A statistical physics approach to the $\text{Sine}_\beta$ process

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Joint work with David Dereudre, Adrien Hardy (U. Lille) and Thomas Leblé (Courant Institute-NYU)

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Outline of the talk

- One-dimensional log-gases and the Sine $\beta$ process
- Dobrushin-Lanford-Ruelle (DLR) equations for the Sine $\beta$ process
- Applications of the DLR equations
- Perspectives
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Log-gases

Configuration $\gamma = \{x_1, \ldots, x_n\}$ of $n$ points in $\mathbb{R}$ (or $\mathbb{U}$)

The energy of the configuration is $H_n(\gamma) := \frac{1}{2} \sum_{i \neq j} -\log |x_i - x_j| + \sum_{i=1}^n V(x_i)$, with a confining potential $V(x)$.

We denote by $P_{nV,\beta}$ the Gibbs measure on $\mathbb{R}^n$ or $\mathbb{U}^n$ associated to this energy:

$$dP_{nV,\beta}(x_1, \ldots, x_n) = \frac{1}{Z_{nV,\beta}} e^{-\beta H_n(x_1, \ldots, x_n)} dx_1 \ldots dx_n$$

On $\mathbb{R}$, if $V(x) = x^2/2$ and $\beta > 0$, we recover the $G_{\beta E}$ (tridiagonal model).

(On $\mathbb{U}$, if $V = 0$, we recover the $C_{\beta E}$ (pentadiagonal model)).
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Microscopic behavior of the log-gas

Valkó-Virág and Killip-Stoiciu independently showed existence of a limit point process for zoomed $G^\beta_E$ and $C^\beta_E$ respectively. Then Nakano showed that the two are the same, called Sine $\beta$ process. The proofs based on tridiagonal/pentadiagonal matricial model. The description of the process goes through "a coupled family of stochastic differential equations driven by a two-dimensional Brownian motion" (Brownian carousel):

$$d\alpha_\lambda(t) = \lambda \beta 4 e^{-\beta t} dt + \Re((e^{i\alpha_\lambda(t)} - 1) dZ_t), \quad \alpha_\lambda(0) = 0.$$  

The number of points of Sine $\beta$ in $[0, \lambda]$ is $\alpha_\lambda(\infty) / (2\pi)$.

Some properties obtained via the SDE description by Valkó, Virág, Holcomb, Paquette...

Valkó-Virág recently showed that the process can also be seen as the spectrum of a random differential operator. Universality with respect to $V$ obtained (Bourgade-Erdős-Yau-Lin/Bekerman-Figalli-Guionnet).
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“Physical” description of the Sine_{\beta} process?
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We started with

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We look at the rescaled configuration $\gamma_n := \sum_{i=1}^n \delta_{nx_i}$. 

This is false! We have to use DLR formalism for Gibbs measures.
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We look at the rescaled configuration $\gamma_n := \sum_{i=1}^n \delta_{nx_i}$. As $n$ goes to infinity, we may expect $\gamma_n \rightarrow C$ in the infinite configuration. $H_n(\gamma_n)$ converges to some function $H(C)$. The limiting process may satisfy

$$d\text{Sine}_\beta(C) = \frac{1}{Z^{\infty}_{V,\beta}} \exp(-\beta H(C)) d\Pi(C),$$

with $\Pi$ the Poisson process.

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Canonical DLR equations for $\text{Sine}_\beta$
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**Theorem** (Dereudre-Hardy-Leblé-M.)

Given a compact set $\Lambda$ and a configuration $\gamma$, the law of the configuration $\eta$ in $\Lambda$ knowing $\gamma$ is given by a Gibbs measure with density

$$d\text{Sine}_\beta(\eta | \gamma_{\Lambda^c}, |\gamma_{\Lambda}|) \propto \exp(-\beta(\mathcal{H}(\eta) + \mathcal{M}(\eta, \gamma_{\Lambda^c}))d\mathcal{B}_{|\gamma_{\Lambda}|}(\eta),$$

where $\mathcal{H}(\eta)$ represents the interaction of $\eta$ with itself and $\mathcal{M}(\eta, \gamma_{\Lambda^c})$ the interaction of $\eta$ with the exterior configuration and $\mathcal{B}$ is the Bernoulli process (with a fixed number of points).
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This has been shown by Bufetov for $\beta = 2$ (see also Kuijlaars-Miña-Diaz)
For any bounded measurable function $f$ on the set of configurations,

$$E_{\text{Sine}_\beta}(f) = \int \left[ \int f(\{x_1, \ldots, x_{|\gamma\Lambda|}\} \cup \gamma\Lambda^c) \rho\Lambda^c(x_1, \ldots, x_{|\gamma\Lambda|}) \prod_{i=1}^{\frac{|\gamma\Lambda|}{2}} dx_i \right] \text{Sine}_\beta(d\gamma),$$
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where

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\rho\Lambda^c(x_1, \ldots, x_{|\gamma\Lambda|}) := \frac{1}{Z(\Lambda, \gamma\Lambda^c)} \prod_{j<k} |x_j - x_k|^\beta \prod_{i=1}^{\gamma\Lambda} \omega_\beta(x_i, \gamma\Lambda^c).
$$
Existence of the Move functions

\[ \mathcal{M}(\eta, \gamma^\Lambda^c) := 2 \int\int_{x \neq y} -\log |x - y| d\eta(x) d\gamma^\Lambda^c(y) \]
Existence of the Move functions

\[ M(\eta, \gamma_{\Lambda^c}) := 2 \int\int_{x \neq y} - \log |x - y| \, d\eta(x) \, d\gamma_{\Lambda^c}(y) = 2 \int \psi(y) \, d\gamma_{\Lambda^c}(y) \]

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Better option: fix a reference configuration \( |\eta_0| \) in \( \Lambda \) with \( |\eta_0| = |\eta| \) and let

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\[ \psi_0(y) := \int_{x \neq y} - \log |x - y| d(\eta - \eta_0)(x) \propto -\frac{1}{y} \text{ as } |y| \to \infty. \]
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We need to compare $\gamma_{\Lambda^c}$ with the Lebesgue measure: discrepancy estimates:

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Putting every thing together, we get that \( \mathcal{M}(\eta, \gamma_{\Lambda^c}) \) is well defined.
DLR for a reference model

We use the $C\beta E$ as a reference model:

\[ \log|\sin(x-y\pi/N)| \]

Showing DLR is easy for this model and we then use the convergence to $Sine\beta$ due to Killip-Stoiciu + Nakano. We obtain Canonical DLR equations (when both the outside configuration and the number of points in $\Lambda$ are fixed).
DLR for a reference model

We use the C\(\beta\)E as a reference model: can be written as a log-gas on the unit circle with periodic pairwise interactions

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The Poisson process is not number-rigid.
Application to (number) rigidity

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The Poisson process is not number-rigid.

A few examples of (D)PP are known to be rigid. In particular Sine is rigid (Bufetov) and Sine$_\beta$ also (Chhaibi-Najnudel).
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All proofs of rigidity that we know rely on the following result (Ghosh-Peres) : Assume that for any $\Lambda$ and $\varepsilon > 0$, there exists a compactly supported function $f_{\Lambda,\varepsilon}$ such that on $\Lambda$, $f_{\Lambda,\varepsilon} = 1$ and $\text{Var}_{\mathbb{P}}(\sum_{x \in \gamma} f_{\Lambda,\varepsilon}(x)) \leq \varepsilon$, then $\mathbb{P}$ is rigid.
Our approach of number-rigidity

Theorem (Dereudre-Hardy-Leblé-M.)

Any process $P$ satisfying the canonical DLR equation

$$dP(\eta|\gamma,\Lambda_c,|\gamma,\lambda) \propto \exp(-\beta(H(\eta) + M(\eta,\gamma,\Lambda_c)))dB|\gamma,\Lambda|\eta)$$

is number-rigid.

In particular, $\sin\beta$ is number-rigid (and tolerant).

From there, we get full (grand canonical) DLR equations by getting rid of the conditioning on the number of points in $\Lambda$. 
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Let $P$ be a point process. Its Campbell measure $C_P^1$ is the joint distribution of a typical point $x$ in the configuration $\gamma$ and its neighborhood $\gamma \setminus \{x\}$:

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We show that if $P$ satisfies canonical DLR, there exists $Q$ such that

$$dC_P^1(x, \gamma) = e^{-\beta h(x, \gamma)} \text{Leb}(x) \otimes Q(d\gamma),$$

with

$$h(x, \gamma) = \lim_{R \to \infty} \sum_{y \in \gamma_{[-R,R]}} (\log(|y|) - \log |x - y|)$$
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and the same holds for $C_P^n$. 
We then show that if $P$ is not number-rigid, there exists $n \geq 1$, such that $Q_n$ is absolutely continuous with respect to $P$. 

By writing $C_2P$ in two different ways, one can check the compatibility relation:

$$\psi(\gamma \cup y) = \psi(\gamma \cup x) + \log |x| - \log |y|.$$
We then show that if $P$ is not number-rigid, there exists $n \geq 1$, such that $Q_n$ is absolutely continuous with respect to $P$ (it means that if we remove $n$ points from $\gamma$, the distribution looks like $P$ with different weights.)
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Let us assume for simplicity for $n = 1$. It means that

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On one side, the quantity

\[ a_k := C_P^2(1_{[0,1]}(x)1_{[k,k+1]}(y)) = E_P(|\gamma_{[0,1]}||\gamma_{[k,k+1]}|) \leq M \]

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On the other hand,
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\begin{align*}
\quad a_k &= E_P \left( \int_0^1 dx \int_k^{k+1} dy e^{-\beta(h(y,\gamma)+\psi(\gamma))} e^{-\beta(h(x,\gamma\cup y)+\psi(\gamma\cup y))} \right) \\
&\geq c k^\beta E_P \left( \int_0^1 dx \int_0^1 dy e^{-\beta(h(y,\gamma-k)+\psi(\gamma-k))} e^{-\beta(h(x,\gamma\cup 1)+\psi(\gamma\cup 1))} \right)
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By ergodicity, we get that \( \frac{1}{n} \sum_{k=n}^{2n} a_k \) converges to infinity, which leads to a contradiction.
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Perspectives

Let $\phi$ be a compactly supported function. Then if $\gamma$ is distributed according to Sine $\beta$, $\int \phi(x) \gamma(dx) \to G$ as $\ell \to \infty$, where $G$ is a centred Gaussian with variance $\frac{1}{2\beta \pi} \int \int (\phi(x) - \phi(y))(x - y)^2 dx dy$.

Rigidity for other Gibbs point processes? Two dimensional Coulomb gases (work in progress?)

Unicity of the solutions of DLR? (see Kuijlaars and Minà-Díaz if $\beta = 2$)
Perspectives

- Application of DLR to get CLT (Leblé)

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- Application of DLR to get CLT (Leblé) Let $\varphi$ be $C^4$, compactly supported function. Then if $\gamma$ is distributed according to $\text{Sine}_\beta$,

$$\int \varphi \left( \frac{x}{\ell} \right) (\gamma(dx) - dx) \to G \text{ as } \ell \to \infty,$$

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