A tale of Pfaffian persistence tails told by a Bonnet-Painlevé VI transcendent

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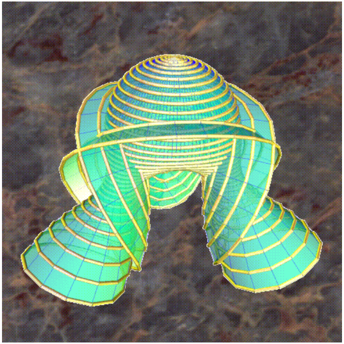
ID, *Pfaffian Persistence, Universal Bonnet-Painlevé VI Probability Distributions, and Ising model Criticalities in* 1+1 *dimensions*, arXiv:1810.06957 (under rev. for J. Stat. Phys.)
Robert Conte & ID, *Persistence, Painlevé VI, Chazy C.V, and Bonnet Surfaces*, in prep. ($<$ 2020) — for a genuine (applied) maths journal

- Let $P_L(x) = \sum_{k=0}^{L} a_k x^k$ the usual Kac polynomial with *real* Gaussian random coeffs, and $\mathcal{N} = \mathcal{N}(L)$ the number of its real roots on $[0,1]$. Then for $L \gg 1$ $\mathbb{E}[\mathcal{N}] \sim \frac{1}{2\pi} \ln L$ (Kac 1943, Thm)
- What about the full distribution: $\mathbb{E}[m^{\mathcal{N}}]$, with $0 < m < 1$? My claim: $\exists$ *scaling function* of $T \equiv \ln L \in (0, +\infty)$:

$$\mathbb{E}[m^{\mathcal{N}}] \to \exp\left(-\frac{1}{2}\int_0^T \left[H(\ell) + \sqrt{H'(\ell)}\right]\right)$$



- $H = H(T;m)$ the (extrinsic) *mean curvature* of the above, also the sole $\nearrow$ regular solution on $\mathbb{R}_+$ with $H(0) = \frac{1-m^2}{2\pi}$, $H'(0) = H(0)^2$ of

$$\left(\frac{1}{2}\right)^2 = \left(\frac{H''}{2H'} + \coth T\right)^2 + \frac{H^2}{H'}\left[1 - \left(\frac{H'}{H} + \coth T\right)^2\right]$$

Appetizer (cont'd)

- and having a finite limit for large times T:

$$\theta(m) = \lim_{T\to+\infty} H(T;m) = \frac{1}{2}\left[\kappa_1(m)+\kappa_2(m)\right] \quad (1)$$

$$= \frac{1}{2}\left(\left[\frac{2}{\pi}\arccos\left(\frac{m}{\sqrt{2}}\right)\right]^2 - \left[\frac{2}{\pi}\arccos\left(\frac{1}{\sqrt{2}}\right)\right]^2\right) \quad (2)$$

- This recovers — independently and by completely different methods — a result just obtained (and before ...) by Poplavsky & Schehr':

$$\mathbb{E}[m^{\mathcal{N}}(L)] \propto L^{-\theta(m)/2} \propto e^{-\theta(m)T/2}, \quad T\text{"} = \text{"} \ln L \gg 1$$

Exact persistence exponent for the 2d-diffusion equation and related Kac polynomials, Phys. Rev. Lett. 2018 (arXiv:1806.11275)

- Both answering a famous question by Dembo et al about *Random polynomials having few or no real zeros* (J. AMS 2002)

$$\lim_{m\to 0^+} \mathbb{E}[m^{\mathcal{N}(L)}] \propto L^{-\theta(0)/2}, \quad \frac{\theta(0)}{2} = \frac{3}{16} \equiv \text{Gauss (intrinsic) curvature}$$

Appetizer (bonuses)

- Yet here bonus: H is a *tau-function* for a $\mathrm{P_{VI}}$ with *monodromy exponents* (up to perm./signs) or parameters

$$\{\vartheta_\infty, \vartheta_0, \vartheta_1, \vartheta_s\} = \left\{\frac{1}{2}, \frac{1}{2}, 0, 0\right\}, \quad \{\alpha, \beta, \gamma, \delta\} = \left\{\frac{1}{8}, -\frac{1}{8}, 0, \frac{1}{2}\right\}$$

- *Universal* à la Tracy-Widom: it appears in four different model systems of interest for nonequilibrium statistical physics
- Halfway (through quadratic+ Okamoto transformations) between some other famous $\mathrm{P_{VI}}$: Picard/Hitchin, & Manin
- $\Rightarrow$ related to Jimbo-Miwa's characterization of the *diagonal* correlations of the planar equilibrium Ising model at all temperatures

$$\frac{\theta(0)}{2} = \frac{3}{16} = \frac{\eta + \beta}{2} = \frac{1/4 + 1/8}{2}, \quad \tanh\left(T/2\right) = \left(\frac{\sinh 2E}{\sinh 2E^*}\right)^2$$

- The reason for all this: *Pfaffian* structure with an integrable kernel, the sech-kernel $K(x,y) = \frac{1}{2\pi}\,\mathrm{sech}\,(x-y)/2$, on $\mathbb{L}^2(-T/2, T/2)$

Introduction: First-passage properties of a random process

- What is the chance for a fluctuating quantity Y to have *always* remained up to a certain time above a given level (say $\langle Y \rangle$), or to *first* cross it a certain instant ?
- Time-honored and basic subject of (applied) probability
- Usual playground: $\{Y(T)\}_T$ Gaussian stationary process, thus $A(T_2 - T_1) = \langle Y(T_1)Y(T_2)\rangle$ determines everything *a priori*
- $P_0(T) =$ No-crossing proba. at zero level ($=\langle Y \rangle$), up to (fixed) T:

$$P_0(T) \propto e^{-\theta T}, \quad \theta = decay \quad rate$$

  $-\mathrm{d}P_0(T)/\mathrm{d}T =$ *first-passage* proba. at time T
- Simple pb to state but extremely hard to solve unless for Markovian (memoryless) processes ...
  The latter have necessarily $A(T) = e^{-\theta T}$ $\forall T$ (Slepian's theorem), hence $Y \equiv$ rescaled Brownian: $P_0(T) = (2/\pi) \arcsin A(T) \propto e^{-\theta T}$

Modern incarnation: persistence in phase-ordering systems

- How a *local* degree of freedom can maintain its initial orientation as domains of globally aligned spins grow as $L(t) \propto t^{1/z}$?
- Simplest situation: quench $\pm$ Ising spins from infinite to zero temperature. Introduce somewhat natural *geometric* definition: $p_0(t) =$ fraction of spins which have never flipped up to time t: $-\mathrm{d}p_0(t)/\mathrm{d}t =$ first-passage probability. of a domain wall at a particular location in space.
- For large times algebraic decay $p_0(t) \propto t^{-\theta}$, $\theta =$ *persistence exponent*

FIG. 2.1 - *Paysage des domaines dans le modèle d'Ising 2d évoluant selon la dynamique de Glauber à $T = 0$, pour des temps $t = 256, 1024, 4096$ dans un système de $N = (512)^2$ sites (avec des conditions aux limites périodiques).*

Two early climaxes in the physical literature *circa* 1995

- Simple diffusion equation $\partial_t \phi(r,t) = \nabla_r^2 \phi(r,t)$, with $\phi(r,0) =$ white noise. A popular model of phase ordering because $\phi(r,t)$ Gaussian. Local "spin" variable (at $r=0$ say) $\mathrm{sgn}[X(t)]$, $X(t) = \phi(0,t)$
- Yet non-trivial $\tilde{\theta}(d)$ in all space dimensions d !
(Majumdar, Sire, Bray, Cornell & Derrida, Hakim, Zeitak, PRLs '95)
- "Simply" because non-Markovian correlator for the associated process $Y(T) = X(e^T)/[\langle X^2(e^T)\rangle]^{1/2}$ (normalized and rendered stationary on the logarithmic timescale $T = \ln t$)
- In particular in $d=2$, $\tilde{\theta}(2) = 0.1875(10)$ (num.) with a correlator

$$A(T) = \langle Y(0)Y(T)\rangle = \mathrm{sech}(T/2) \quad (\mathrm{sech} = 1/\cosh)$$

- Later realized (Dembo et al., Schehr-Majumdar, Forrester ...) that the very same Gaussian $\{Y(T)\}_T$ also describes the number of real roots of random Kac's polynomials or the eigenvalues of truncated random orthogonal matrices, both *Pfaffian* point processes

Second climax: Derrida, Hakim, Pasquier's *tour-de-force*

- An exact expression for the persistence proba. $p_0^{\mathrm{Potts}}(t_1, t_2; q)$ that a q-state Potts spin on a $1d$ chain with zero-temperature Glauber dynamics has not flipped between (arbitrary) times t_1, t_2
- After tremendous technicalities, $p_0^{\mathrm{Potts}}(t_1, t_2; q) \propto (t_2/t_1)^{-\hat{\theta}(q)}$ with

$$\hat{\theta}(q) = -\frac{1}{8} + \frac{2}{\pi^2}\left[\arccos\left(\frac{2-q}{\sqrt{2}\,q}\right)\right]^2 \Longrightarrow \hat{\theta}(2) = 3/8 \quad (\mathrm{Ising\,spins})$$

- Their crucial insight: the pers. proba. for the particular spin located at the origin of a *semi-infinite* chain is determined by the *Pfaffian* formed by the no-meeting proba. $c(s,t)$ between two random walkers

$$c(s,t) = \sum_{0 \le x \le y} [p(x;s)p(y;t) - p(x;t)p(y;s)] \approx 1 - \frac{4}{\pi}\mathrm{arctan}\sqrt{\frac{s}{t}}$$

- Enough since $p_0^{\mathrm{SemiP}}(t_1, t_2; q) = [p_0^{\mathrm{Potts}}(t_1, t_2; q)]^{1/2} \propto (t_2/t_1)^{-\hat{\theta}(q)/2}$

Yet . . .

- Verbatim from the conclusions of DHP (J.Stat.Phys.'96):
  This probably means that there are simpler ways of rewriting our expression (for $p_0^{\mathrm{SemiP}}(t_1, t_2; q)$) where all the cases can be treated in the same manner. Unfortunately, we did not find these simpler expressions.
- Puzzling numerical proximity

$$\hat{\theta}(q=2)/2 = \frac{1}{2}(3/8) \quad \textit{vs.} \quad \tilde{\theta}(d=2) = 0.1875(10)$$

  But how these two model systems, apparently so dissimilar, and that do not even live in an ambient space with the same physical dimension, could possibly be related at the level of a quantity so sensitive to details as the persistence exponent?
- :-(Just proved by M. Poplavskyi & G. Schehr! *Exact persistence exponent for the 2d-diffusion equation and related Kac polynomials*, arXiv:1806.11275 (PRL in press)

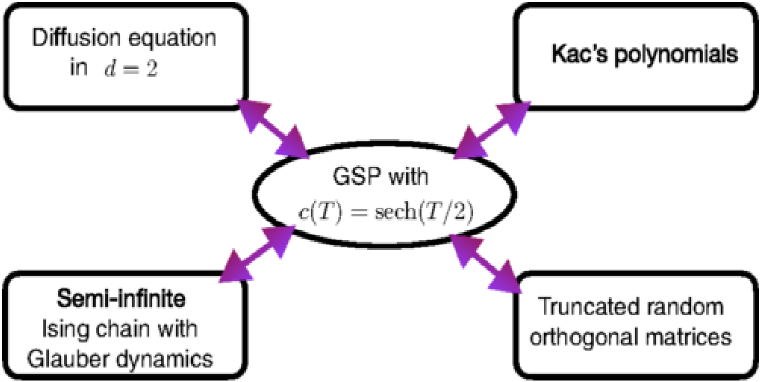
$$\mathrm{sgn} X^{\mathrm{Diff2d}}(t) \equiv S_0^{\mathrm{SemiIsing}}(t) \quad \text{(as processes)}$$

What else ? Different and More (if possible. . .)

I had (vague) indications of a Painlevé VI lurking in the background

and my hope was that this could allow to rederive PS's result "somehow"

Angle of attack: SouthWest face (©Fig.1 from PS's PRL)

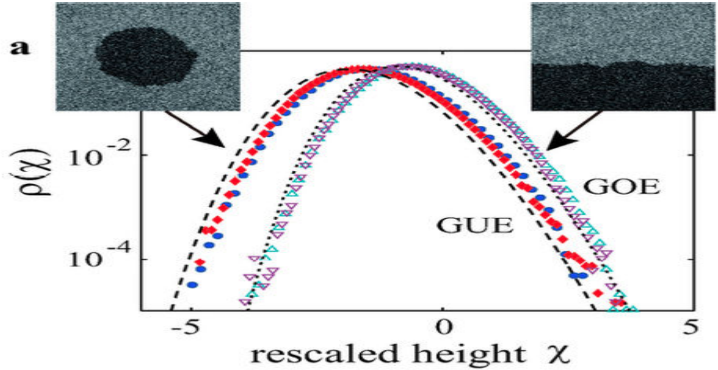


Bottom-up, pedestrian approach ("alpine style"): no representation theory, no Riemann-Hilbert, no isomonodromy, no conformal field theory, no algebraic geometry. Essentially (by now) classical Tracy-Widom + old school Painlevé VI (with a generous seasoned guide for the latter)

Hidden in the persistence probas. are new non-trivial and universal limit distributions for correlated random variables

$\equiv$ The exact analog for the sech-kernel and a $\mathrm{P_{VI}}$ transcendent of the famous Tracy-Widom $\mathrm{P_{II}}$ distributions for the Airy kernel (G-U/O-E at the edge), or the Jimbo-Miwa-Mori-Satô $\mathrm{P_V}$ found for the Gaudin-Mehta sine kernel (GOE in the bulk).

Ex.: KPZ universal interface growth (Takeuchi et al., 2010)



Relevant literature

⚠ As far as I can tell, none of the results I present relies on any of PS

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- C. A. Tracy, H. Widom, *Fredholm determinants, differential equations, and matrix models*, Commun. Math. Phys. **163**, 33-72 (1994).
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- A.I. Bobenko, U. Eitner, *Bonnet Surfaces and Painlevé Equations*, J. Reine Angew. Math. **499**, 47-79 (1998).
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Results 0/4 (preparatory notations)

- Best expressed for the persistence probability(ies) on the log. timescale $T = \ln t_2 - \ln t_1$ AND by trading q-state Potts spins for $\pm$ Ising spins on an arbitrarily m-magnetized *half-space* chain:

$$P_0^+(T = T_2 - T_1; m) = \frac{1}{q} p_0^{\mathrm{HalfP}}(e^{T_1}, e^{T_2}; q)\big|_{\frac{1}{q} = \frac{1+m}{2}}$$

⚠ sum-rule $\forall T, m$ (reversing globally the initial condition):

$$P_0^{\mathrm{HalfI}}(T; m) = P_0^+(T; m) + P_0^-(T; m), \quad P_0^-(T; m) = P_0^+(T; -m)$$

- Consider the even-difference sech-kernel

$$K(x, y) = K(x - y) = \frac{1}{2\pi} \operatorname{sech}[(x - y)/2] \quad (= \rho_0 A^{\mathrm{2Diff}}(x - y)),$$

($\rho_0 = 1/(2\pi)$ also density of zero-crossings for the $2d$ diffusing field)

- On log. scale $s = e^x, t = e^y$, the no-meeting proba. $c(s, t)$ is:

$$C(x, y) = c(e^x, e^y) = \frac{2}{\pi} \int_{e^{(x-y)/2}}^{e^{(y-x)/2}} \frac{\mathrm{d}u}{1 + u^2} \overset{u = e^{v/2}}{=} \int_{x-y}^{y-x} \mathrm{d}v\, K(v)$$

Results 1/4

- Consider the solutions $1 > \lambda_0(T) > \lambda_1(T) > \cdots > 0$ of the eigenvalue integral equation for $K_T = K\big|_{[-T/2,T/2]}$:

$$\int_{-T/2}^{T/2} \mathrm{d}y\, K(x-y)\phi(y) = \lambda\phi(x)$$

and the associated Fredholm determinants generating functions $\mathcal{D}_{e,o}(T;\xi) = \prod_{k\ even/odd}[1-\xi\lambda_k(T)]$ for the even/odd part of K.

$$P_0^{\pm}(T;m) = \frac{\mathcal{D}_e(T;\xi) \pm m\,\mathcal{D}_o(T;\xi)}{2}\bigg|_{\xi=1-m^2}, \quad P_0^{\pm}(0;m) = \frac{1\pm m}{2}.$$

- The pers. proba. is a Fredholm Pfaffian *gap probability* gen. function:

$$P_0^{\mathrm{HalfI}}(T;m) = \mathcal{D}_e(T,\xi) = \exp\left\{-\int_0^{T/2}\mathrm{d}x[R(x,x)+R(x,-x)]\right\}$$

with $R(x,y) = \langle x|\mathbb{R}|y\rangle$ the matrix elements of the *resolvent operator* $\mathbb{R}$ for ξK_T, i.e. $\mathbb{1}+\mathbb{R} = (\mathbb{1}-\xi K_T)^{-1}$, (and $\delta(x-y) = \langle x|\mathbb{1}|y\rangle$)

Proof: recast DHP in the framework of TW-GOSE

- Start from the central result of DHP:

$$P_0^{\rm SemiP}(t_1,t_2;q) = \left(\sqrt{1-\mu\tilde{c}(t_2,t_2)} - \lambda\sqrt{-\mu\tilde{c}(t_2,t_2)}\right) e^{\frac{1}{2}{\rm TrLog}M}$$

  where $\lambda = q-1, \mu = (1-q)/q$, and $M, \tilde{c} = cM^{-1}$ are two operators defined in terms of $c(s,t) = 1-(4/\pi)\arctan\sqrt{s/t}$:

$$M(s,t)dt = \left[\delta(s-t) + \frac{2(1-q)}{q}\frac{{\rm d}c}{{\rm d}s}\right]{\rm d}t = [\mathbb{1}-(1-m^2)K(x-y)]{\rm d}y$$

  after $s=e^x, t=e^y, 1/q=(1+m)/2$. This gives the (easy) 1st piece:

$$e^{\frac{1}{2}{\rm TrLog}M} = \sqrt{\det(\mathbb{1}-\xi K_T)} = e^{-\frac{1}{2}\int_0^{T/2}{\rm d}x[R(x,x)+R(-x,-x)]}$$

- For the ominous-looking "amplitude", $\tilde{c} = -\mathbb{C}(\mathbb{1}-\xi K_T)^{-1}$ with $\mathbb{C}$ the antisymmetric operator with matrix elements $C(x,y) = c(e^x,e^y)$

$$\mathbb{C} = -2\varepsilon K, \quad \varepsilon = D^{-1} \equiv \frac{1}{2}{\rm sgn}(x-y) \quad ({\rm sgn}'(x) = 2\delta(x))$$

- *Intrinsic* computation valid for any even difference kernel on a symmetric interval: just relies on the Pfaffian structure

Time for a (first) old Reminder

SUR LA LOI LIMITE DE L'ESPACEMENT DES VALEURS PROPRES D'UNE MATRICE ALÉATOIRE

MICHEL GAUDIN

Centre d'Études Nucléaires de Saclay, Gif-sur-Yvette (S. et O.), France

Reçu le 16 Janvier 1961

Abstract: The distribution function of the level spacings for a random matrix in the limit of large dimensions is expressed by means of a rapidly converging infinite product which has been used for a numerical calculation. Comparison with Wigner's hypothesis gives a very good agreement.

inférieure à 0.0066 dans la région $S < 3D$. La fig. 2 représente les fonctions p et p_W dont la différence relative est inférieure à 5 % pour $S < 2D$, et l'écart moindre que 0.0162.

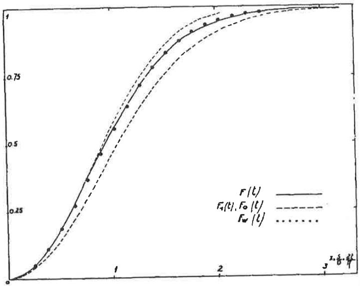

Fig. 1. La distribution de Wigner $F_W(S)$ et la fonction exacte $F(S)$ comprise entre F_0 et F_1.

In a "modern" language

RELATIONSHIPS BETWEEN τ-FUNCTION AND FREDHOLM DETERMINANT EXPRESSIONS FOR GAP PROBABILITIES IN RANDOM MATRIX THEORY

PATRICK DESROSIERS AND PETER J. FORRESTER

Abstract. The gap probability at the hard and soft edges of scaled random matrix ensembles with orthogonal symmetry are known in terms of τ-functions. Extending recent work relating to the soft edge, it is shown that these τ-functions, and their generalizations to contain a generating function parameter, can be expressed as Fredholm determinants. These same Fredholm determinants occur in exact expressions for the same gap probabilities in scaled random matrix ensembles with unitary and symplectic symmetry.

1. Introduction

In the 1950's Wigner introduced random real symmetric matrices to model the highly excited energy levels of heavy nuclei (see [13]). From the experimental data, a natural statistic to calculate empirically is the distribution of the spacing between consecutive levels, normalized so that the spacing is unity. For random real symmetric matrices X with independent Gaussian entries such that the joint probability density function (p.d.f.) for the elements is proportional to $e^{-\mathrm{Tr}(X^2)/2}$ (such matrices are said to form the Gaussian orthogonal ensemble, abbreviated GOE), Wigner used heuristic reasoning to surmise that the spacing distribution is well approximated by the functional form

$$p_1^{\mathrm{W}}(s) := \frac{\pi s}{2} e^{-\pi s^2/4}. \tag{1.1}$$

In the limit of infinite matrix size, it was subsequently proved by Gaudin that the exact spacing distribution is given by

$$p_1(s) = \frac{d^2}{ds^2} \det(\mathbb{I} - K^{\mathrm{bulk},+}_{(0,s)}), \tag{1.2}$$

where $\mathbb{I}$ stands for the identity operator and where $K^{\mathrm{bulk},+}_{(0,s)}$ is the integral operator supported on $(0, s)$ with kernel

$$\frac{\sin \pi(x-y)}{\pi(x-y)} \tag{1.3}$$

restricted to its even eigenfunctions. It was shown that this integral operator commutes with

Bilinear representation for an integrable integral operator

- For the Airy, Bessel, or sine kernels there exists a representation

$$K(x,y) = \frac{\phi(x)\psi(y) - \phi(y)\psi(x)}{x-y} = \int_0^\infty \mathrm{d}z\, \Omega(x+z)\Omega(y+z)$$

Ex.: For the Airy kernel K_{Airy}, $\phi = \mathrm{Ai}$, $\psi = \mathrm{Ai}'$, and $\Omega = \mathrm{Ai}$ itself.

- (very) useful for determination of limiting distrib. in RMT: allows to rewrite $K_{\mathrm{Airy}}\big|_{[s,+\infty)}(x,y)$ as the *square* of $\mathrm{Ai}(x+y-s)$, and to find a differential operator $\mathcal{L}$ commuting with K (also WKB techniques)
- Sech-kernel self-dual in Fourier space:

$$\widehat{K}(q) = \mathrm{sech}\,(\pi q) = \frac{1}{\pi}\Gamma(1/2 + \imath q)\Gamma(1/2 - \imath q)$$

Complement formula for Gamma function $\equiv$ Wiener-Hopf factorization for the sech-kernel. Allows to derive the asymptotic decay of $P_0^{\mathrm{HalfI}}(T;\xi)$, but no obvious $\mathcal{L}$ (yet there exists sthg else...)

Results 2/4: Compute the Fredholm dets.

- The sech-kernel is an *integrable* integral operator:

$$\frac{1}{\cosh[(x-y)/2]} = \frac{2\sinh[(x-y)/2]}{\sinh[(x-y)]} = \frac{e^{3x/2}e^{y/2} - e^{x/2}e^{3y/2}}{e^{2x} - e^{2y}}$$

  the same Christoffel-Darboux like identity as for the finite-N sine kernel (Circular Unitary Ensemble of RMT),

$$K_N(x,y) = \frac{1}{2\pi N}\frac{\sin[N(x-y)/2]}{\sin[(x-y)/2]},$$

  known to give rise to a $\mathrm{P_{VI}}$, up to $x, y \to 2ix, 2iy$, and $N = \pm 1/2$ (!)
- Output for the two resolvent functions $G(T) = R(T/2, -T/2)$ and $H(T) = R(T/2, T/2)$: coupled 1st order quadratic non-linear ODEs:

$$H' = G^2 \quad ({}' = \mathrm{d}/\mathrm{d}T, \mathrm{Gaudin's\,relation}), \qquad \Theta^2 = N^2 = 1/4$$
$$\Theta^2(G\sinh T)^2 + [(H\sinh T)']^2 = (H\sinh T)^2 + [(G\sinh T)']^2$$

Results 2/4 (cont'd)

- Eliminating differentially (without square-root !) G, one obtains a *closed* 2nd order 2nd degree nonlinear ODE for H

$$(H''+2H'\coth T)^2-4H'\left[(H'+H\coth T)^2-H^2+\Theta^2 H'\right]=0$$

- (Local) Cauchy problem at $T=0$: $H(T)=h_0+h_0'T+\dots$, coeffs. determined through Neumann expansion of the resolvent: $h_0=\rho_0\xi=\frac{1-m^2}{2\pi}$, $\quad h_0'=h_0^2$
$\Longrightarrow$ there should exist a *unique regular* solution for H on $[0,+\infty)$ connecting a finite limit $H(T)\to h_\infty$ to have a pers. exponent
- This regular sol. $H(T;(h_0,h_0'))$ should be the equivalent of the $\mathrm{P_{II}}$ Hastings-McLeod sol. for the GβE-like *tail distribution functions*:

$$\begin{aligned}\det\left[\mathbb{1}-(1-m^2)K_T\right] &= E_2(T)=\int_T^{+\infty}\mathrm{d}\ell\, p_2(\ell)=\exp\left[-\int_0^T \mathrm{d}\ell\, H(\ell)\right]\\ P_0^{\mathrm{HalfI}}(T;m) &= E_1(T)=\left[E_2(T)\right]^{1/2}\exp\left[-\frac{1}{2}\int_0^T\mathrm{d}\ell\sqrt{H'(\ell)}\right]\end{aligned}$$

The explicit $\mathrm{P_{VI}}$ and its monodromy exponents

- $H(T) = \mathcal{H}_{\mathrm{VI}}(p,q,s)$ evaluated on Hamilton's equations of motion:

$$H(T) = -\frac{(x-1)}{2}\frac{x^2y'^2-\Theta^2y^2}{y(y-1)(y-x)}, \quad y(x)=q(s), x=s=e^{2T}$$

  (Chazy 1911-Jimbo-Miwa-Okamoto form of $\mathrm{P_{VI}}$)
- The distribution functions $F(T)$ or $P_0^{\mathrm{HalfI}}(T;m)$ are *τ-functions*, as for critical scaling correlations of e.g. the $2d$ Ising model. Here best viewed as exact Kramers' formula for an explicitly time-dependent Hamiltonian, where persistence exponent asymptotic decay rate!
- Nice-looking parameters $\mathrm{P_{VI}}[y(x);\alpha,\beta,\gamma,\delta]$ with $\Theta^2=N^2=1/4\ldots$

$$y'' = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)y'^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y'+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\times$$

$$\times\left[\underbrace{0}_{\alpha}\times 1+\underbrace{0}_{\beta}\times\frac{x}{y^2}+\underbrace{\frac{\Theta^2}{2}}_{\gamma=1/8}\times\frac{x-1}{(y-1)^2}-\underbrace{\frac{\Theta(\Theta+2)}{2}}_{\delta=3/8}\times\frac{x(x-1)}{(y-x)^2}\right]$$

How this ? Ask a local/global expert for indications

Bonnet 1867/Hazzidakis 1897, found the *same* 3rd/2nd order nonlinear ODE for $H(T)$ in a different context ...

JOURNAL

DE

L'ÉCOLE IMPÉRIALE POLYTECHNIQUE.

MÉMOIRE

SUR

LA THÉORIE DES SURFACES APPLICABLES SUR UNE SURFACE DONNÉE[*],

PAR M. OSSIAN BONNET,

Membre de l'Institut.

DEUXIÈME PARTIE.

Détermination de toutes les surfaces applicables sur une surface donnée.

17. Supposons l'élément linéaire de la surface donnée S, évalué en fonction des variables particulières que nous avons appelées x et y dans la première partie, et soit

$$ds^2 = 4\varphi^2\,dx\,dy$$

- Monodromy exponents for $\mathrm{P_{VI}}$ Bonnet surfaces (up to homographic transformations):

$$(\vartheta_\infty^2, \vartheta_0^2, \vartheta_1^2, (\vartheta_x - 1)^2) = (0, \Theta^2, \Theta^2, 0)$$

- That RC had just extrapolated to the full $\mathrm{P_{VI}}$ (Gauss-Codazzi moving frame equations $\equiv$ "best" Lax pair) ...

CrossMark

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Generalized Bonnet surfaces and Lax pairs of $\mathrm{P_{VI}}$

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We build analytic surfaces in $\mathbb{R}^3(c)$ represented by the most general sixth Painlevé equation $\mathrm{P_{VI}}$ in two steps. First, the moving frame of the surfaces built by Bonnet in 1867 is extrapolated to a new, second order, isomonodromic matrix Lax pair of $\mathrm{P_{VI}}$,

Results 3/4: Αγεωμετητοζ μηδειζ εισιιτω

- Bonnet surfaces in conformal coordinates $(z, \bar{z})$ uniquely determined by the 2nd order 2nd degree nonlinear ODE satisfied by their *mean curvature function* $\mathbb{H}_m \equiv$ a $\mathrm{P_{VI}}$-Hamiltonian (Bobenko & Eitner)

$$\mathbb{H}_m(\Re z = T) = -H(T)/2$$

Reincarnation of the coarsening motto "motion by mean-curvature" !

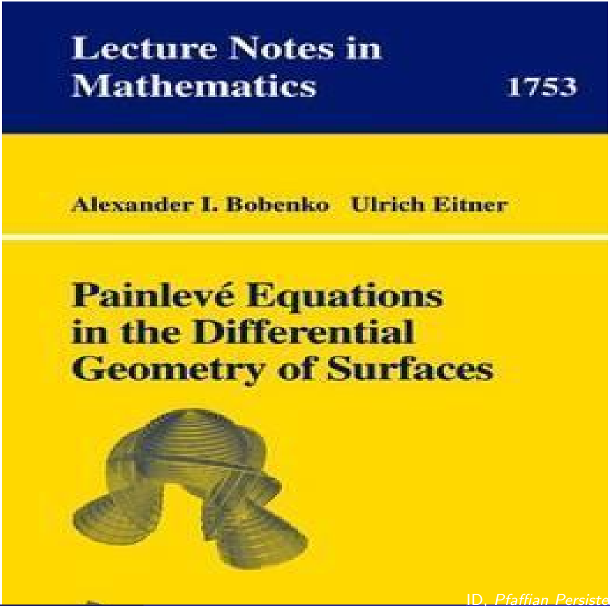
- Persistence exponent $\theta(m)(= \hat{\theta}(q = 2/(1+m))$ simply related to the asymptotic average curvatures of the underlying Bonnet-B surface:

$$\frac{\kappa_1 + \kappa_2}{2} = -\frac{\theta(m)}{2} = \frac{1}{4}\left\{\left[\frac{2}{\pi}\arccos\left(\frac{1}{\sqrt{2}}\right)\right]^2 - \left[\frac{2}{\pi}\arccos\left(\frac{m}{\sqrt{2}}\right)\right]^2\right\}$$

(kind of non-linear Buffon's needle formula, in the spirit of random geometry of Edelman & Kostlan, *How many zeros of a random polynomial are real?*, Bull. Amer. Math. Soc. **32**, 1 (1995))

- ⚠ Expression for $\hat{\theta}(q)$ or $\theta(m)$ buried somewhere in Jimbo '82, who solved completely the *connexion problem* for $\mathrm{P_{VI}}$ (RC & ID, in prep.)

Results 3/4 cont'd



Results 3/4 (last but not least) geometric interpretation

- Recall that

$$P_0^{\mathrm{HalfI}}(T;m) = \exp\left[-\frac{1}{2}\int_0^T \mathrm{d}s[H(s) + \sqrt{H'(s)}]\right]$$

  If $-H/2 = \mathbb{H}_m$, $\sqrt{H'}$ is also some length ...
- For Bonnet surfaces, the metric (first fundamental quadratic form of Gauss) is given by

$$\mathrm{d}\ell^2 = e^u \mathrm{d}z\mathrm{d}\bar{z} = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{H'(T)\sinh^2 T}, \quad T = \Re z, \quad \frac{1}{\sinh^2 T} = \text{Hopf factor}$$

- Recall that (from DHP), the amplitude of the persistence proba $e^{-\int_0^T \sqrt{H'}/2} \propto \sqrt{q(2-q)} \propto \sqrt{m}$ as $q \to 2^-$ or $m \to 0^+ \implies$ singularity in the metric: *umbilic* point where curvatures coincide

$$\mathbb{H}_m = \sqrt{\mathbb{K}_{\mathrm{Gauss}}} = \kappa = -\frac{\theta(0)}{2} = -\frac{3}{16}$$

  By Gauss' *Theorema Egregium*, $\mathbb{K}_{\mathrm{Gauss}}$ intrinsic: Persistence exponent has topological content for symmetric Ising spins ?!

Results 4/4: Universality

- Expanding the Pfaffian Fredholm generating function, and using Matsumoto-Shirai:

$$\forall T,\, P_0^{\mathrm{HalfI}}(T; m=0) = P_0^{\mathrm{2Diff}}(T|Y(0)=0) \Longrightarrow \tilde{\theta}(d=2) = \frac{3}{16}$$

(conditioning due to $E_2(T) = \int_T^{+\infty} \mathrm{d}\ell p_2(\ell)$: once-conditioned spacing proba., guaranteed by a choice of the origin on the stationary timescale)

- Conjecture: due to its intrinsic geometric content, and given that

$$\theta_{\mathrm{I}sing2d} = \theta_{\mathrm{M}odelA} = 0.19(1),$$

$\theta_{\mathrm{2Diff}} = 3/16$ could even be the *universal* critical exponent for curvature-driven growth of a non-conserved scalar order parameter in two space dimensions.

If true, needs to understand why autocorrelation exponents are distinct: $\lambda_{\mathrm{Diff2d}} = 1/2$ while $\lambda_{\mathrm{Ising2d}} \approx 5/8$ (Fisher-Huse). Large cancellations in $\langle S(0)S(T)\rangle = \sum_n (-1)^n p_n(T)$ (p_n n-flip proba.) ?

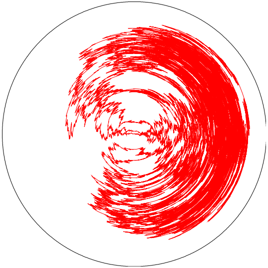
Conclusions

- Take-home message: Painlevé transcendents capable of fulfilling, on the exemplary value of the persistence probability, the Holy Grail of statistical physics: the exact integration through a local non-Markovian temporal process of the remaining spatial interacting degrees of freedom
- feasible because of a lot of underlying structure: harmonious interplay — with $\mathrm{P_{VI}}$ at the center — between algebra, geometry, probability, analysis. *stochastic integrability* (H. Spohn) or *integrable probability* (A. Borodin et al.)
- Phenomenon generic for all Painlevé, with a lot of universal non-trivial limit distributions to discover (cf. RC & ID, *The master* $\mathrm{P_{VI}}$ *heat equation*, CRAS Maths. (2014)

Another example: Phase-noise distribution, imaginary exponential functional of Brownian Motion, and the Sine-Gordon $\mathrm{P_{III}}$ transcendent (ID, in prep.)

- What is the distribution (in the complex plane) of

$$Z_\sigma = \int_0^{+\infty} \mathrm{d}s \exp\left[-s + 2\imath\sigma B(s)\right] \equiv R e^{\imath\theta}, \quad B \text{ 1d Brownian ?}$$



- Related to the solution $w(r)$ of

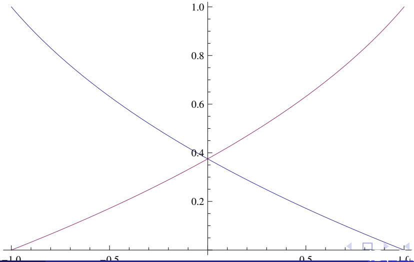
$$\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w}{dr} + \frac{1}{2}\sin(2w) = 0$$

(Broken) m-symmetry: not one but two pers. exponents

- Reversing globally the initial condition (leaving unaffected the dual dynamics of coalescing random walkers):
  $\theta_+(m) = \theta_-(-m) = \theta(m) = \hat{\theta}(q)|_{q=2/(1+m)}$, with

$$\theta(m) = \frac{1}{2}\left\{\left[\frac{2}{\pi}\arccos\left(\frac{m}{\sqrt{2}}\right)\right]^2 - \left[\frac{2}{\pi}\arccos\left(\frac{1}{\sqrt{2}}\right)\right]^2\right\}$$

⚠ NOT even in m: asymp. behavior of $P_0^{\mathrm{HalfI}}(T;m) = P_0^+(T;m) + P_0^-(T;m)$ dictated by slower decay rate, i.e. smaller exponent



Paradox: Just the branch $m \geq 0$ of $\theta(m)$ observable ?!

- Asymptotic expression for $P_0^{\mathrm{HalfI}}(T; m)$ can be checked (along with computation of amplitudes: "Widom's constant problem") using results in the math. literature on truncated Wiener-Hopf+Hankel Fredholm determinants
- Cusp due to the singular behavior of the "symbol" for the sech-kernel, whose (self-dual) Fourier transform is $\mathcal{F}[K](q) = \mathrm{sech}\,(\pi q)$. Hence largest eigenvalue $\lambda_0(T) \to 1$ and logarithm of $\det[1-(1-m^2)K_T]$ has pbs for $T \gg 1$ AND $m \to 0\ldots$
- Somewhat spurious: disappears if conditioning $P_0^{\pm}$ also w.r.t. the value the Ising spin at the origin had in the "initial" condition