A statistical physics approach to the Sine_β process

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Joint work with David Dereudre, Adrien Hardy (U. Lille) and Thomas Leblé (Courant Institute-NYU)

CIRM, Luminy - April 2019

Outline of the talk

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- One-dimensional log-gases and the Sine_β process

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- Perspectives

Log-gases

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We denote by $\mathbb{P}^n_{V,\beta}$ the Gibbs measure on $\mathbb{R}^n$ or $\mathbb{U}^n$ associated to this energy :

$$\mathrm{d}\mathbb{P}^n_{V,\beta}(x_1, \dots, x_n) = \frac{1}{Z^n_{V,\beta}} \mathrm{e}^{-\frac{\beta}{2} H_n(x_1,\dots,x_n)} \mathrm{d}x_1 \dots \mathrm{d}x_n$$

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(On $\mathbb{U}$, if $V = 0$, we recover the CβE (pentadiagonal model)).

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- Universality with respect to V obtained (Bourgade-Erdös-Yau-Lin/Bekerman-Figalli-Guionnet)

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Theorem (Dereudre-Hardy-Leblé-M.)
Given a compact set Λ and a configuration γ, the law of the configuration η in Λ knowing γ is given by a Gibbs measure with density

$$\mathrm{dSine}_\beta(\eta|\gamma_{\Lambda^c}, |\gamma_\Lambda|) \propto \exp(-\beta(\mathcal{H}(\eta) + \mathcal{M}(\eta, \gamma_{\Lambda^c}))\mathrm{d}\mathbf{B}_{|\gamma_\Lambda|}(\eta),$$

where $\mathcal{H}(\eta)$ represents the interaction of η with itself and $\mathcal{M}(\eta, \gamma_{\Lambda^c})$ the interaction of η with the exterior configuration and $\mathbf{B}$ is the Bernoulli process (with a fixed number of points).

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This has been shown by Bufetov for $\beta = 2$ (see also Kuijlaars-Miña-Diaz)

For any bounded measurable function f on the set of configurations,

$$\mathbb{E}_{\mathrm{Sine}_\beta}(f) = \int \left[\int f(\{x_1, \dots, x_{|\gamma_\Lambda|}\} \cup \gamma_{\Lambda^c}) \rho_{\Lambda^c}(x_1, \dots, x_{|\gamma_\Lambda|}) \prod_{i=1}^{|\gamma_\Lambda|} \mathrm{d}x_i \right] \mathrm{Sine}_\beta(\mathrm{d}\gamma),$$

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where

$$\rho_{\Lambda^c}(x_1, \ldots, x_{|\gamma_\Lambda|}) := \frac{1}{Z(\Lambda, \gamma_{\Lambda^c})} \prod_{j<k}^{|\gamma_\Lambda|} |x_j - x_k|^\beta \prod_{i=1}^{|\gamma_\Lambda|} \omega_\beta(x_i, \gamma_{\Lambda^c}).$$

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Putting every thing together, we get that $\mathcal{M}(\eta, \gamma_{\Lambda^c})$ is well defined.

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We obtain Canonical DLR equations (when both the outside configuration and the number of points in Λ are fixed).

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All proofs of rigidity that we know rely on the following result (Ghosh-Peres) : Assume that for any Λ and $\varepsilon > 0$, there exists a compactly supported function $f_{\lambda,\varepsilon}$ such that on Λ, $f_{\lambda,\varepsilon} = 1$ and $Var_{\mathbb{P}}(\sum_{x\in\gamma} f_{\lambda,\varepsilon}(x)) \leq \varepsilon$, then $\mathbb{P}$ is rigid.

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Any process $\mathbb{P}$ satisfying the canonical DLR equation

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is number-rigid. In particular, Sine_β is number-rigid (and tolerant).

From there, we get full (grand canonical) DLR equations by getting rid of the conditioning on the number of points in Λ.

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We show that if P satisfies canonical DLR, there exists Q such that

$$\mathrm{d}C_P^1(x,\gamma) = \mathrm{e}^{-\beta h(x,\gamma)}\mathrm{Leb}(x) \otimes Q(\mathrm{d}\gamma),$$

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By writing C_P^2 in two different ways, one can check the compatibility relation :

$$\psi(\gamma \cup y) = \psi(\gamma \cup x) + \log|x| - \log|y|.$$

On one side, the quantity

$$a_k := C_P^2(\mathbf{1}_{[0,1]}(x)\mathbf{1}_{[k,k+1]}(y)) = E_P(|\gamma_{[0,1]}||\gamma_{[k,k+1]}|) \leq M$$

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$$\ge ck^\beta E_P\left(\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \mathrm{e}^{-\beta(h(y,\gamma-k)+\psi(\gamma-k))}\mathrm{e}^{-\beta(h(x,\gamma\cup 1)+\psi(\gamma\cup 1))}\right)$$

By ergodicity, we get that $\frac{1}{n}\sum_{k=n}^{2n} a_k$ converges to infinity, which leads to a contradiction.

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$$\int \varphi\left(\frac{x}{\ell}\right)(\gamma(\mathrm{d}x) - \mathrm{d}x) \to G \text{ as } \ell \to \infty,$$

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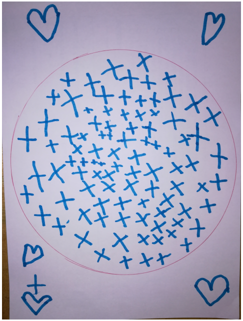

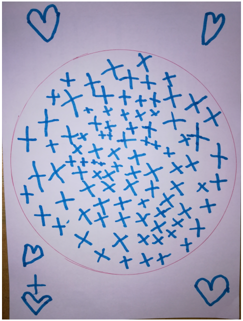

Thanks for your attention !