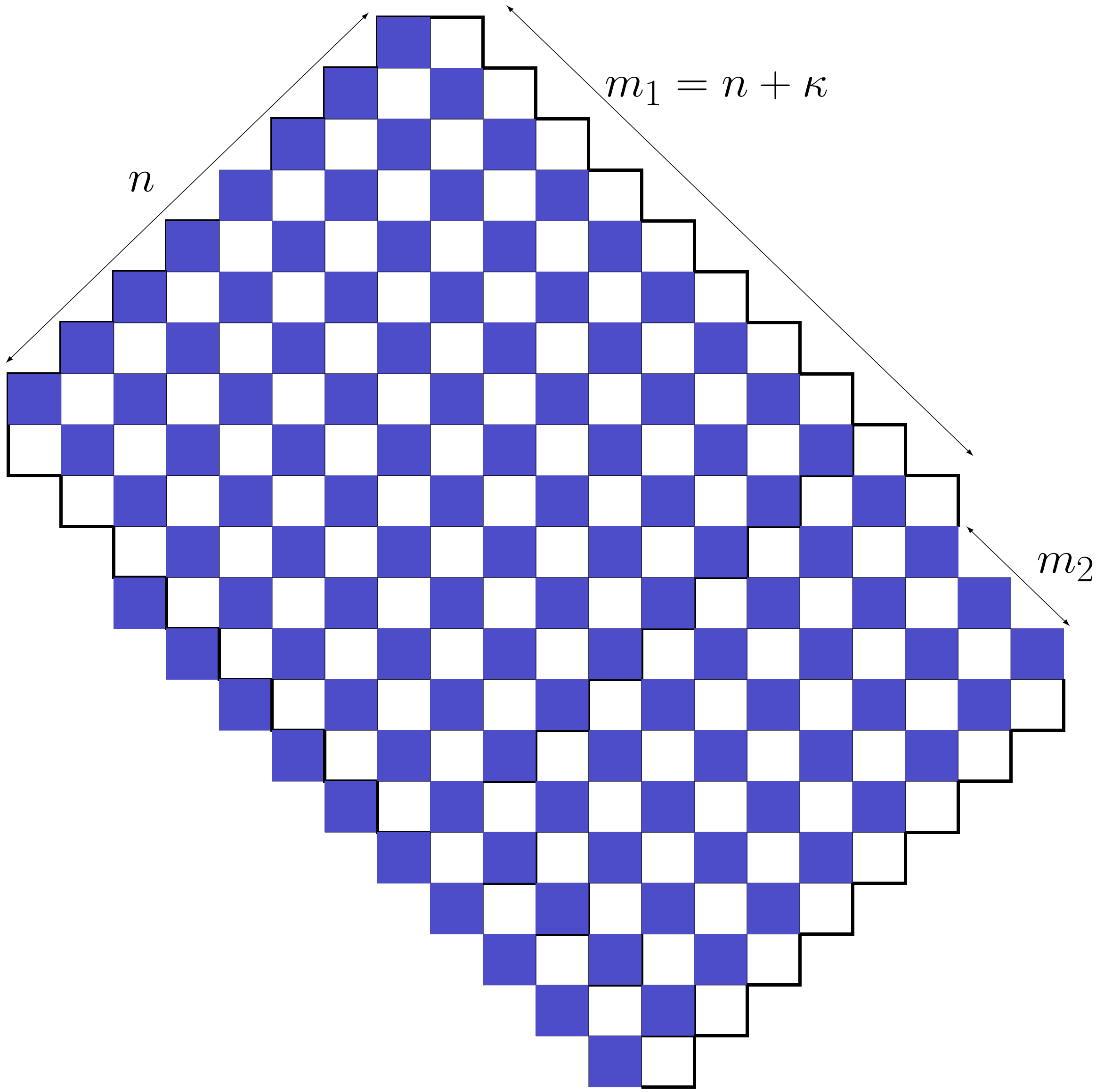


$a = 1/2$
Arctic ellipse (n=100)

- Airy Process (stationary process version of the Tracy-Widom distrib.)
- GUE-minor process (interlacing spectra of the minors of GUE-matrix)
- Gaussian Free Field

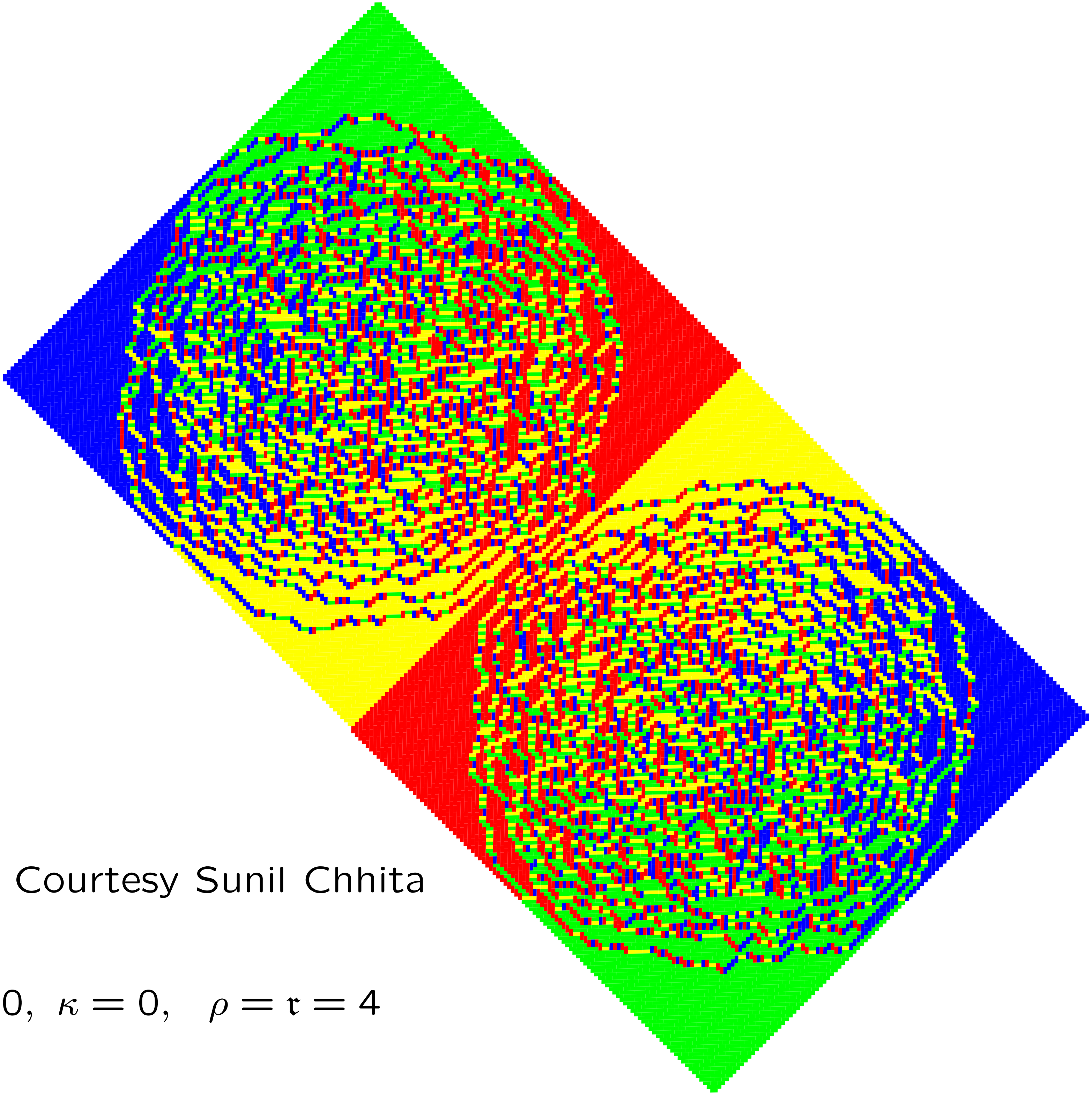
(courtesy Sunil Chhita)

Rectangular Aztec diamond: Covering with domino's



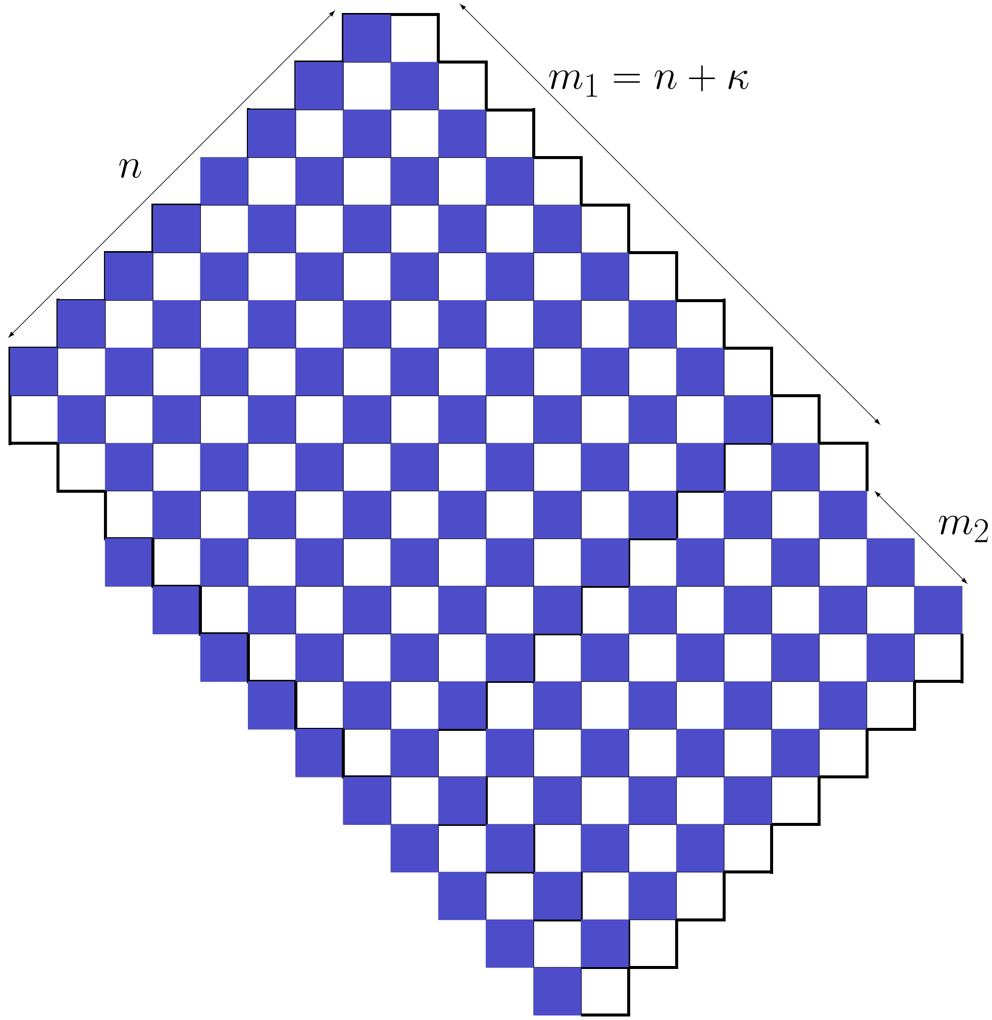
$$n = 8,\ m_1 = 10,\ m_2 = 3,\ \kappa = 2.$$

Imagine n, m_1, m_2 very large, keeping κ finite!



Courtesy Sunil Chhita

$n = 100,\ \kappa = 0, \quad \rho = \mathfrak{r} = 4$



Proposition: an Aztec rectangle is tilable iff

$$m_1 \geq 0, \quad m_2 \leq n+1$$

Throughout assume $\kappa \geq 0$!

• Weights:

– vertical domino's have weight a
– horizontal domino's have weight 1

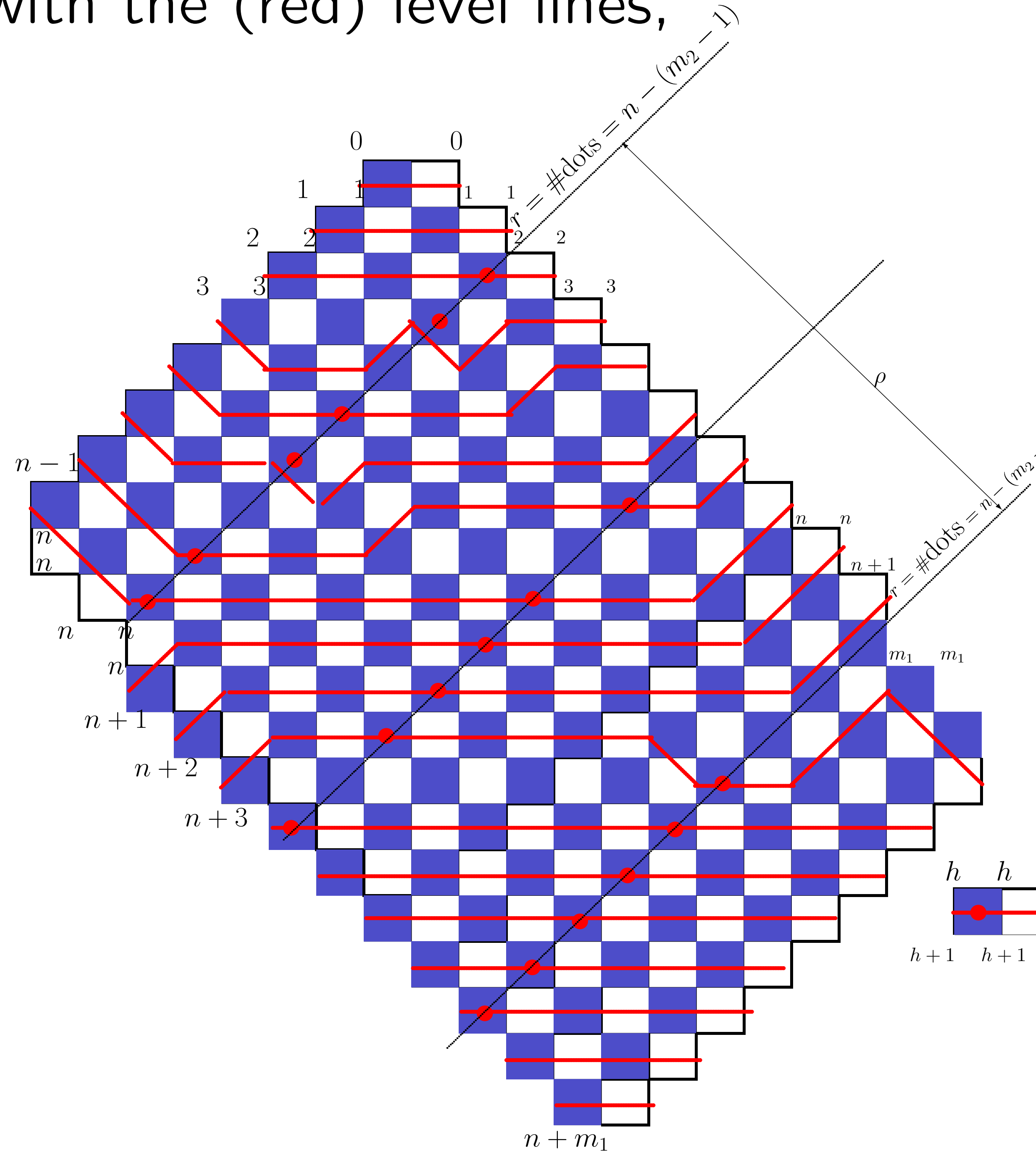
• Probability on tilings:

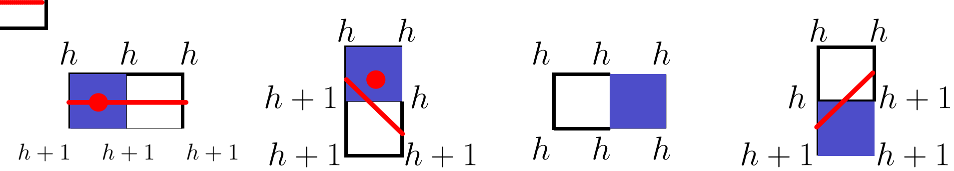
$$\mathbb{P}(\text{domino tiling } T) = \frac{a^{\#\text{vertical domino's in } T}}{\sum_{\text{all possible tilings } T} a^{\#\text{vertical domino's in } T}}$$

• GOAL: study the following random point process with kernel $\mathbb{K}^{red}$:

• Random surface and its level lines, corresponding to levels $1/2,\ 3/2,\ \ldots, n+1/2$, etc....

• The point process of the intersections of the oblique lines with the (red) level lines,





Random surface and its level lines: two important numbers ρ and $\mathfrak{r}$.

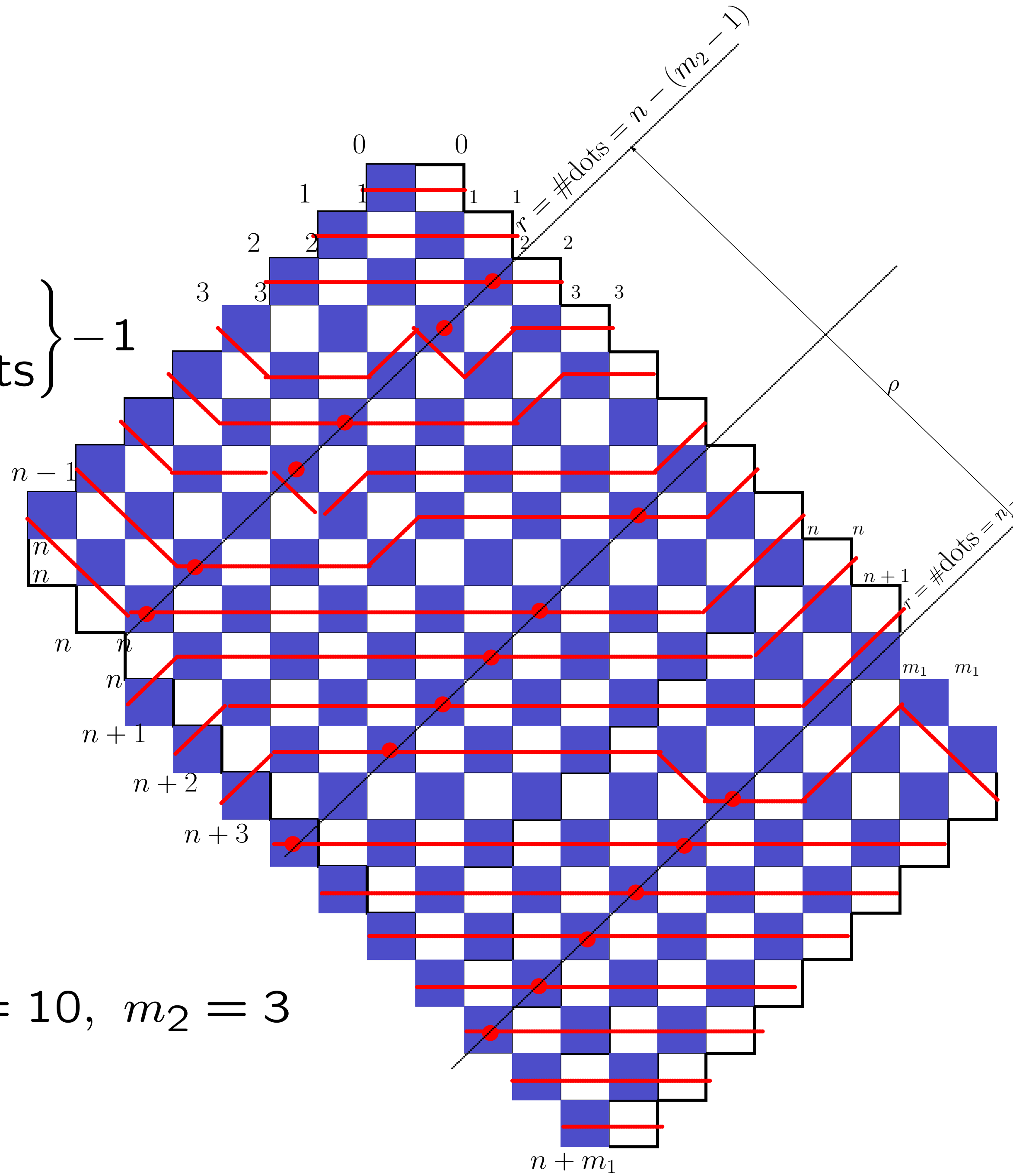
Imagine $n,\ m_1,\ m_2$ very large,

keeping ρ and $\mathfrak{r}$ finite!

$$\rho = \#\left\{\begin{array}{l}\text{oblique lines with}\\ \text{minimal \# of red dots}\end{array}\right\} - 1$$

$$= m_1 - m_2 + 1$$

$$= \text{width of strip} = 8$$



In this example $n = 8,\ m_1 = 10,\ m_2 = 3$

Random surface and its level lines: two important numbers ρ and $\mathfrak{r}$.

Imagine $n, \; m_1, \; m_2$ very large,

keeping ρ and $\mathfrak{r}$ finite!

$$\rho = \# \left\{ \begin{array}{l} \text{oblique lines with} \\ \text{minimal \# of red dots} \end{array} \right\} - 1$$

$$= m_1 - m_2 + 1$$

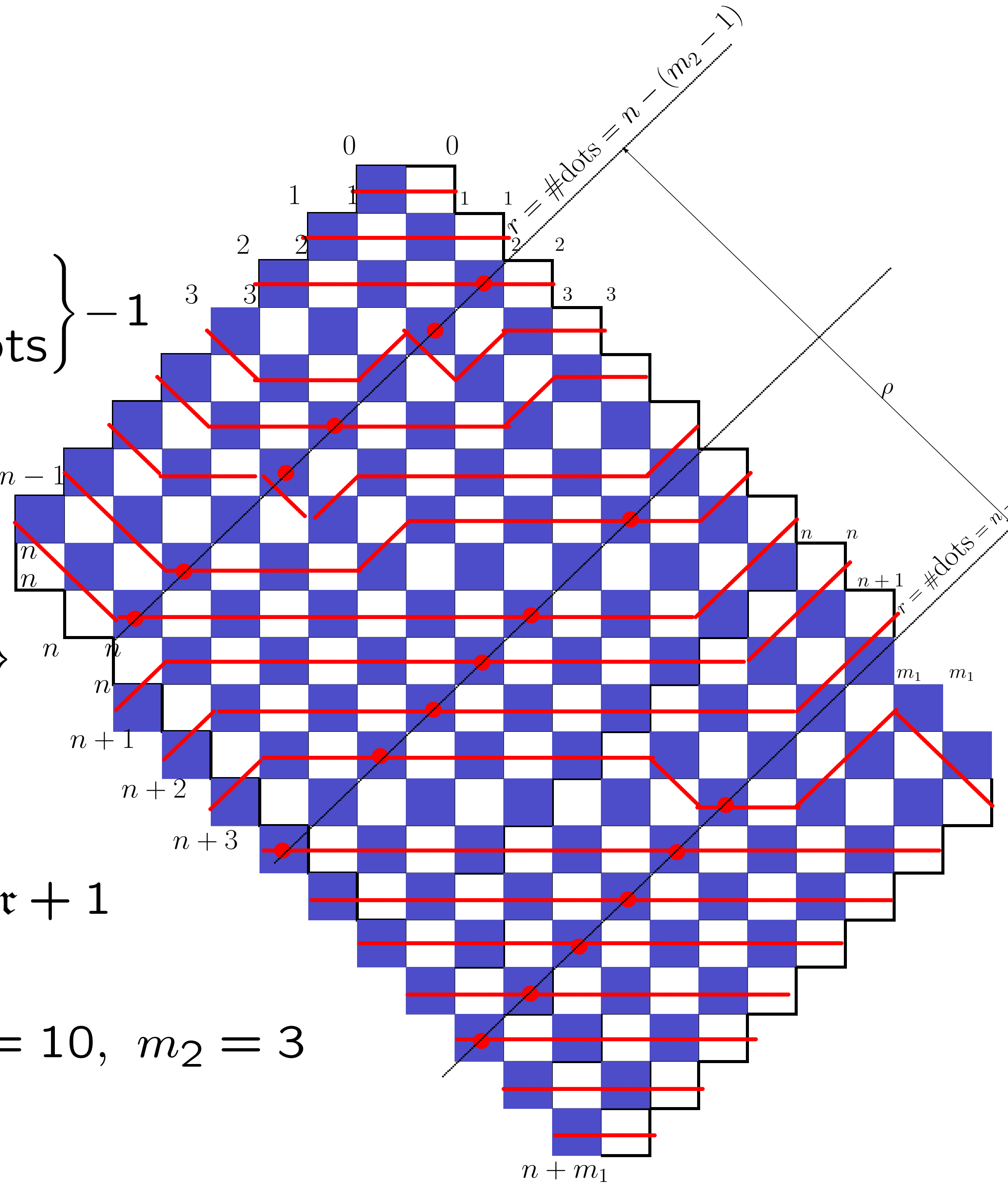
$$= \text{width of strip} = 8$$

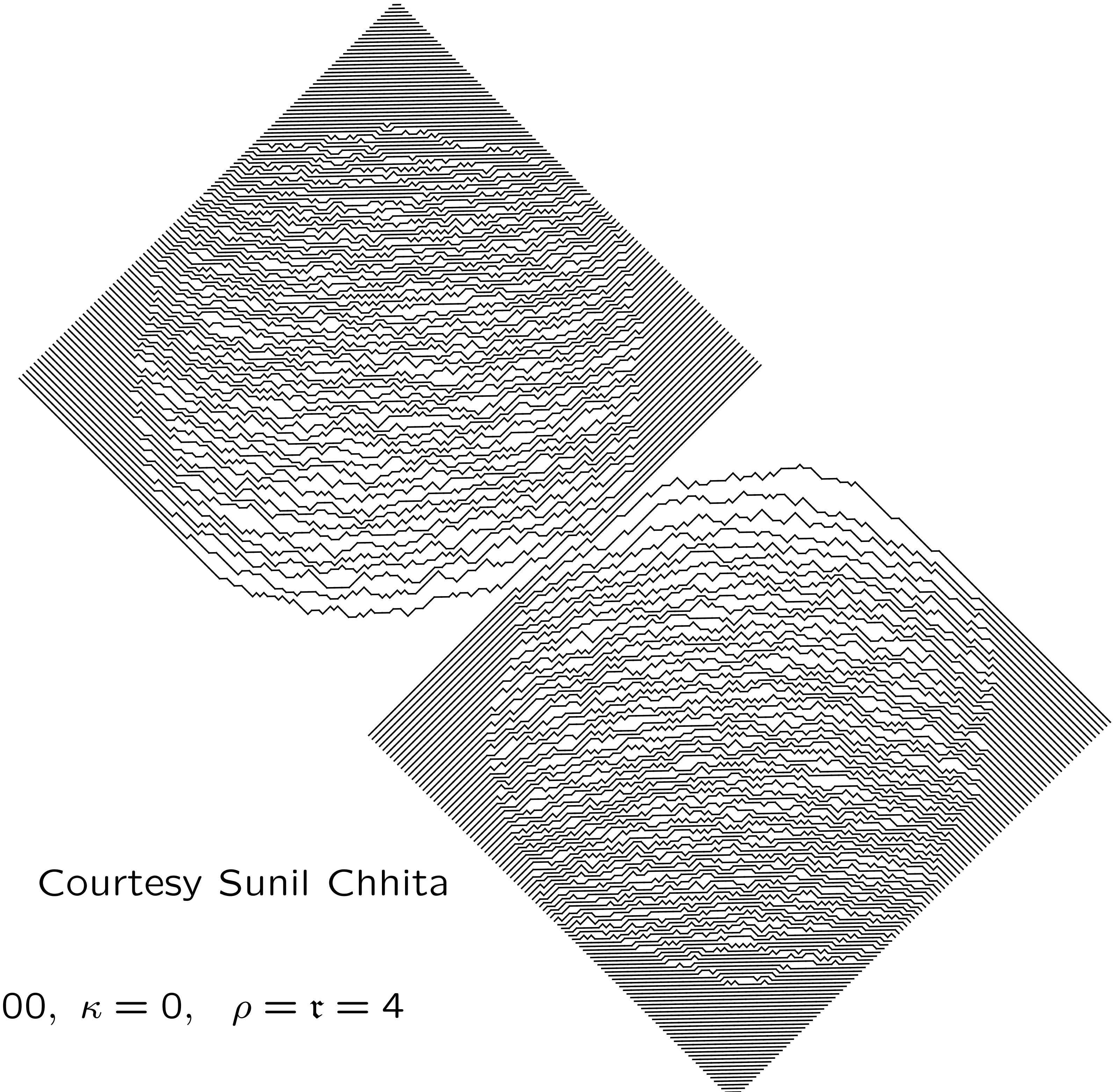
$$\mathfrak{r} = \# \left\{ \begin{array}{l} \text{red dots along the} \\ \rho + 1 \text{ oblique lines} \end{array} \right\}$$

$$= n - m_2 + 1 = \rho - \kappa = 6,$$

$$m_1 = n - \mathfrak{r} + \rho, \quad m_2 = n - \mathfrak{r} + 1$$

In this example $n = 8, \; m_1 = 10, \; m_2 = 3$



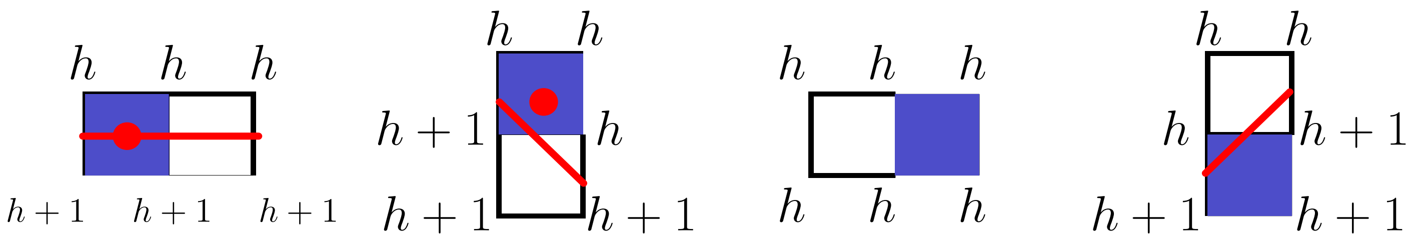


Courtesy Sunil Chhita

$n = 100, \ \kappa = 0, \quad \rho = \mathfrak{r} = 4$

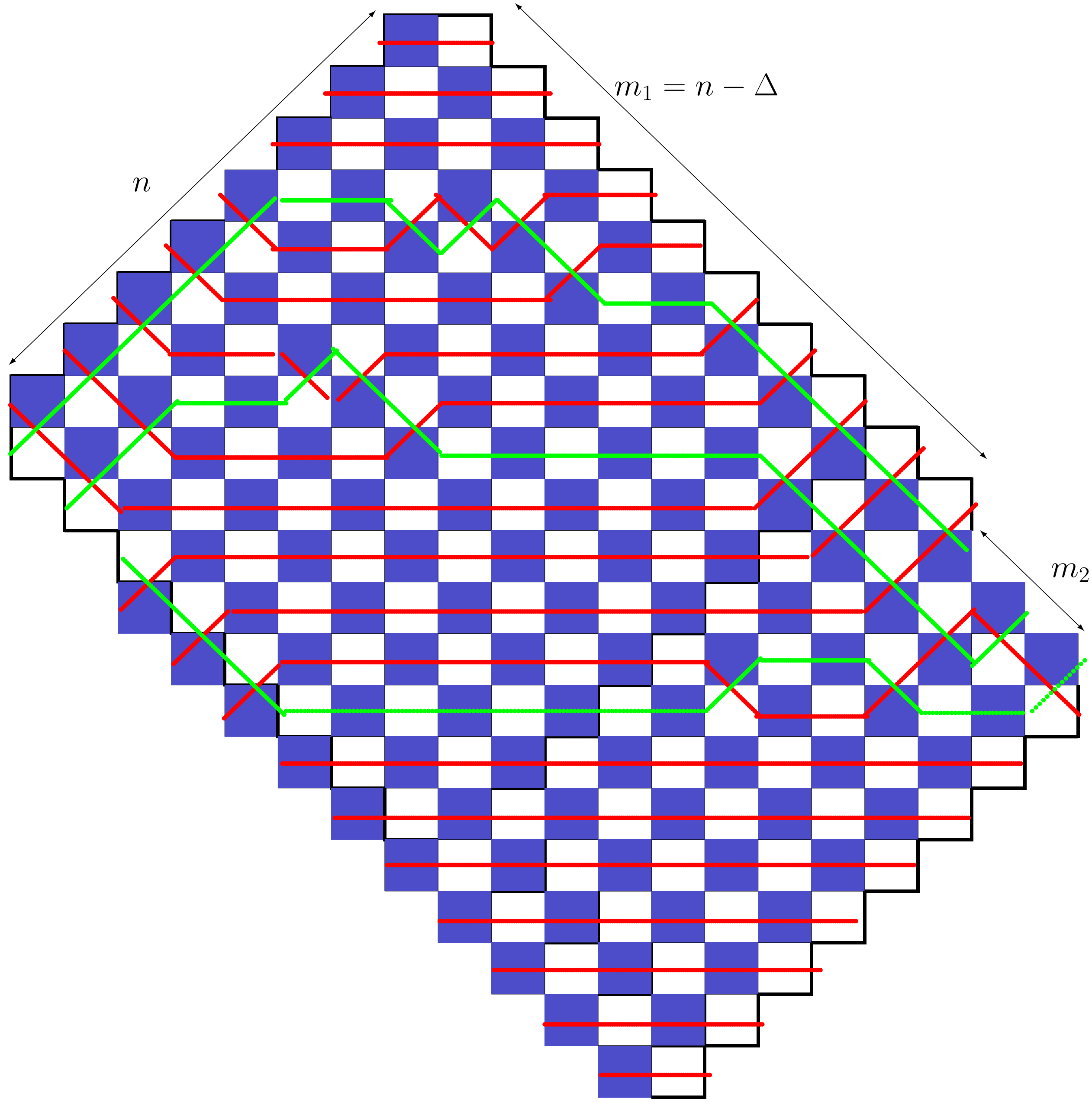
Dual paths to the level lines (green paths):

Remember the red paths coming from the following tiles:



Construct the dual paths (green) to the red paths :

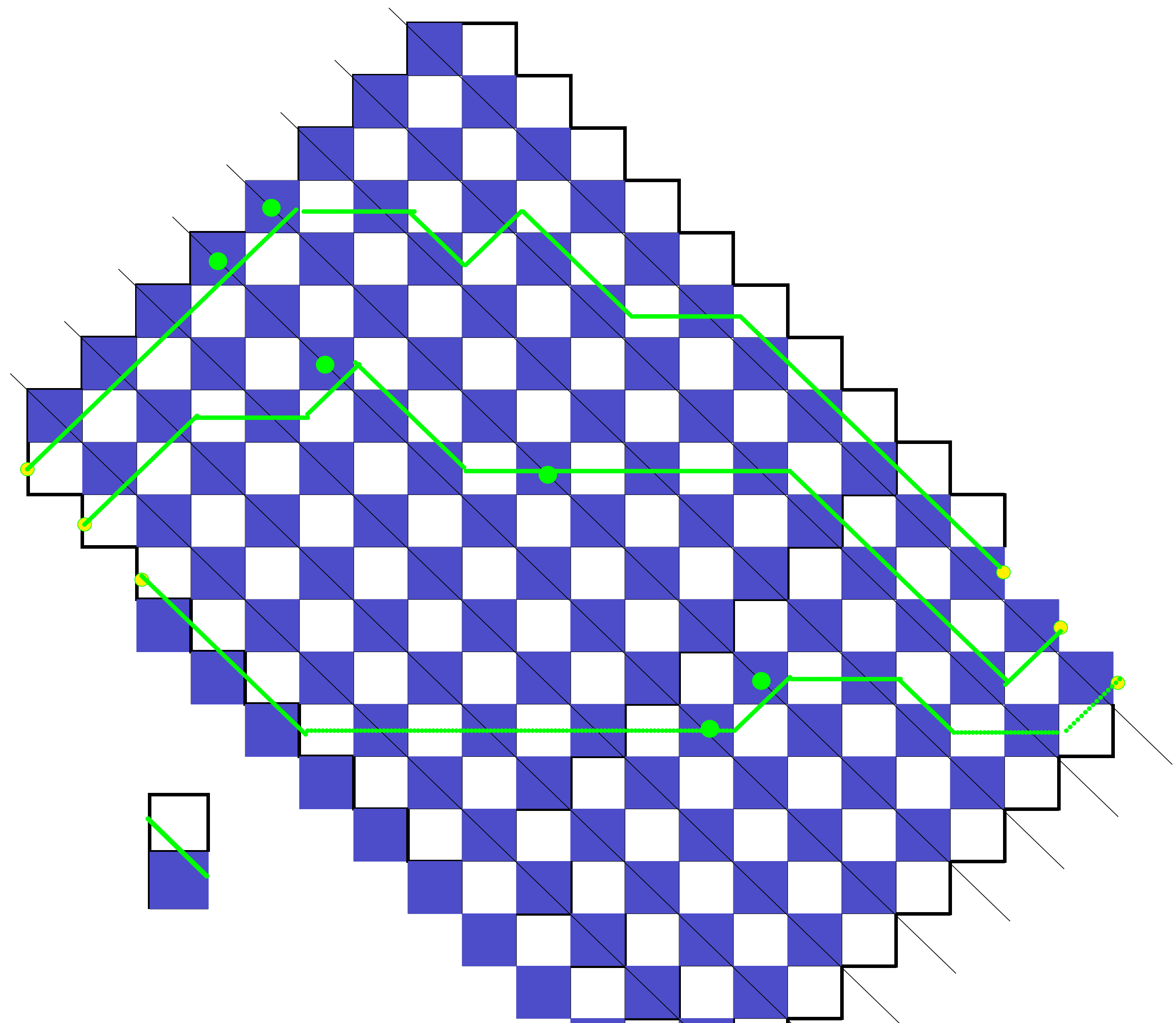
(Green) Dual paths to the (red) paths:



STEP 1: Point process defined by the intersection of the dual paths (green) with oblique lines:

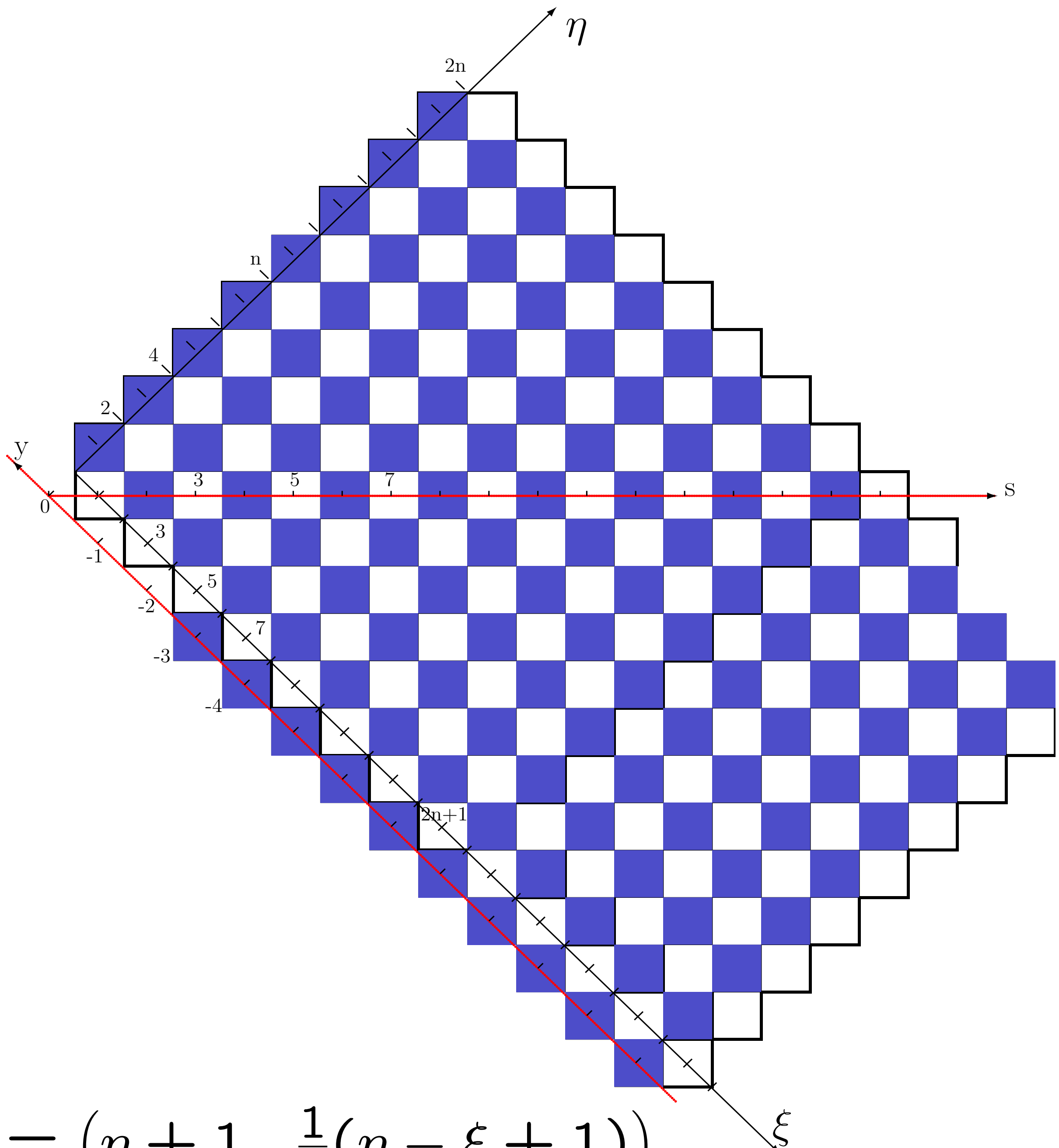
(1) the green paths connect m_2 contiguous points

(2) $\#\{$ dots per oblique line$\} = m_2$



Kernel $\mathbb{K}^{\textbf{green}}$ for the Green paths , where $s_i,\ y_i \in \mathbb{Z}$.

Coordinates (ξ, η) and (s, y):



$$(\xi, \eta) \to (s, y) = \left(\eta + 1,\ \frac{1}{2}(\eta - \xi + 1)\right)$$

$$\mathbb{K}^{\mathsf{green}}(s_1, y_1; s_2, y_2)$$
$$= -\mathbb{1}_{s_1<s_2} p_{s_1;s_2}(y_1, y_2) + \widetilde{\mathbb{K}}^{\mathsf{green}}(s_1, y_1; s_2, y_2)$$

where (with radii $a < \rho_1 < \sigma_1 < \sigma_2 < \rho_2 < a^{-1}$)

$$\widetilde{\mathbb{K}}^{\mathsf{green}}(s_1, y_1; s_2, y_2)$$
$$= \int_{\gamma_{\rho_1}} \frac{z^{y_1} dz}{2\pi \mathrm{i} z} \int_{\gamma_{\rho_2}} \frac{w^{-y_2} dw}{2\pi \mathrm{i} w} \psi_{s_1,2n+1}(z) \psi_{0,s_2}(w)$$
$$\times \frac{D_{m_2-1}\left[\psi_{0,2n+1}(\zeta)\left(1-\frac{\zeta}{w}\right)\left(1-\frac{z}{\zeta}\right)\right]}{D_{m_2}\left[\psi_{0,2n+1}(\zeta)\right]}.$$

where

$$D_n(f) = \det\left(\hat{f}_{i-j}\right)_{1\le i,j\le n}, \quad \text{with} \quad \hat{f}_k = \oint_{S^1} \frac{dz}{2\pi \mathrm{i} z} \frac{f(z)}{z^k}$$

Toeplitz determinant with singularity!

Transition functions:

$$\varphi_{s_1,s_2}(\zeta) = \frac{(1+a\zeta)^{[\frac{s_2}{2}]-[\frac{s_1}{2}]}}{(1-\frac{a}{\zeta})^{s_2-[\frac{s_2}{2}]-s_1+[\frac{s_1}{2}]}}$$

$$\psi_{s_1,s_2}(\zeta) = \begin{cases} \varphi_{s_1,s_2}(\zeta) & \text{for } s_2 < 2n+1 \\ z^{\kappa}\varphi_{s_1,s_2}(\zeta) & \text{for } s_2 = 2n+1 \end{cases}$$

$$p_{s_1,s_2}(x,y) = \int_{\gamma_{\rho_1}} \zeta^{x-y}\varphi_{s_1,s_2}(\zeta)\frac{d\zeta}{2\pi\mathrm{i}\zeta}$$

Key Lemma: The Toeplitz determinant for a singular symbol:

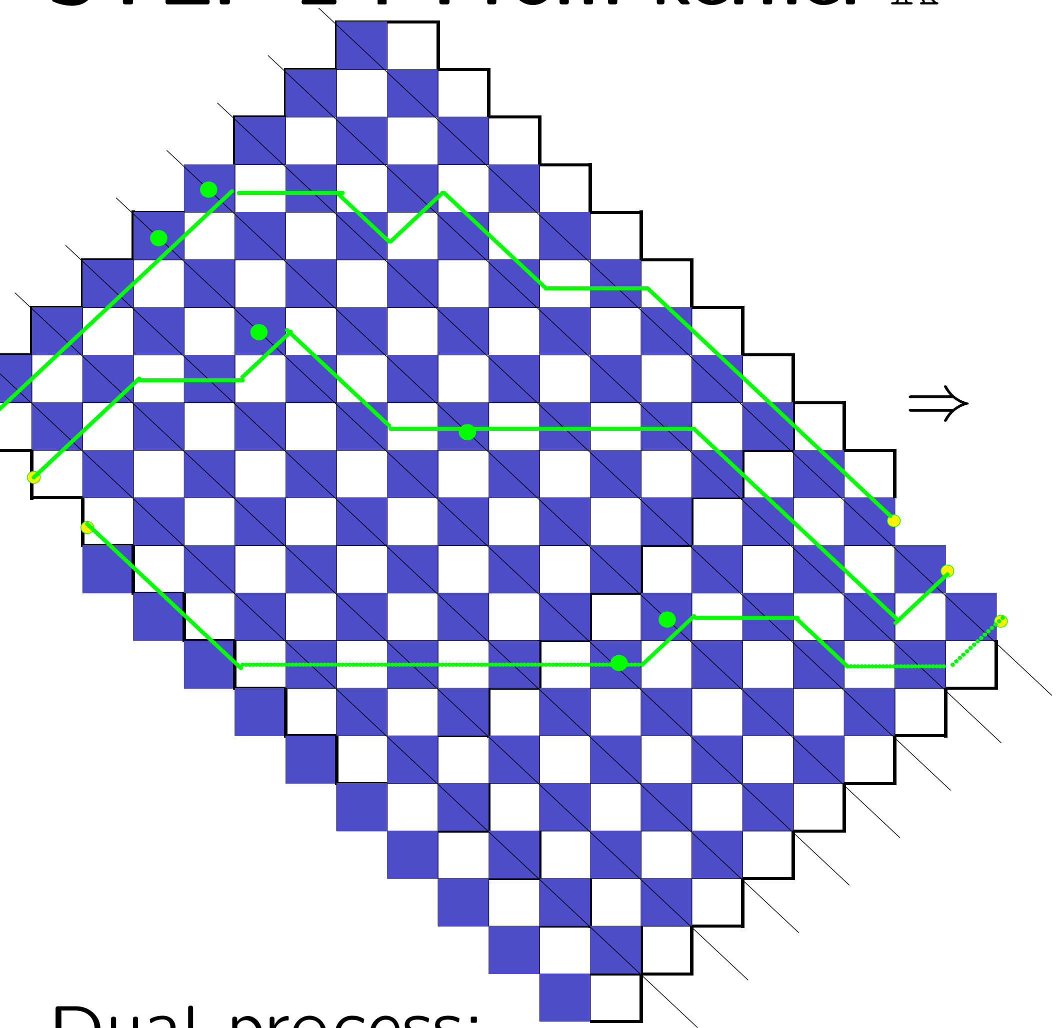
$$D_n[\zeta^{\pm\kappa} f(\zeta)] = (-1)^{\kappa n} \oint_{(\Gamma_R)^\kappa} \prod_1^\kappa \frac{\lambda_j^n d\lambda_j}{2\pi \mathrm{i} \lambda_j} D_n \left[\frac{P^\lambda(\zeta^{\pm 1})}{\prod_1^\kappa \lambda_i} f(\zeta) \right],$$

where $P^\lambda(z) := \prod_{i=1}^\kappa (\lambda_i - z)$.

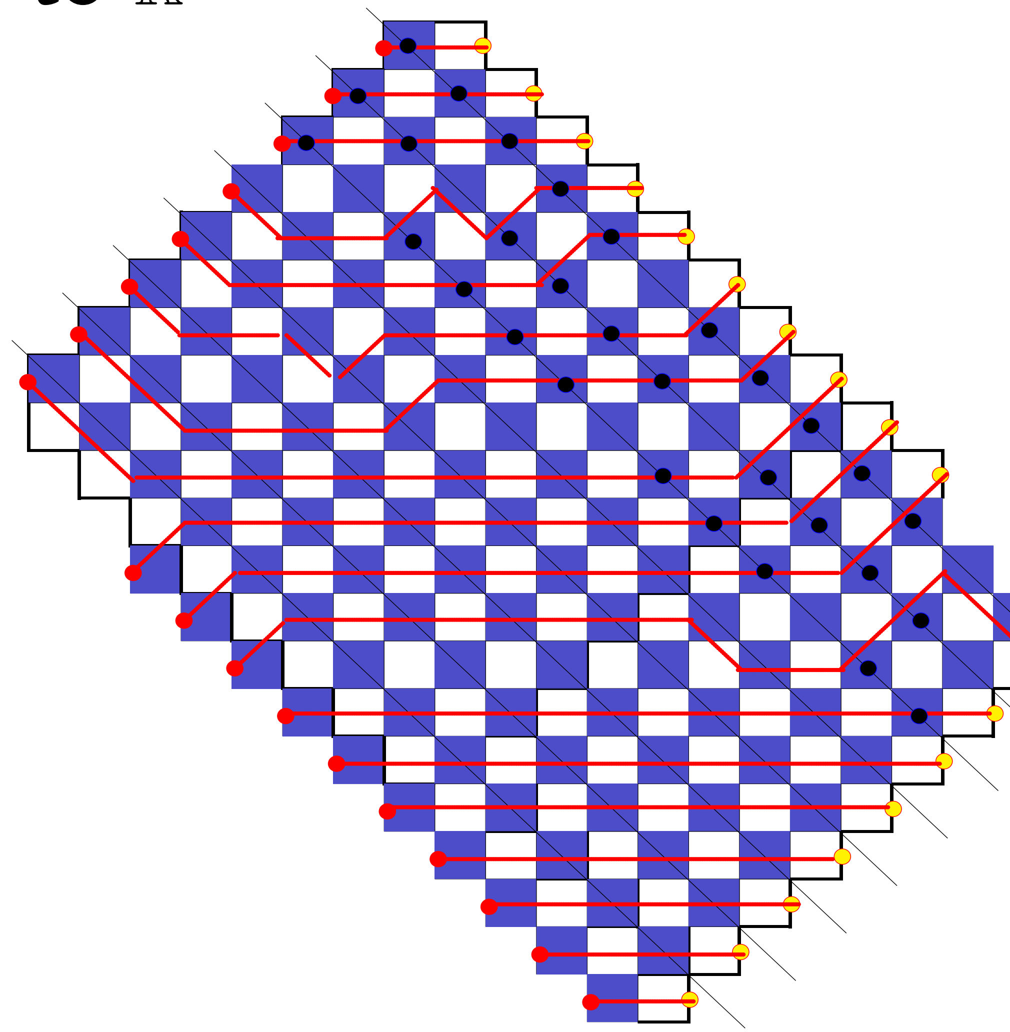
Lemma (Case-Geronimo-Borodin-Okounkov)

$$D_{m_2-1} \left[\psi_{0,2n+1}(\zeta) \left(1 - \frac{\zeta}{w}\right) \left(1 - \frac{z}{\zeta}\right) \right]$$
$$= {\scriptstyle f(w,z)} \oint_{(\Gamma_R)^\kappa} \left[\prod_1^\kappa \frac{\lambda_j^{m_2-1} d\lambda_j}{2\pi \mathrm{i}} \right] \frac{1}{P^\lambda(z)} \det(\mathbb{1} - \mathcal{K}_{k,\ell}^{(\lambda)}(w^{-1}, z))_{\geq m_2 - 1}$$

STEP 2 : From kernel $\mathbb{K}^{\text{green}}$ to $\mathbb{K}^{\text{blue}}$



$\Rightarrow$



Dual process:

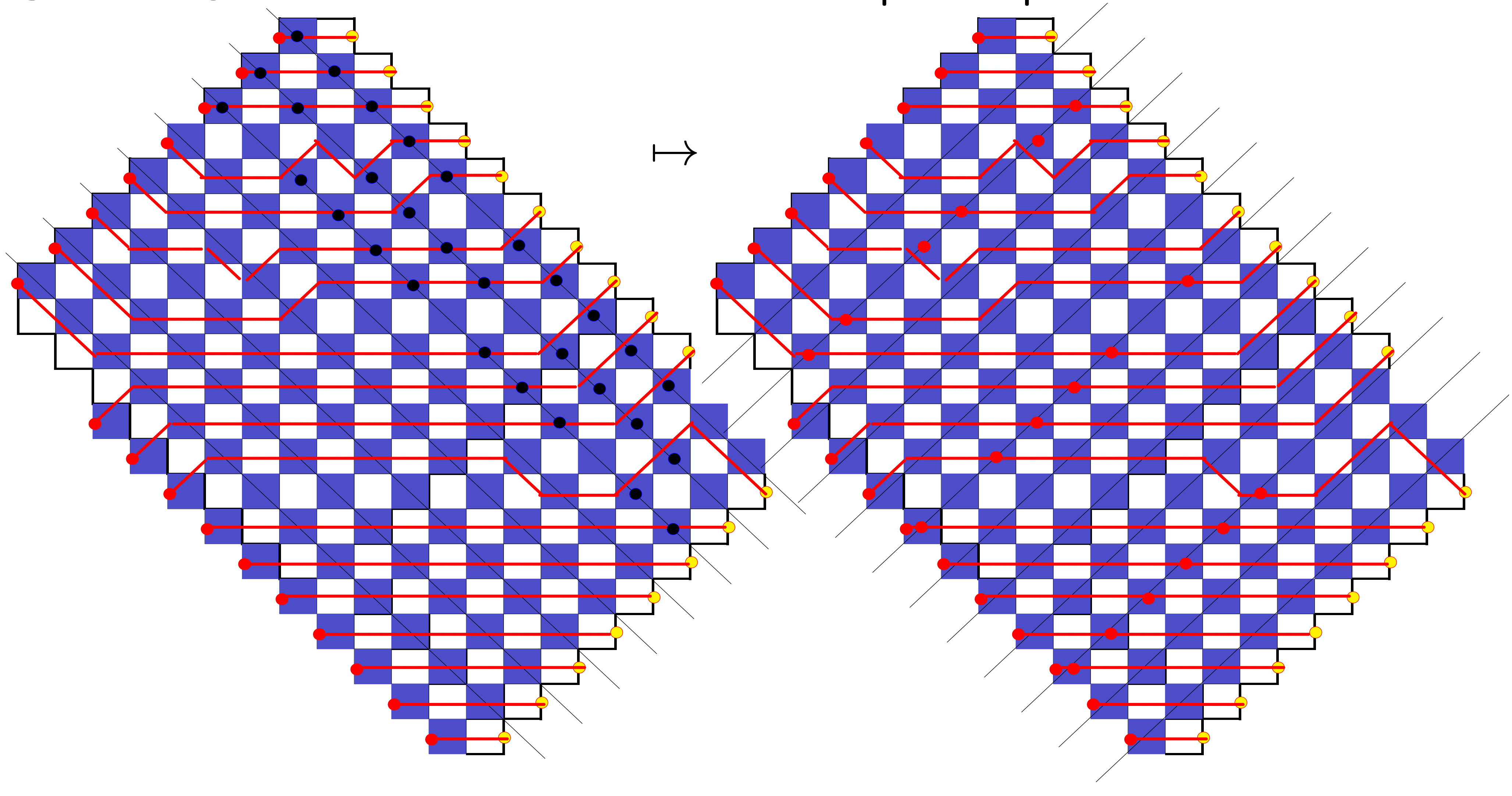
$$\mathbb{K}^{blue}(n,u_1;n,u_2) = \mathbb{1}_{\{u_1=u_2\}} - \mathbb{K}^{green}(n,u_1;n,u_2).$$

For any s_1 and s_2:

$$\begin{aligned}
&\mathbb{K}^{blue}(s_1, u_1; s_2, u_2) \\
&\quad = - \mathbb{1}_{s_2 < s_1} p_{s_1, s_2}(u_1, u_2) \\
&\quad\quad + p_{s_1, n}(u_1, \bullet) *_\bullet \ \mathbb{K}^{blue}(n, \bullet; n, \circ) *_\circ \ p_{n, s_2}(\circ, u_2)
\end{aligned} \tag{1}$$

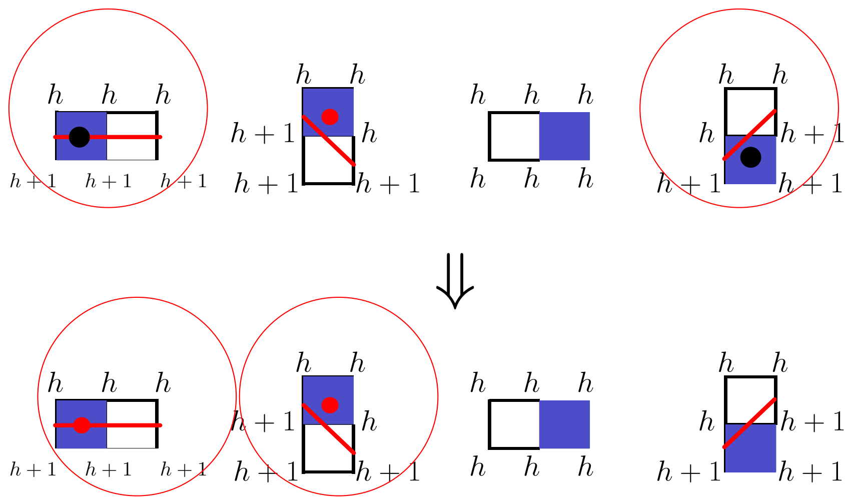
Notice: $f(x) *_x \ g(x) = \sum_{x \in \mathbb{Z}} f(x) g(x)$.

STEP 3: From $\mathbb{K}^{blue}$ to the actual point process $\mathbb{K}^{red}$:

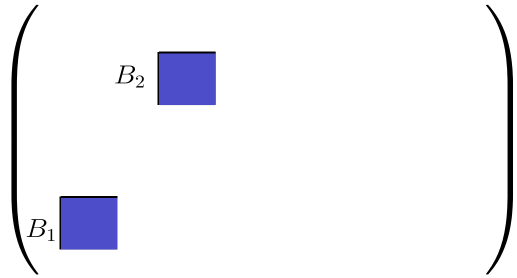


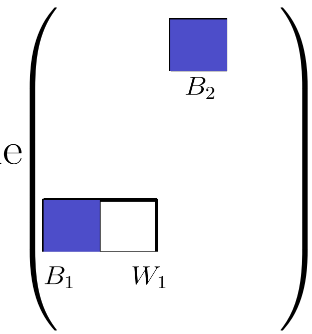
Going from

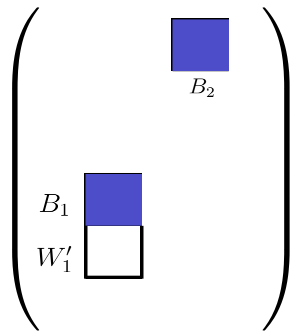
to



Theorem:

$$\mathbb{K}^{\mathrm{red}}(B_1, B_2) = \mathbb{K}^{\mathrm{red}}$$ 

$$=\mathbb{K}^{\mathrm{blue}}$$ 

$$-\,a\mathbb{K}^{\mathrm{blue}}$$ 

$$= \quad \mathbb{K}^{\mathrm{blue}}(B_2, W_1) \quad -a \ \mathbb{K}^{\mathrm{blue}}(B_2, W_1')$$

Proof: the dimer model of the tilings is the dual graph of the Aztec rectangle. It is obtained by

• putting a black circle $(b_1, b_2) \in (2\mathbb{Z}, 2\mathbb{Z}+1)$ in the middle of a black square

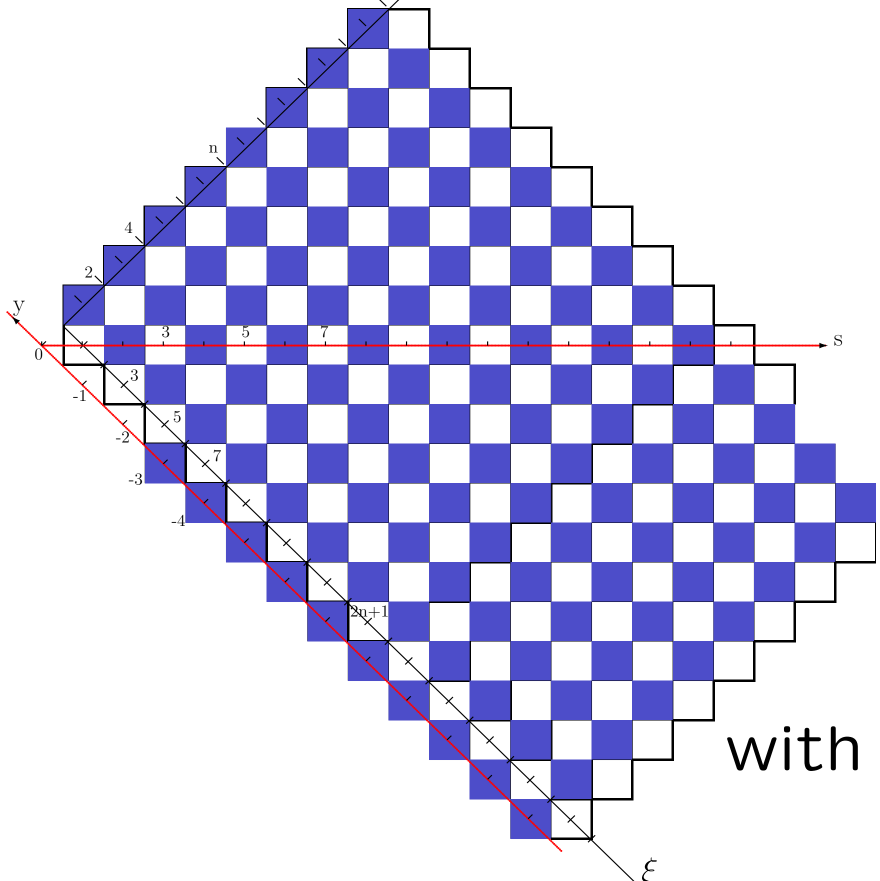
• putting a white circle $(w_1, w_2) \in (2\mathbb{Z}+1, 2\mathbb{Z})$ in the middle of a white square

Kasteleyn matrix K_{Kast} = the adjacency matrix for this dual graph.

One shows:

$$K_{Kast}^{-1}\Big((w_1, w_2), (b_1, b_2)\Big) = -(-1)^{(w_1 - w_2 + b_1 - b_2 + 2)/4}$$
$$\times \mathbb{K}^{blue}\left(b_2 + 1, \frac{1}{2}(b_2 - b_1 + 1), w_2 + 1, \frac{1}{2}(w_2 - w_1 + 1)\right)$$

Then deduce $\mathbb{K}^{red}$ from K_{Kast} and K_{Kast}^{-1} ∎



$$m_2 - 1 \leq n \leq m_1 = n + \kappa$$

Asymptotics about

$\xi \simeq 2n$ and $\eta \simeq n$

with $m_1 = n + \rho - \mathfrak{r}, \quad m_2 = n + 1 - \mathfrak{r}$

STEP 4. SCALING: For $n \to \infty$, $x_i \in \mathbb{Z}$ and $y_i \in \mathbb{R}$,

$$a = 1 + \frac{\beta}{\sqrt{n}}, \; \xi_i := 2n - 2x_i, \quad \eta_i := n + y_i\sqrt{2n} - 1, \; \frac{\Delta\eta}{2} = dy\sqrt{\frac{n}{2}}$$

Putting the scaling in the kernel (before letting $n = \frac{1}{t^2} \to \infty$)

$$\begin{aligned}
&\mathbb{K}^{red}(\xi_1, \eta_1; \xi_2, \eta_2)\frac{\Delta\eta}{2} \\
&\quad = (-1)^{r-s} a^{x-y} t^{x_2 - x_1} \mathbb{L}^{red}(x_1, y_1; x_2, y_2)\sqrt{2}dy\frac{1+a^2}{2}
\end{aligned}$$

$$\mathbb{L}^{red} = -\mathbb{L}_0 + \mathbb{L}_1 + \mathbb{L}_2$$

with $(a < \rho_1 < \sigma_1 < \sigma_2 < R < \rho_2 < \frac{1}{a})$

$$\begin{aligned}
-\mathbb{L}_0(x_1, y_1; x_2, y_2) &= (-\sqrt{2})^{x_1 - x_2 - 1}\mathbb{H}^{(x_1 - x_2)}(y_2 - y_1) + O(t)) \\
\mathbb{L}_1(x_1, y_1; x_2, y_2) &= \frac{\mathcal{L}\left(-\mathcal{S}(\lambda)\right)}{\mathcal{L}\left(1\right)} \\
\mathbb{L}_2(x_1, y_1; x_2, y_2) &= \frac{\mathcal{L}\left(\mathcal{R}(\lambda) + \left\langle (\mathbb{1} - \mathcal{K}^{(\lambda)}(0,0))^{-1}_{\geq 0} A^{\lambda}_{x_1, y_1}, B^{\lambda}_{x_2, y_2} \right\rangle\right)}{\mathcal{L}\left(1\right)}
\end{aligned}$$

$$\mathcal{L}(f) := \oint_{(\Gamma_R)^\kappa} d\mu(\lambda) \det(I - \mathcal{K}^{(\lambda)}(0,0))_{\geq 0} f(\lambda)$$

where

$$d\mu(\lambda) := \frac{1}{\psi} \prod_{j=1}^{\kappa} \frac{(1 - t\lambda_j)^{n+1} d\lambda_j}{2\pi \mathrm{i} \lambda_j{}^{\mathfrak{r}+1}}, \text{ with } \oint_{(\Gamma_R)^\kappa} d\mu(\lambda) = 1 \qquad (2)$$

For a kernel A independent of w, z, we have:

- $\det(I - A + (w-z)a(w) \otimes b(z))$

$$= (w-z)\det(I - A + a(w) \otimes b(z)) + (1-w+z)\det(I-A)$$

- $\oint_{\gamma_{r_2}} \frac{dwF(w)}{2\pi \mathrm{i}} \oint_{\gamma_{r_1}} \frac{dzG(z)}{2\pi \mathrm{i}} \det\Big(I - A + a(w) \otimes b(z)\Big)$

$$= \det\left(I - A + \left(\oint_{\gamma_{r_2}} a(w)\frac{dwF(w)}{2\pi \mathrm{i}}\right) \otimes \left(\oint_{\gamma_{r_1}} b(z)\frac{G(z)dz}{2\pi \mathrm{i}}\right)\right)$$
$$+ \left[\left(\oint_{\gamma_{r_2}} F(w)\frac{dw}{2\pi \mathrm{i}}\right)\left(\oint_{\gamma_{r_1}} G(z)\frac{dz}{2\pi \mathrm{i}}\right) - 1\right]\det(I-A)$$

- For $Q(z) = \prod_1^n (z - v_i) = \sum_{i=1}^n (-1)^i z^{n-i} \varepsilon_i(v_1, \ldots, v_n)$, we have

$$\prod_1^n \varepsilon_i(v_{J_\sigma})^{\gamma_i} = \sum_{\substack{0 \le \lambda_1 \le \sum_1^n \gamma_i \\ 0 < \lambda_1^\top \le \sigma,\ |\lambda| \le \sum_1^n i\gamma_i}} C_\lambda(\gamma_0) S_\lambda(v_1, \ldots, v_n), \quad \text{Schur pol.}$$

Step 5. Limiting kernel $=$ **"Discrete Tacnode Kernel"** $\mathbb{L}^{\mathsf{dTac}}$

For $x_i \in \mathbb{Z}$ and $y_i \in \mathbb{R}$:

$$\lim_{t \to 0} \quad \mathbb{L}^{red}(x_1, y_1; x_2, y_2) = \mathbb{L}^{\mathsf{dTac}}(x_1, y_1; x_2, y_2)$$

For $x_i, \tau_i \in \mathbb{Z}$ and $y_i \in \mathbb{R}$:

$$
\begin{aligned}
&\mathbb{L}^{\mathsf{dTac}}(x_1, y_1; x_2, y_2)\Big|_{x_i = \tau_i - \kappa} \\
&\quad := -\,\mathbb{H}^{\tau_1 - \tau_2}(\sqrt{2}(y_1 - y_2)) \\
&\qquad + \oint_{\Gamma_0} \frac{dV}{(2\pi \mathrm{i})^2} \oint_{\uparrow L_{0+}} \frac{dZ}{Z - V} \frac{V^{\rho - \tau_1}}{Z^{\rho - \tau_2}} \frac{e^{-\frac{V^2}{2} + (\beta + y_1\sqrt{2})V}}{e^{-\frac{Z^2}{2} + (\beta + y_2\sqrt{2})Z}} \frac{\Theta_{\mathfrak{r}}(V, Z)}{\Theta_r(0,0)} \\
&\qquad + \oint_{\Gamma_0} \frac{dV}{(2\pi \mathrm{i})^2} \oint_{\uparrow L_{0+}} \frac{dZ}{Z - V} \frac{V^{\tau_2}}{Z^{\tau_1}} \frac{e^{-\frac{V^2}{2} + (\beta - y_2\sqrt{2})V}}{e^{-\frac{Z^2}{2} + (\beta - y_1\sqrt{2})Z}} \frac{\Theta_{\mathfrak{r}}(V, Z)}{\Theta_r(0,0)} \\
&\qquad + \oint_{\uparrow L_{0+}} \frac{dV}{(2\pi \mathrm{i})^2} \oint_{\uparrow L_{0+}} dZ \frac{V^{-\tau_1}}{Z^{\rho - \tau_2}} \frac{e^{-\frac{V^2}{2} - (\beta - y_1\sqrt{2})V}}{e^{-\frac{Z^2}{2} + (\beta + y_2\sqrt{2})Z}} \frac{\Theta^{+}_{\mathfrak{r}-1}(V, Z)}{\Theta_{\mathfrak{r}}(0,0)} \\
&\qquad - \oint_{\Gamma_0} \frac{dV}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} dZ \frac{V^{\rho - \tau_1}}{Z^{-\tau_2}} \frac{e^{-\frac{V^2}{2} + (\beta + y_1\sqrt{2})V}}{e^{\frac{Z^2}{2} - (\beta - y_2\sqrt{2})Z}} \frac{\Theta^{-}_{\mathfrak{r}+1}(V, Z)}{\Theta_{\mathfrak{r}}(0,0)}
\end{aligned}
$$

where

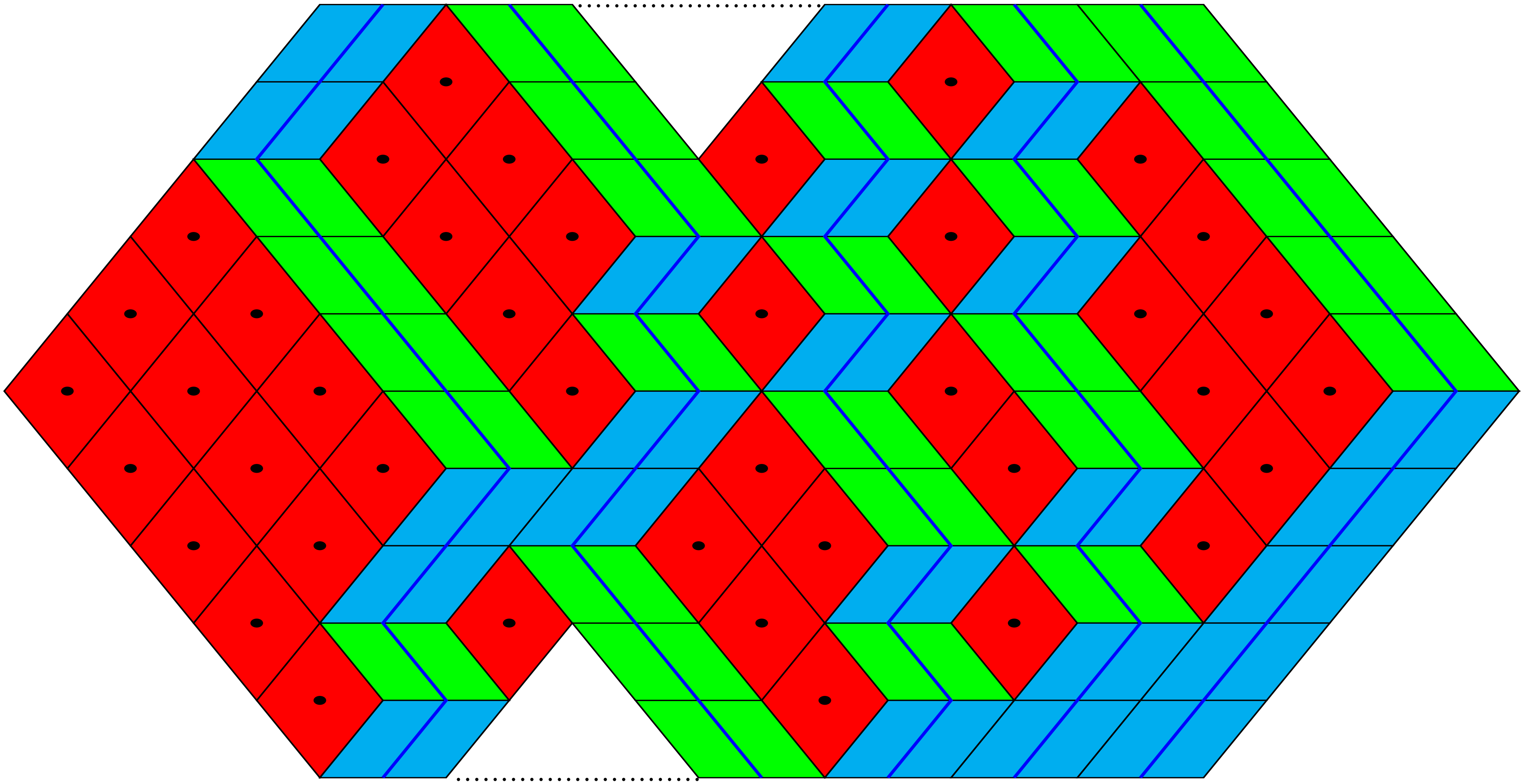
$$\mathbb{H}^m(z) := \frac{z^{m-1}}{(m-1)!}\mathbb{1}_{z\geq 0}\mathbb{1}_{m\geq 1}, \qquad \text{(Heaviside function)}$$

$$\Theta_{\mathfrak{r}}(V,Z) := \frac{1}{\mathfrak{r}!}\left[\prod_1^{\mathfrak{r}}\oint_{\uparrow L_{0+}}\frac{e^{W_\alpha^2-2\beta W_\alpha}dW_\alpha}{2\pi \mathrm{i} W_\alpha^\rho}\left(\frac{Z-W_\alpha}{V-W_\alpha}\right)\right]\Delta_{\mathfrak{r}}^2(W_1,\dots,W_r)$$

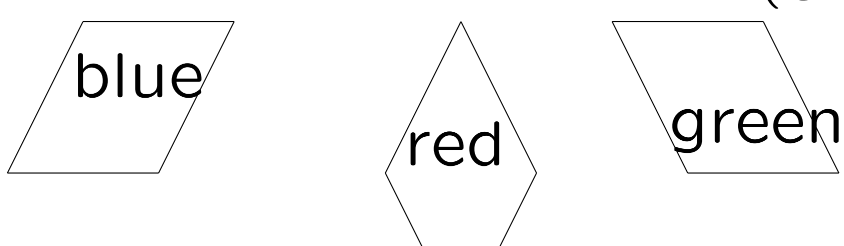
$$\begin{aligned}\Theta^{\pm}_{\mathfrak{r}\mp 1}(V,Z) := {} & \frac{1}{(\mathfrak{r}\mp 1)!} \\ & \times\left[\prod_1^{\mathfrak{r}\mp 1}\oint_{\uparrow L_{0+}}\frac{e^{W_\alpha^2-2\beta W_\alpha}dW_\alpha}{2\pi \mathrm{i} W_\alpha^\rho}\ \left((Z-W_\alpha)\ (V-W_\alpha)\right)^{\pm 1}\right] \\ & \times\ \Delta^2_{\mathfrak{r}\mp 1}(W_1,\dots,W_{\mathfrak{r}\mp 1}).\end{aligned}$$

2. Lozenge tilings of non-convex polygons !

Lozenge tilings of non-convex polygons ! What are the statistical fluctuations of the blue tiles between the two cuts?



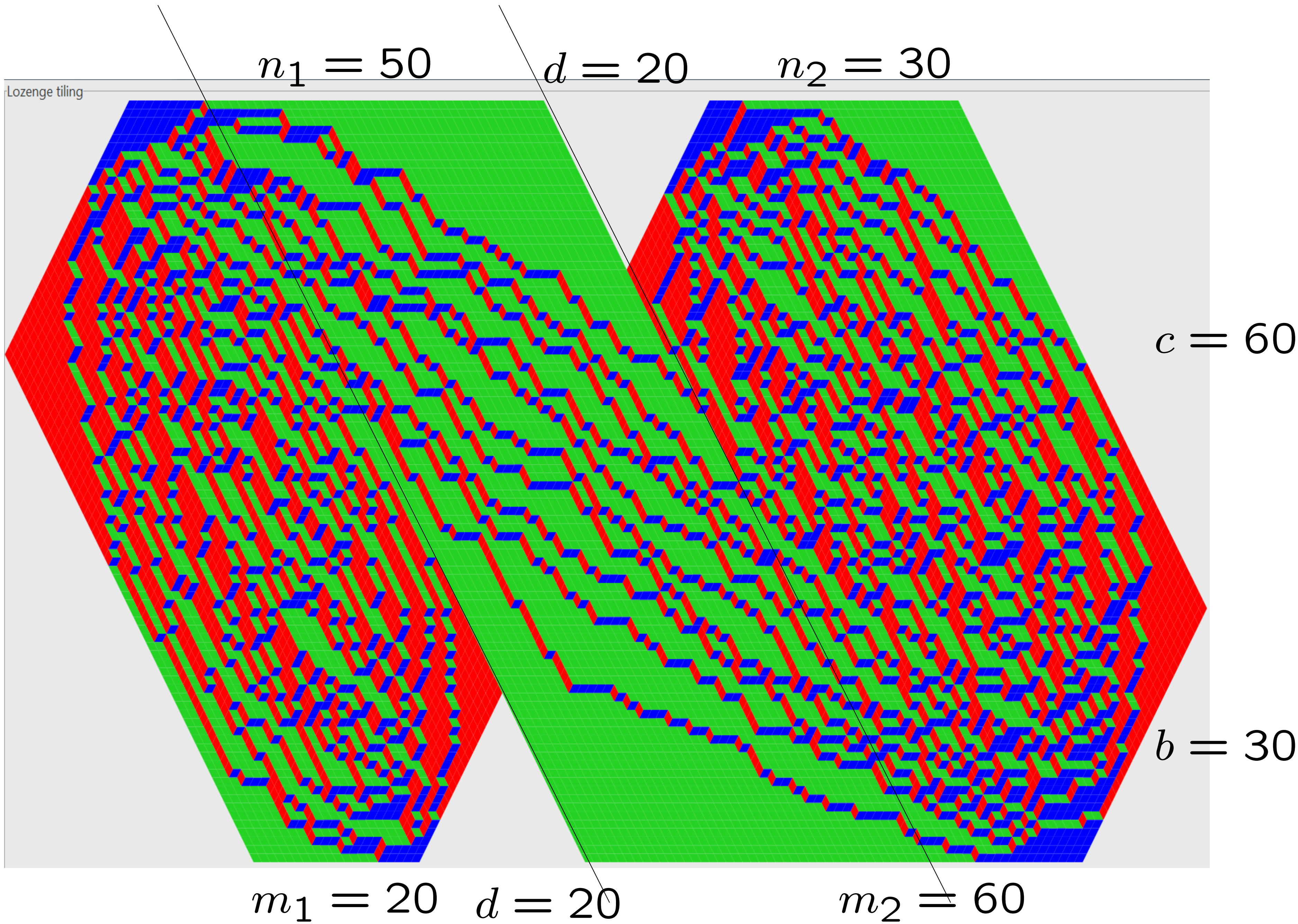
(Courtesy of Antoine Doeraene)



Let the size of the polygon and the cuts tend to infinity, while keeping the strip between the two cuts finite.

Then the double scaling limit of the correlation kernel is also

$$\mathbb{L}^{\mathrm{dTac}}(x_1, y_1; x_2, y_2)$$

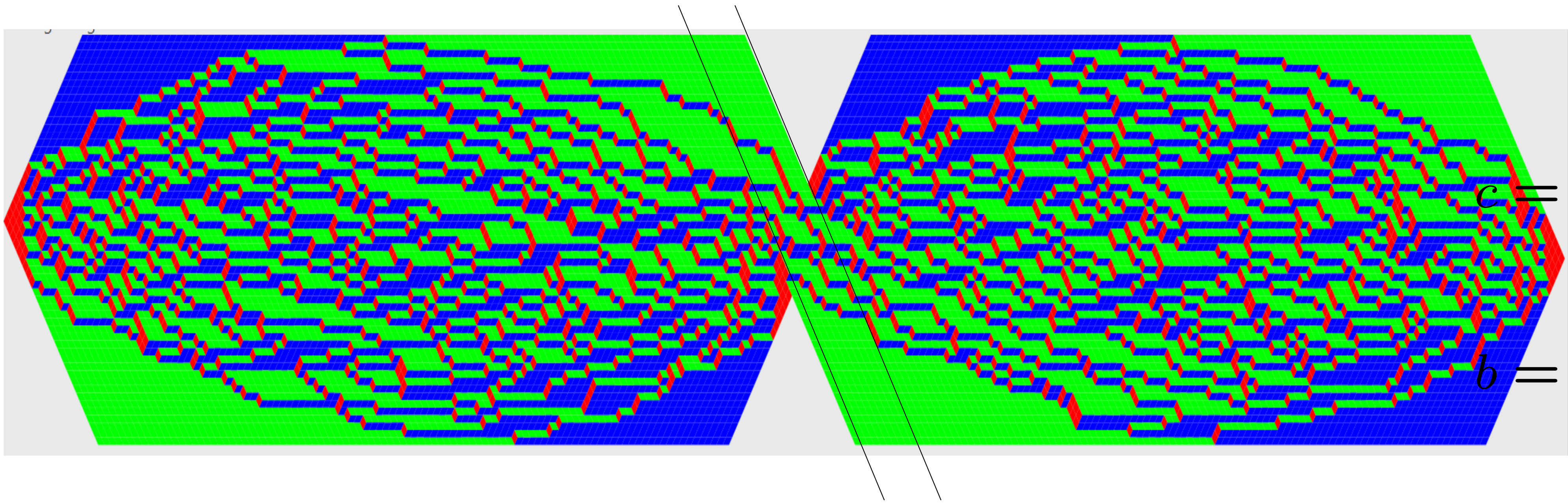


width of strip $= \rho := n_1 - m_1 + b - d = 40$

$\#\{$paths connecting the 2 arctic ellipses $\} = r := b - d = 10$

Other example

$n_1 = 105$ $\quad d = 20$ $\quad n_2 = 95$



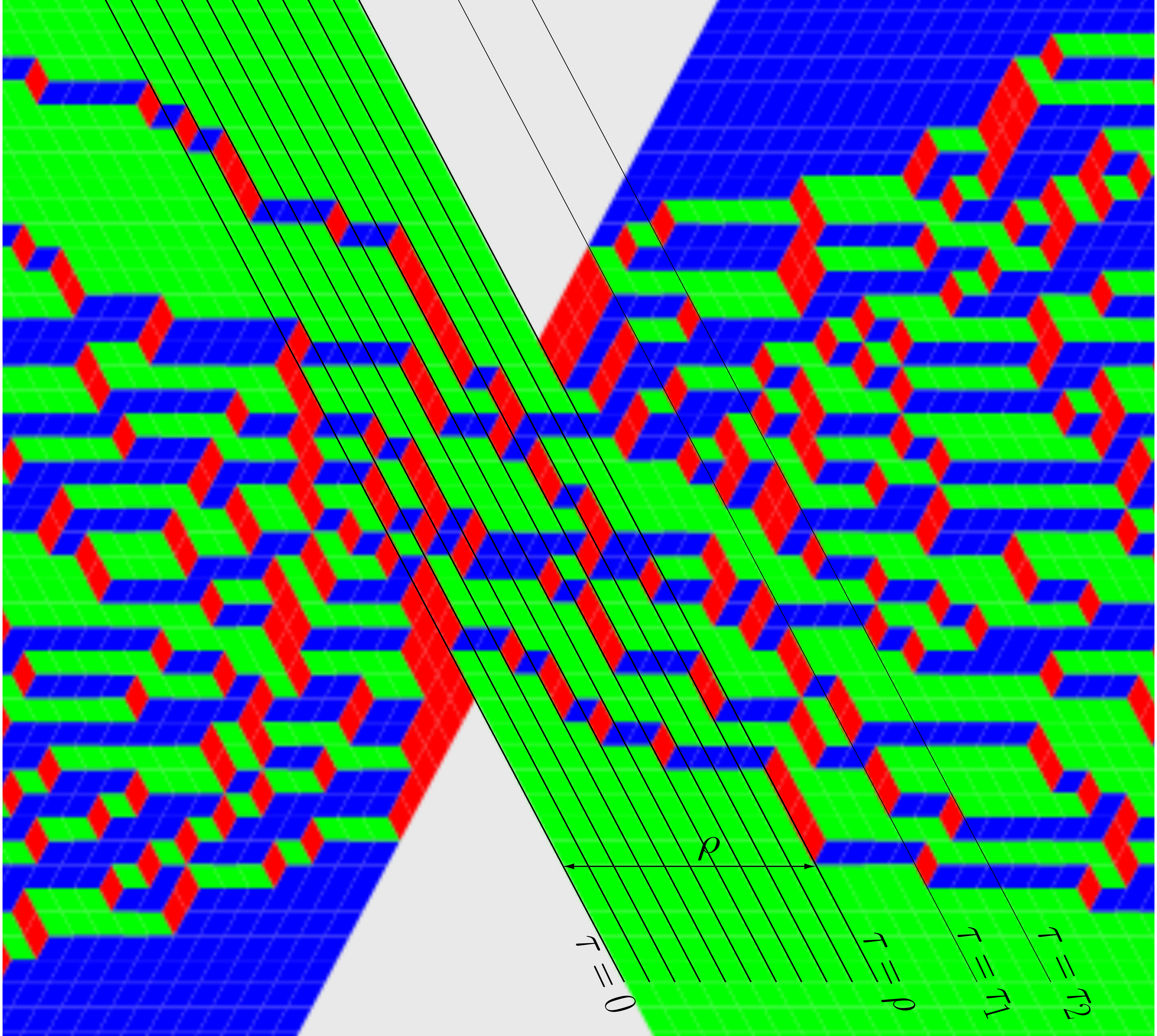
$c = 30$

$b = 25$

$m_1 = 100$ $\quad d = 20$ $\quad m_2 = 100$

$$r = b - d = 5, \qquad \rho = n_1 - m_1 + b - d = 10, \qquad \kappa = \frac{3}{2}$$

Courtesy of Antoine Doeraene



ρ
$\tau = 0$
$\tau = \rho$
$\tau = \tau_1$
$\tau = \tau_2$

- $n = \#\{\text{blue tiles along the oblique line } \tau\} = (\tau-\rho)_{>0} + r$

- $\mathbf{x}^{(\tau)} \in \mathbb{R}^n$

Theorem (*distribution and joint distribution of the blue tiles*)

$$\mathbb{P}\left(\ \mathbf{x}^{(\tau)} \in d\mathbf{x}\ \right) = D(\tau,\mathbf{x};\tau,\mathbf{x})d\mathbf{x}, \qquad \text{for } \tau \geq 0$$

$$\mathbb{P}\left(\ \mathbf{x}^{(\tau_1)} \in d\mathbf{x} \text{ and } \mathbf{y}^{(\tau_2)} \in d\mathbf{y}\ \right)$$
$$= D(\tau_1,\mathbf{x};\tau_2,\mathbf{y})\mathrm{Vol}(\mathcal{C}(\tau_1,\mathbf{x};\tau_2,\mathbf{y}))d\mathbf{x}d\mathbf{y}$$
$$\text{for } 0 \leq \tau_1 < \tau_2 \leq \rho \quad \text{ or } \quad \rho < \tau_1 < \tau_2$$

$\mathrm{Vol}(\mathcal{C}(\tau_1,\mathbf{x};\tau_2,\mathbf{y}))$ = Volume of polytope (Gibbs property!)

Theorem (*distribution and joint distribution of the blue tiles*)

$$\mathbb{P}\left(\ \mathbf{x}^{(\tau)} \in d\mathbf{x}\ \right) = D(\tau, \mathbf{x}; \tau, \mathbf{x}) d\mathbf{x}, \qquad \text{for } \tau \geq 0$$

$$\mathbb{P}\left(\ \mathbf{x}^{(\tau_1)} \in d\mathbf{x} \text{ and } \mathbf{y}^{(\tau_2)} \in d\mathbf{y}\ \right)$$

$$= D(\tau_1, \mathbf{x}; \tau_2, \mathbf{y}) \mathrm{Vol}(\mathcal{C}(\tau_1, \mathbf{x}; \tau_2, \mathbf{y})) d\mathbf{x} d\mathbf{y}$$

$$\text{for } 0 \leq \tau_1 < \tau_2 \leq \rho \qquad \text{or} \qquad \rho < \tau_1 < \tau_2$$

where

- $$D(\tau_1, \mathbf{x}^{(\tau_1)}; \tau_2, \mathbf{y}^{(\tau_2)}) := \begin{cases} C \widetilde{\Delta}_{n_1, \tau_1}(\mathbf{x} + \frac{\beta}{2}) \Delta_{n_2}(\mathbf{y}) \left(\prod_{i=1}^{n_2} \frac{e^{-y_i^2}}{\sqrt{\pi}} \right) \\ \qquad \text{for } \rho < \tau_1 < \tau_2 \\ \\ C' \widetilde{\Delta}_{n_1, \tau_1}(\mathbf{x} + \frac{\beta}{2}) \widetilde{\Delta}_{n_2, \rho - \tau_2}(-\mathbf{y}) \\ \\ \qquad \text{for } 0 \leq \tau_1 \leq \tau_2 \leq \rho \end{cases}$$

- For $n := (\tau - \rho)_{>0} + r$, define:

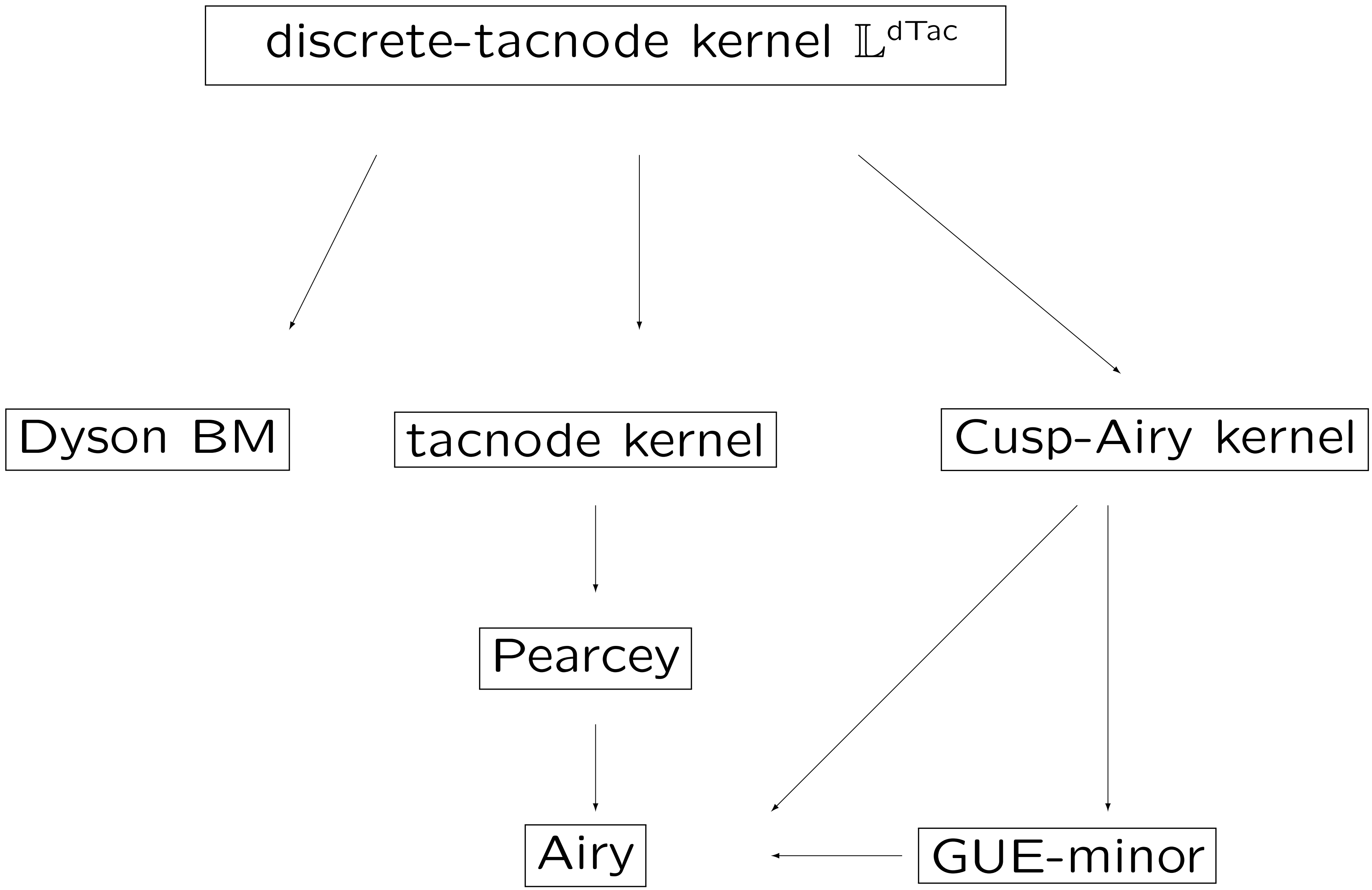
$$\widetilde{\Delta}_{n,\tau}(x) = \begin{cases} \det \begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \\ \vdots & & \vdots \\ x_1^{\tau-\rho-1} & \dots & x_n^{\tau-\rho-1} \\ \Phi_{\tau-1}(x_1) & \dots & \Phi_{\tau-1}(x_n) \\ \vdots & & \\ \Phi_{\tau-r}(x_1) & \dots & \Phi_{\tau-r}(x_n) \end{pmatrix}, & \text{for } \rho < \tau \\ \\ \det \begin{pmatrix} \Phi_{\tau-1}(x_1) & \dots & \Phi_{\tau-1}(x_n) \\ \vdots & & \\ \Phi_{\tau-r}(x_1) & \dots & \Phi_{\tau-r}(x_n) \end{pmatrix}, & \text{for } 0 \le \tau \le \rho \end{cases}$$

with

$$\Phi_k(\eta) := \frac{1}{2\pi \mathrm{i}} \int_L \frac{e^{v^2+2\eta v}}{v^{k+1}} dv$$

$$= \frac{e^{-\eta^2}}{2^{-k}\sqrt{\pi}} \times \begin{cases} \int_0^\infty \frac{\xi^k}{k!} e^{-\xi^2+2\xi\eta} d\xi & , k \geq 0 \\ \\ H_{-k-1}(-\eta) & , \quad k \leq -1 \end{cases}$$

$\mathbb{L}^{\mathsf{dTac}}(x_1, y_1; x_2, y_2)$ is a master kernel



THANK YOU !