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Zeros, moments and determinants

Integrability and Randomness in Mathematical Physics and Geometry, April 8th, 2019

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$$\det\begin{pmatrix} \binom{1}{K-1} & \binom{3}{K-1} & \binom{5}{K-1} & \cdots & \binom{2K-1}{K-1} \\ \binom{2}{K-1} & \binom{4}{K-1} & \binom{6}{K-1} & \cdots & \binom{2K}{K-1} \\ \binom{3}{K-1} & \binom{5}{K-1} & \binom{7}{K-1} & \cdots & \binom{2K-1}{K-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{K}{K-1} & \binom{K+2}{K-1} & \binom{K+4}{K-1} & \cdots & \binom{3K-2}{K-1} \end{pmatrix} = (-1)^{K+1} 2^{\binom{K}{2}}$$

Emilia Alvarez (thesis)

$$\det_{K\times K}\begin{pmatrix} \frac{1}{\Gamma(2K)} & \frac{1}{\Gamma(2K-2)} & \frac{1}{\Gamma(2K-4)} & \cdots & \frac{1}{\Gamma(2)} \\ \frac{1}{\Gamma(2K-1)} & \frac{1}{\Gamma(2K-3)} & \frac{1}{\Gamma(2K-5)} & \cdots & \frac{1}{\Gamma(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\Gamma(K+2)} & \frac{1}{\Gamma(K)} & \frac{1}{\Gamma(K-2)} & \cdots & \frac{1}{\Gamma(-K+4)} \\ \frac{1}{\Gamma(K)} & \frac{1}{\Gamma(K-2)} & \frac{1}{\Gamma(K-4)} & \cdots & \frac{1}{\Gamma(-K+2)} \end{pmatrix}$$

$$= \frac{K(K-1)}{2}\det_{K\times K}\begin{pmatrix} \frac{1}{\Gamma(2K-1)} & \frac{1}{\Gamma(2K-3)} & \frac{1}{\Gamma(2K-5)} & \cdots & \frac{1}{\Gamma(1)} \\ \frac{1}{\Gamma(2K-2)} & \frac{1}{\Gamma(2K-4)} & \frac{1}{\Gamma(2K-6)} & \cdots & \frac{1}{\Gamma(0)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\Gamma(K)} & \frac{1}{\Gamma(K-2)} & \frac{1}{\Gamma(K-4)} & \cdots & \frac{1}{\Gamma(-K+2)} \end{pmatrix}$$

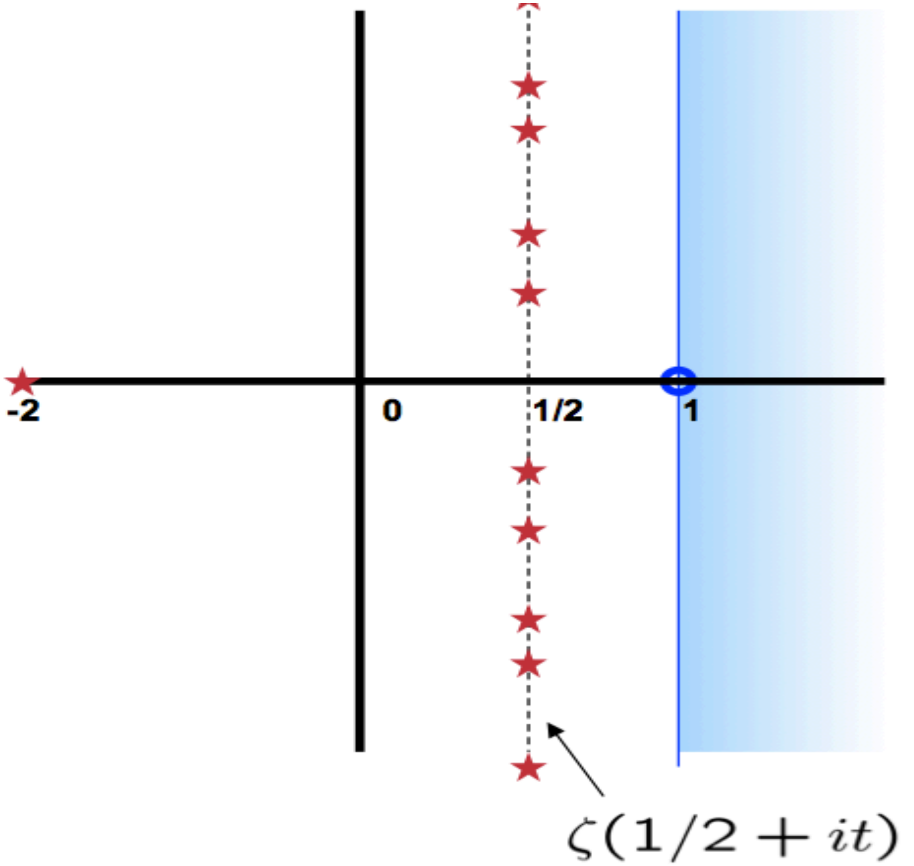
Ian Cooper (thesis)

$$\left|\begin{matrix} \frac{1}{\Gamma(2k)} & \frac{1}{\Gamma(2k-1)} & \cdots & \frac{1}{\Gamma(k+1)} & \frac{1}{\Gamma(2k)} & \frac{-1}{\Gamma(2k-1)} & \cdots & \frac{(-1)^{k-1}}{\Gamma(k+1)} \\ \frac{1}{\Gamma(2k-1)} & \frac{1}{\Gamma(2k-2)} & \cdots & \frac{1}{\Gamma(k)} & \frac{-1}{\Gamma(2k-1)} & \frac{1}{\Gamma(2k-2)} & \cdots & \frac{(-1)^{k}}{\Gamma(k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\Gamma(1)} & \frac{1}{\Gamma(0)} & \cdots & \frac{1}{\Gamma(2-k)} & \frac{-1}{\Gamma(1)} & \frac{1}{\Gamma(0)} & \cdots & \frac{(-1)^{3k-2}}{\Gamma(2-k)} \end{matrix}\right|$$

$$= (-1)^k 2^{k^2} \prod_{\ell=1}^{k-1} \frac{\ell!}{(k+\ell)!}$$

Conrey, Farmer, Keating, Rubinstein, Snaith. Proc. London. Math. Soc. (3) 91 (2005)

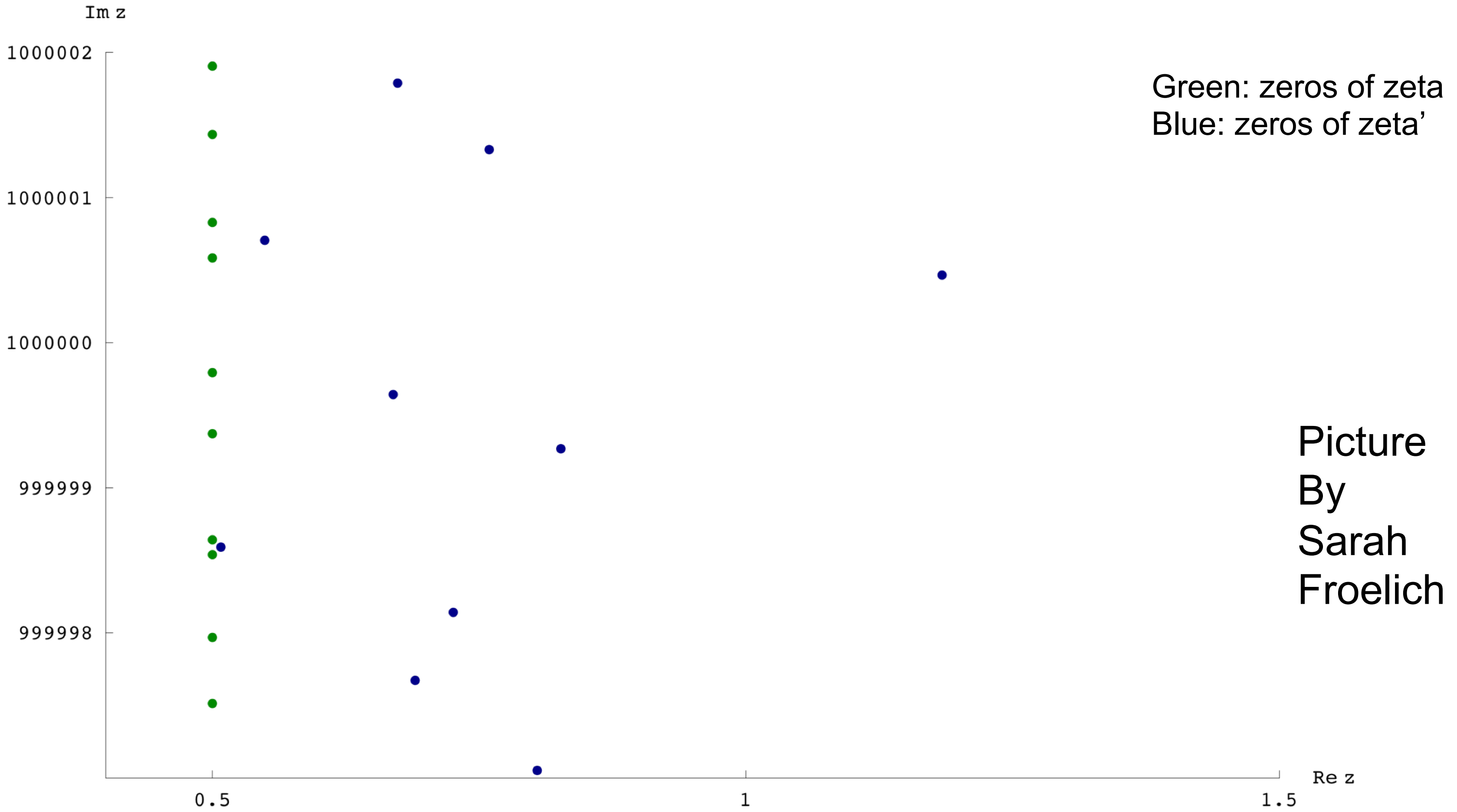
The Riemann Zeta Function



$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}s > 1$$

$$= \prod_p (1 - 1/p^s)^{-1}$$

Speiser: RH is equivalent to all complex zeros of $\zeta'(s)$ having real part at least 1/2



Picture By Sarah Froelich

Levinson, Montgomery (1974): $\zeta(s)$ and $\zeta'(s)$ have essentially the same number of zeros in $\Re(s) < 1/2$.

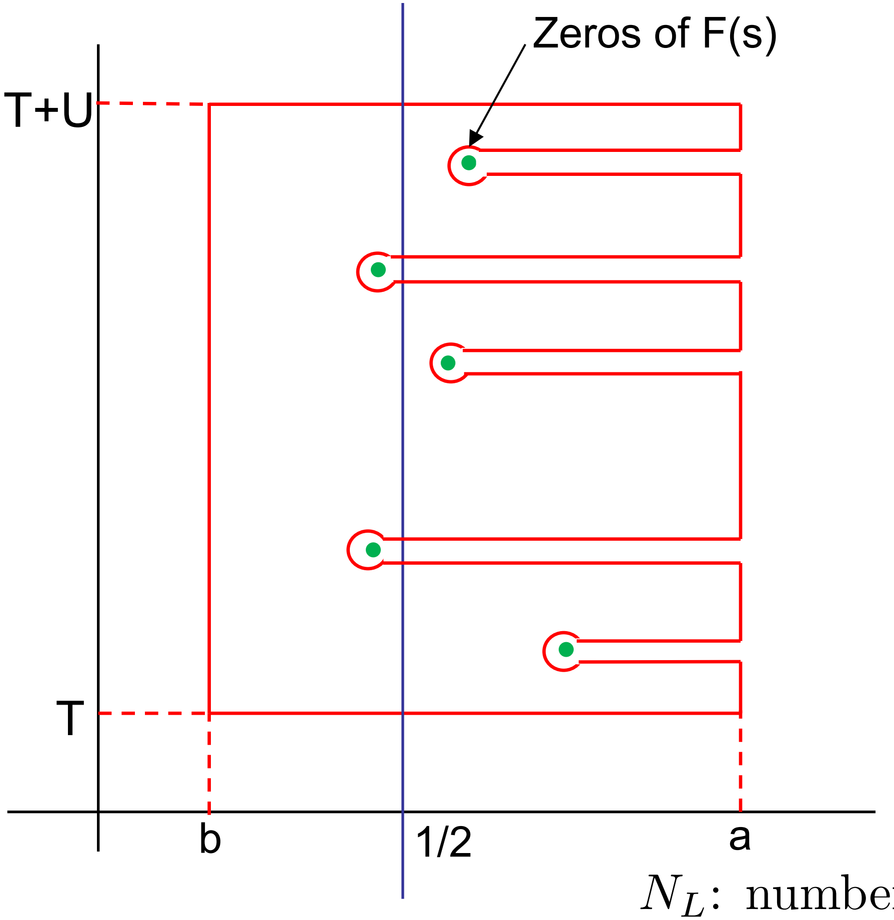
Levinson (1974): More than one third of the zeros of the Riemann zeta function are on the critical line

Conrey (1989): More than two fifths of the zeros of the Riemann zeta function are on the critical line

Bui, Conrey, Young (2011): More than 41% of the zeros of the Riemann zeta function are on the critical line

Horizontal distribution of zeros of the derivative of the Riemann zeta function

Littlewood's lemma (1924): For an analytic function $F(s)$, (Eg. $\zeta'(s)$)



$$\int_T^{T+U} \log|F(b+it)|dt - \int_T^{T+U} \log|F(a+it)|dt + \int_b^a \arg F(\sigma + i(T+U))d\sigma - \int_b^a \arg F(\sigma + iT)d\sigma$$

$$= 2\pi \sum \text{dist} > 2\pi(a - 1/2)\ N_L$$

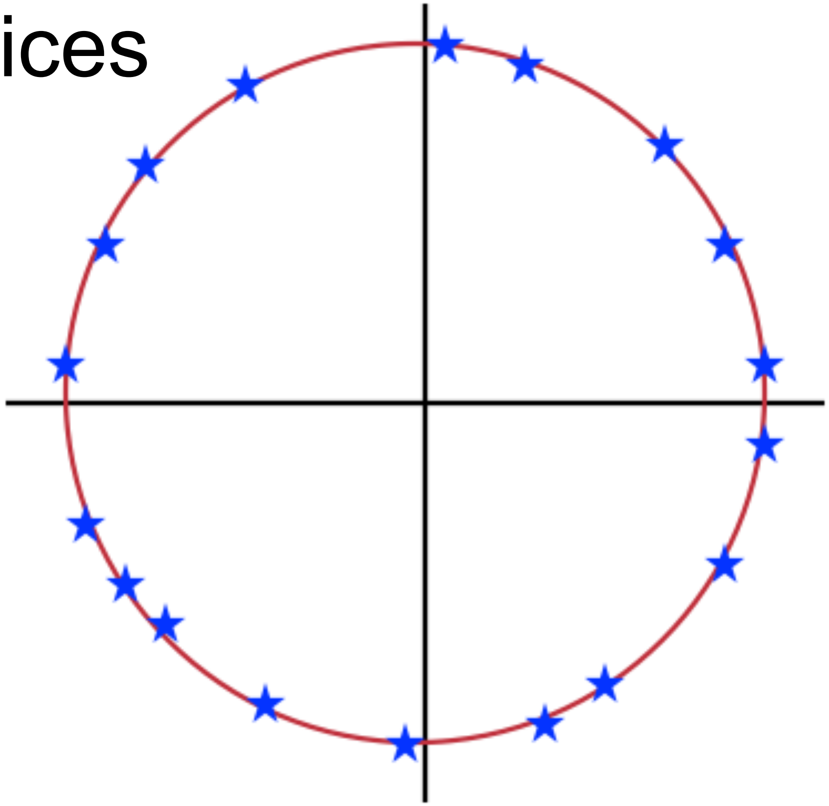
N_L: number of zeros of $\zeta'(s)$ to the left of the 1/2-line

So number theorists are interested in:

$$\int_0^T \log|\zeta'(a+it)|dt$$

$$= \frac{d}{dv}\int_0^T |\zeta'(a+it)|^v dt\Big|_{v=0}$$

Random Unitary Matrices



A is an $N \times N$ unitary matrix:

$$AA^\dagger = A^\dagger A = I$$

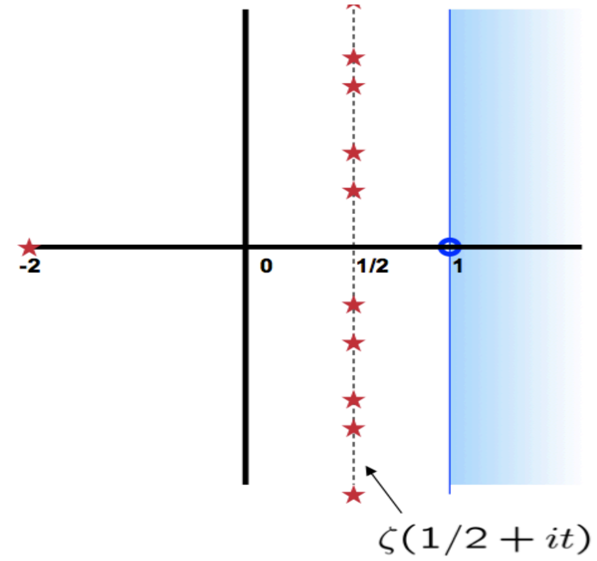
eigenvalues of A: $e^{i\theta_n}$

A is chosen randomly with respect to Haar measure on $U(N)$

density of eigenphases: $\frac{N}{2\pi}$

Density of zeros:

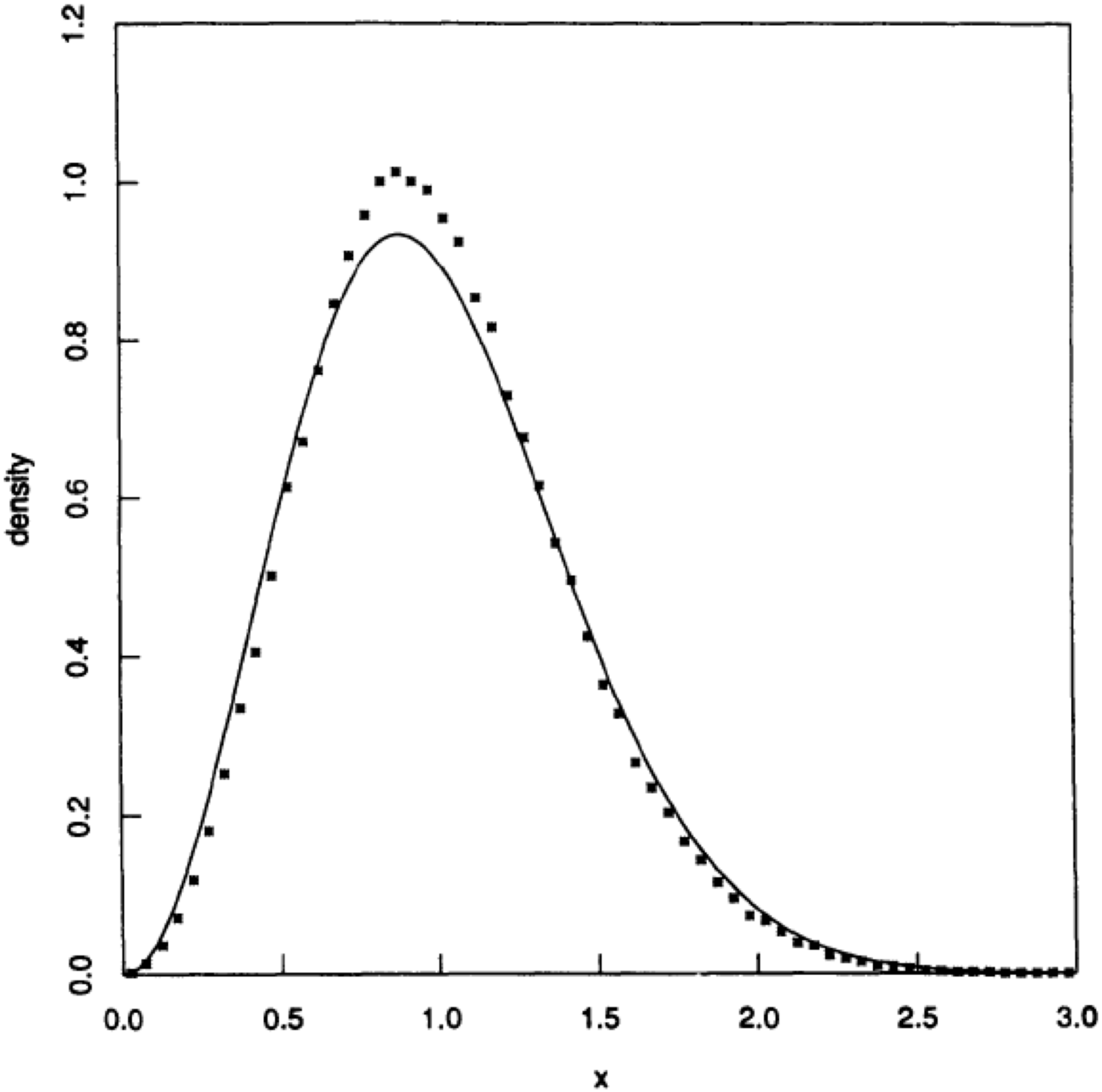
$$d(t) \sim \frac{1}{2\pi} \log \frac{t}{2\pi}$$



$$w_n = t_n \frac{1}{2\pi} \log \frac{t_n}{2\pi}, \quad t_n = \text{n}^{\text{th}} \text{ Riemann zero}$$

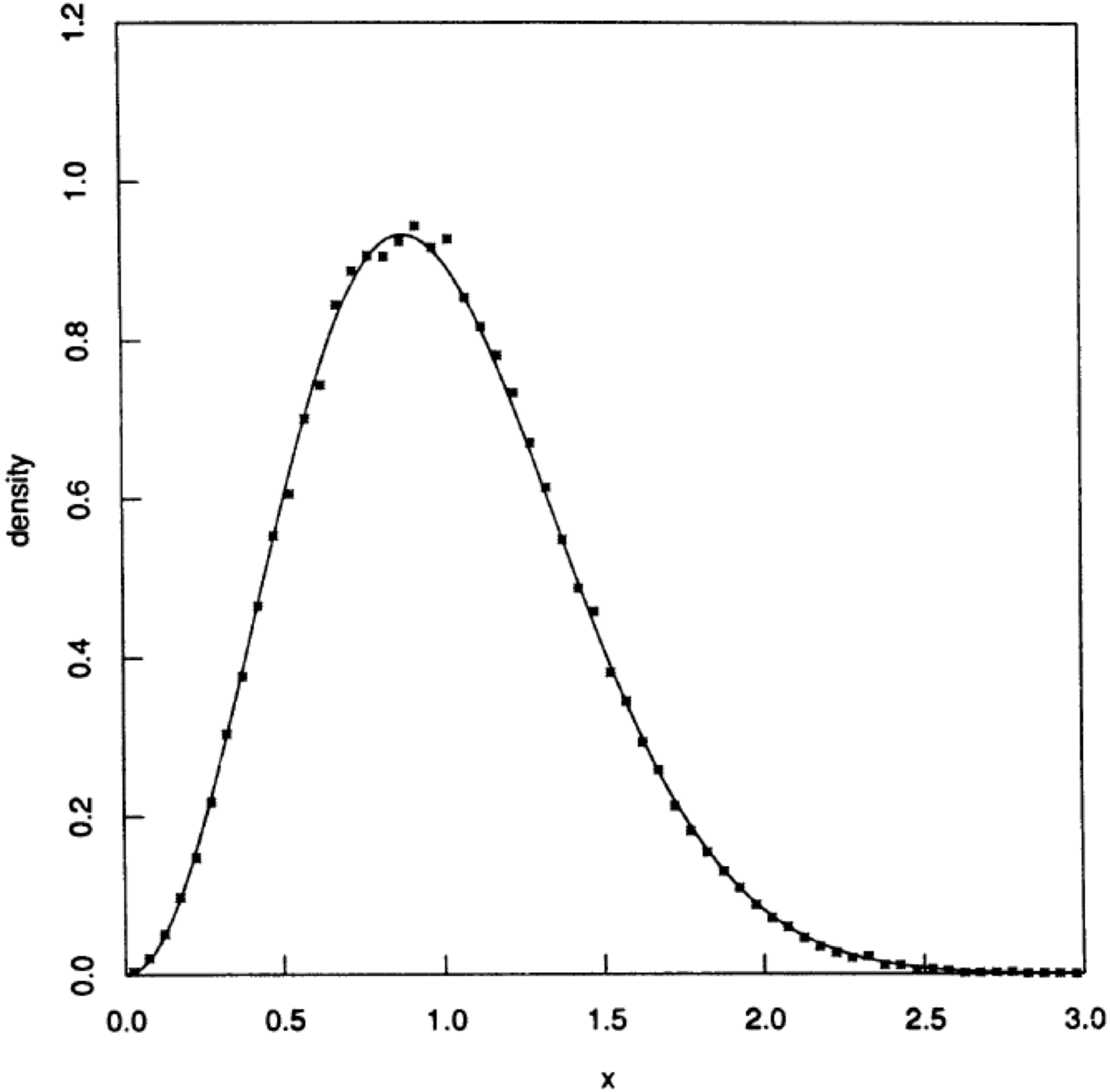
scale the Riemann zeros so that their average spacing is 1

Nearest Neighbor Spacing

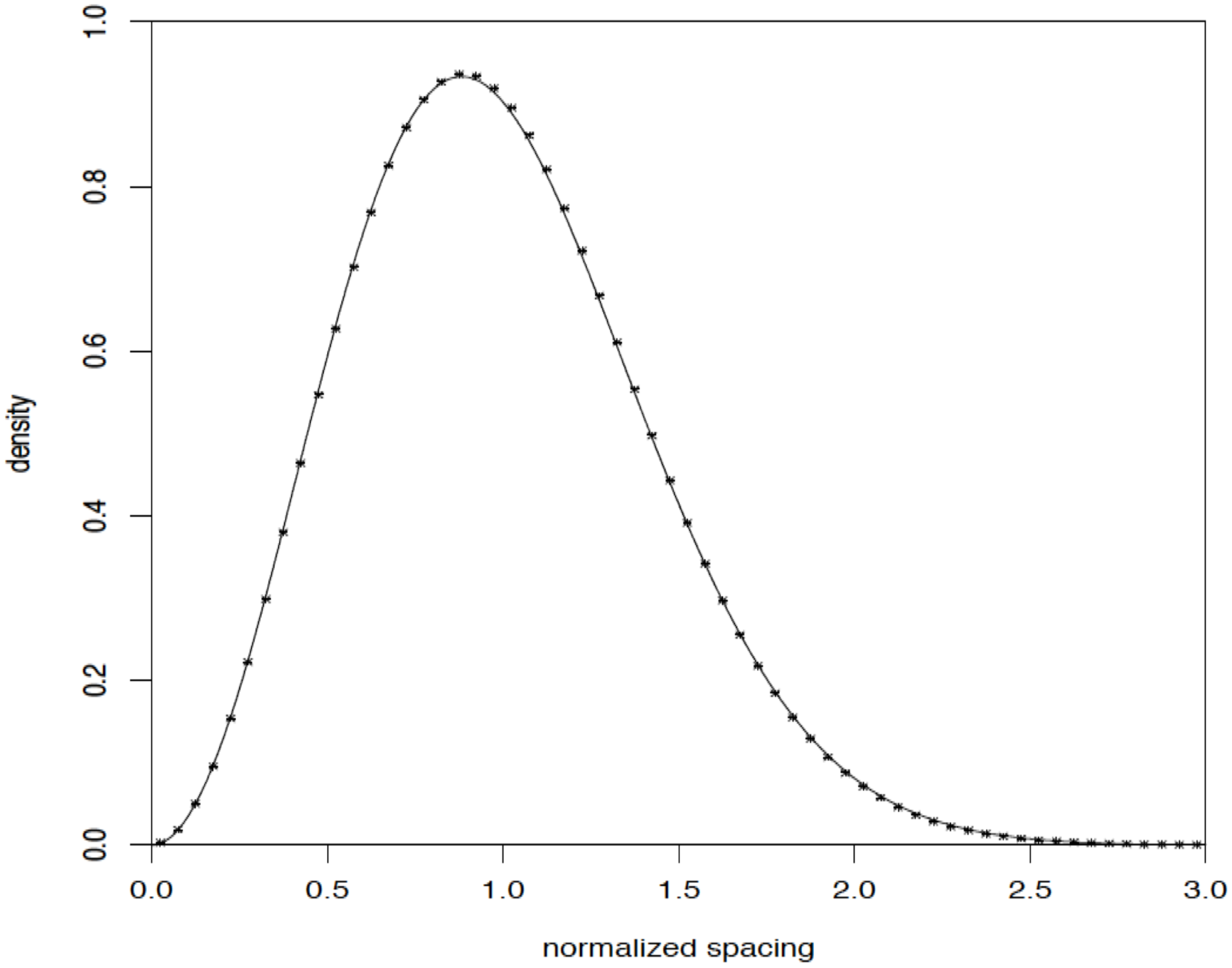


Probability density for distances between consecutive eigenvalues/ zeros

Using the first 100000 Riemann zeros – Picture by Andrew Odlyzko



10^5 zeros around the 10^{12}th zero



Picture by
A. Odlyzko

billion zeros around the 10^{16} th zero

Characteristic polynomial:

$$\Lambda_A(s) = \prod_{n=1}^{N} \left(1 - se^{-i\theta_n}\right) = \det\left(I - A^{\dagger} s\right)$$

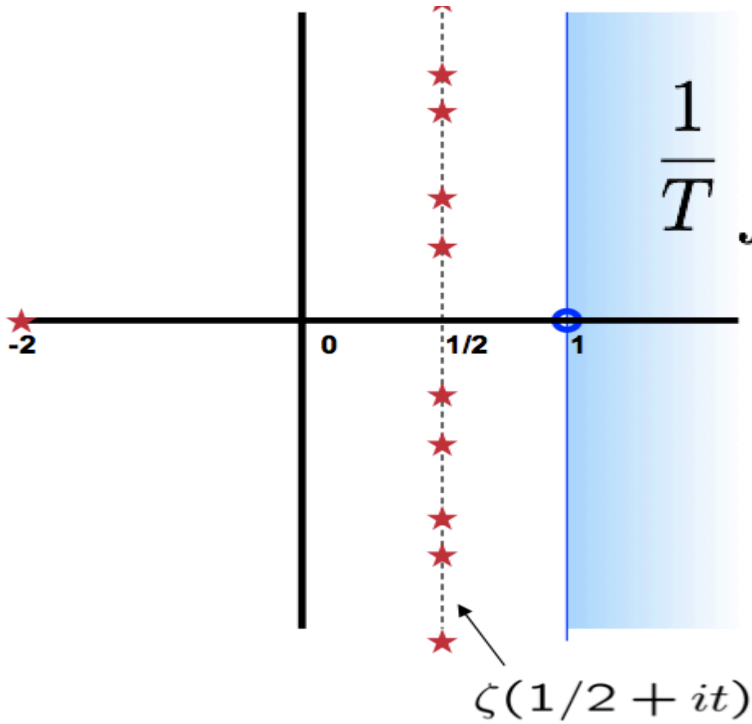
Equate densities of zeros:

$$\overset{\text{RZF}}{\frac{1}{2\pi}\log\frac{T}{2\pi}} = \overset{\text{RMT}}{\frac{N}{2\pi}}$$

Moments of the Riemann Zeta Function

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^2 dt \sim \log T$$

(Hardy and Littlewood, 1918)



$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4 dt \sim \frac{1}{2\pi^2}\log^4 T$$

(Ingham, 1926)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}\, s > 1$$
$$= \prod_p (1 - 1/p^s)^{-1}$$

Moments of the Riemann Zeta Function: Conjecture

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\lambda} dt \sim a_\lambda f_\lambda \log^{\lambda^2} T$$

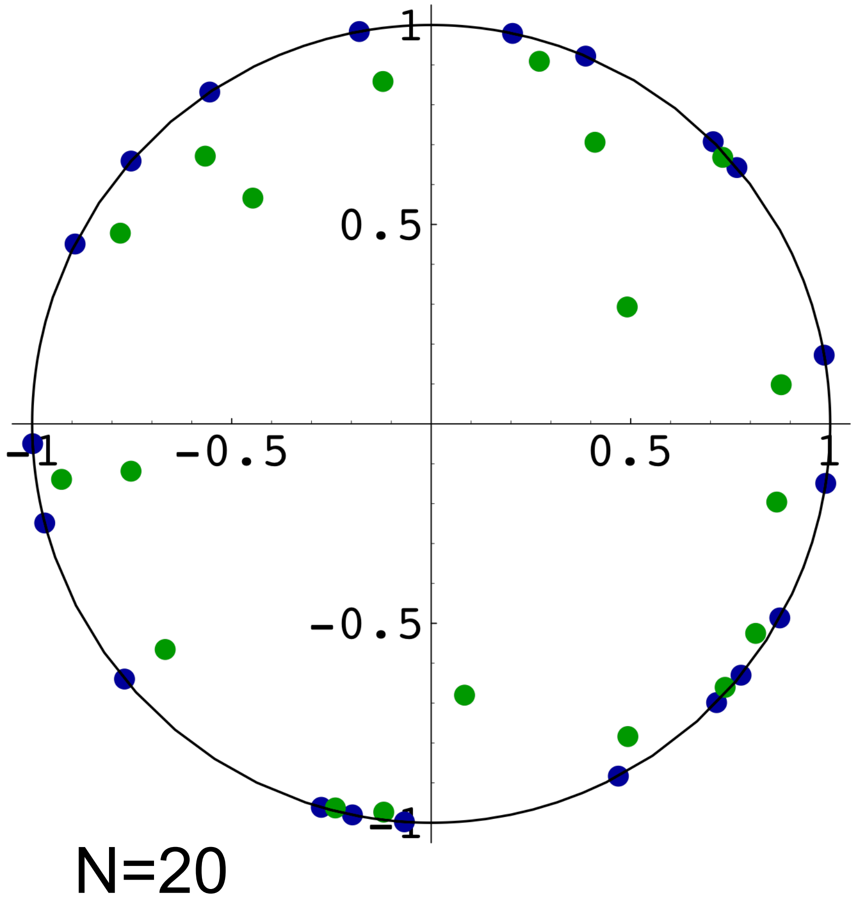
Moments of characteristic polynomials: Theorem

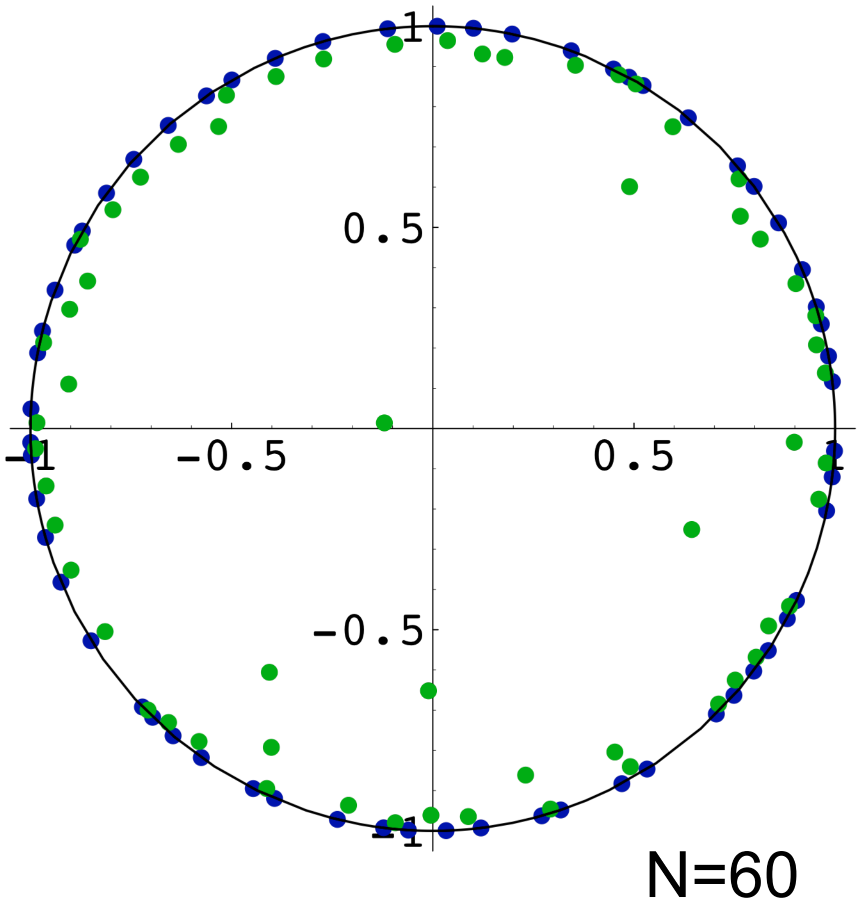
$$\int_{U(N)} |\Lambda_A(1)|^{2\lambda} dA_{Haar} \sim f_{\lambda,U(N)} N^{\lambda^2}$$

$$N \sim \log T$$

Conjecture (Keating and Snaith, 2000): $f_\lambda = f_{\lambda,U(N)}$

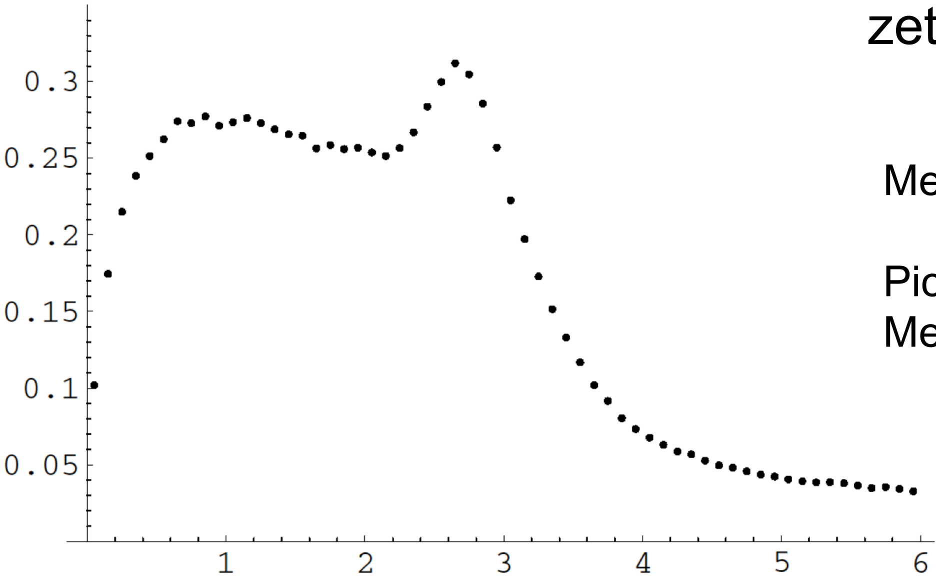
Zeros of a characteristic polynomial (blue) and its derivative (green)





Pictures by Phan Toan

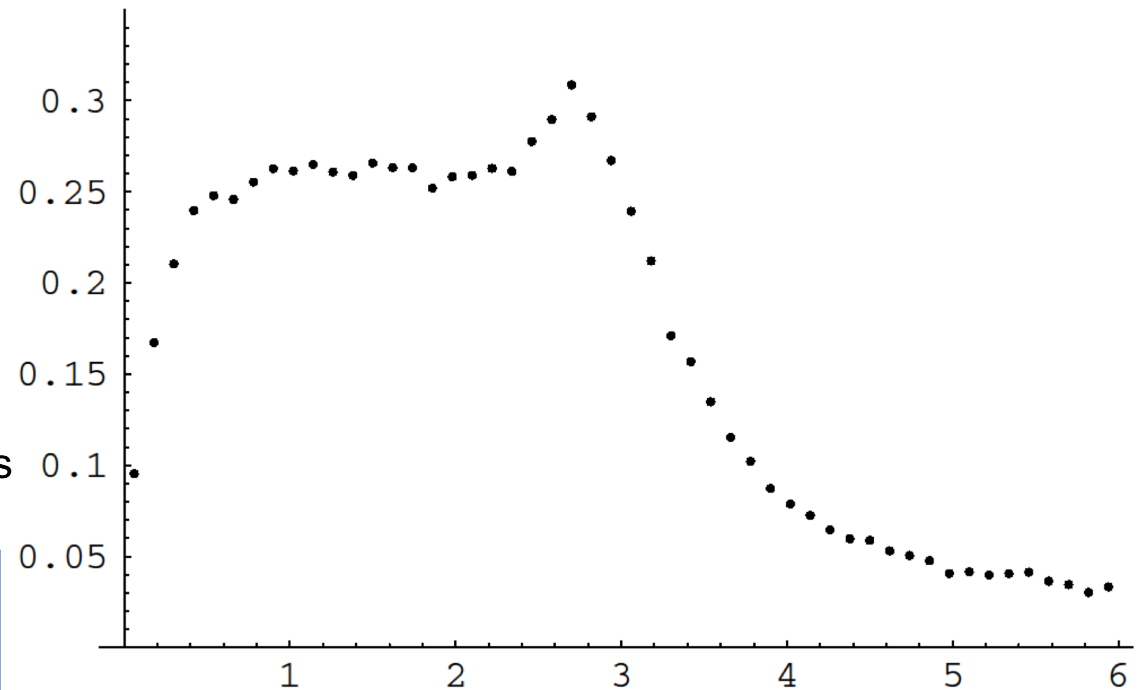
Horizontal (radial) distribution of zeros of the derivative of zeta (characteristic polynomial)



Mezzadri: J. Phys. A 36 (2003)

Pictures from Dueñez, Farmer, Froelich, Hughes, Mezzadri, Phan: Nonlinearity 23 (2010)

Distribution of distance from the unit circle of zeros of $\Lambda'_A(s)$, N=100



Distribution of real part of the zeros of $\zeta'(s)$ for 100000 zeros around T=1000000.

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Important to number theorists:

$$\int_0^T |\zeta'(a+it)|^v dt$$

We would like to calculate:

$$\int_{U(N)} |\Lambda_A'(s)|^v dA_{Haar}$$

v non-integer

$v = 1$ Winn: Commun. Math. Phys. 315 (2012)

Important to number theorists:

$$\int_0^T |\zeta'(a+it)|^v dt$$

We would like to calculate:

$$\int_{U(N)} |\Lambda_A'(s)|^v dA_{Haar}$$

v non-integer

We can calculate:

$$\int_{U(N)} |\Lambda_A'(e^{i\theta})|^{2K-2M} |\Lambda_A(e^{i\theta})|^{2M} dA_{Haar}$$

K-M integer

Hughes (2001); Conrey, Rubinstein, Snaith (2006); Dehaye (2008);
Bailey, Bettin, Blower, Conrey, Prokhorov, Rubinstein, Snaith (preprint);
Basor, Bleher, Buckingham, Grava, Its, Its, Keating (preprint)

Important to number theorists:

$$\int_0^T |\zeta'(a+it)|^v dt$$

We would like to calculate:

$$\int_{U(N)} |\Lambda'_A(s)|^v dA_{Haar}$$ v non-integer

We can calculate:

$$\int_{U(N)} \left| \frac{\Lambda'_A}{\Lambda_A}(e^{-\alpha}) \right|^{2K} dA_{Haar}, \; K \text{ integer}$$

Conrey, Snaith: Commun. Number Th. and Physics. Vol. 2, Num.3 (2008)

Bailey, Bettin, Blower, Conrey, Prokhorov, Rubinstein, Snaith: preprint

Important to number theorists:

$$\int_0^T |\zeta'(a+it)|^v dt$$

We would like to calculate:

$$\int_{U(N)} |\Lambda_A'(s)|^v dA_{Haar}$$ v non-integer

We can calculate:

$$\int_{SO(2N)} (\Lambda''(1))^K dA_{Haar}, \ K \text{ integer}$$

Altuğ, Bettin, Petrow, Rishikesh, Whitehead (2014)

Important to number theorists:

$$\int_0^T |\zeta'(a+it)|^v dt$$

We would like to calculate:

$$\int_{U(N)} |\Lambda_A'(s)|^v dA_{Haar}$$

v non-integer

We can calculate:

$$\int_{SO(2N)} \frac{\Lambda(1)^r \Lambda'(e^{-\phi})}{\Lambda(e^{-\phi})} dA_{Haar}, \ \Re(r) > 1/2$$

Ian Cooper (thesis)

Important to number theorists:

$$\int_0^T |\zeta'(a+it)|^v dt$$

We would like to calculate:

$$\int_{U(N)} |\Lambda'_A(s)|^v dA_{Haar}$$

v non-integer

We can calculate:

$$\int_{SO(2N)} \left(\frac{\Lambda'_A}{\Lambda_A}(e^{-\alpha}) \right)^K dA_{Haar}, \text{ for integer } K$$

Mason and Snaith (2018)
Emilia Alvarez (thesis)

For large N

$$\int_{SO(2N)} \left(\frac{\Lambda'_A}{\Lambda_A}(e^{-a/N} \right)^K dA_{Haar}$$

$$\to \oint \cdots \oint \Delta(u_1^2, u_2^2, \ldots, u_K^2)\Delta(u_1, u_2, \ldots, u_K) f(u_1) \cdots f(u_K) du_1 \cdots du_K$$

Vandermonde:

$$\Delta(u_1, u_2, \ldots, u_K) = \det \begin{vmatrix} 1 & u_1 & u_1^2 & \cdots & u_1^{K-1} \\ 1 & u_2 & u_2^2 & \cdots & u_2^{K-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_K & u_K^2 & \cdots & u_K^{K-1} \end{vmatrix}$$

$$\Delta(u_1^2, u_2^2, \ldots, u_K^2)\Delta(u_1, u_2, \ldots, u_K)$$

$$= \left(\sum_{\sigma\in S_K} sgn(\sigma) \prod_{i=1}^{K} u_i^{2\sigma(i)-2}\right) \left(\sum_{\tau\in S_K} sgn(\tau) \prod_{k=1}^{K} u_k^{\tau(k)-1}\right)$$

$$\to K! \left(\sum_{\sigma\in S_K} sgn(\sigma) u_1^{2\sigma(1)-2} u_2^{2\sigma(2)-1} u_3^{2\sigma(3)} \cdots u_K^{2\sigma(K)+K-3}\right)$$

$$= K! \det(u_i^{2j+i-3})_{i,j=1}^{K}$$

For large N

$$\int_{SO(2N)} \left(\frac{\Lambda'_A}{\Lambda_A}(e^{-a/N} \right)^K dA_{Haar}$$

$$\to \oint \cdots \oint K! \det(u_i^{2j+i-3})_{i,j=1}^K f(u_1)\cdots f(u_K) du_1 \cdots du_K$$

$$\to K! \det \left(\oint f(u_i) u_i^{2j+i-3} du_i \right)_{i,j=1}^K$$

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$$\det \begin{pmatrix} \binom{1}{K-1} & \binom{3}{K-1} & \binom{5}{K-1} & \cdots & \binom{2K-1}{K-1} \\ \binom{2}{K-1} & \binom{4}{K-1} & \binom{6}{K-1} & \cdots & \binom{2K}{K-1} \\ \binom{3}{K-1} & \binom{5}{K-1} & \binom{7}{K-1} & \cdots & \binom{2K-1}{K-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{K}{K-1} & \binom{K+2}{K-1} & \binom{K+4}{K-1} & \cdots & \binom{3K-2}{K-1} \end{pmatrix}$$

$$\binom{n}{r} - \binom{n-1}{r}$$

$$= \binom{n-1}{r-1}$$

$$\det \begin{pmatrix} \binom{1}{K-1} & \binom{3}{K-1} & \binom{5}{K-1} & \cdots & \binom{2K-1}{K-1} \\ \binom{1}{K-2} & \binom{3}{K-2} & \binom{5}{K-2} & \cdots & \binom{2K-1}{K-2} \\ \binom{2}{K-2} & \binom{4}{K-2} & \binom{6}{K-2} & \cdots & \binom{2K}{K-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{K-1}{K-2} & \binom{K+1}{K-2} & \binom{K+3}{K-2} & \cdots & \binom{3K-1}{K-2} \end{pmatrix}$$

Emilia Alvarez (thesis)

$$\det \begin{pmatrix} \binom{1}{K-1} & \binom{3}{K-1} & \binom{5}{K-1} & \cdots & \binom{2K-1}{K-1} \\ \binom{1}{K-2} & \binom{3}{K-2} & \binom{5}{K-2} & \cdots & \binom{2K-1}{K-2} \\ \binom{1}{K-3} & \binom{3}{K-3} & \binom{5}{K-3} & \cdots & \binom{2K-1}{K-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{1}{1} & \binom{3}{1} & \binom{5}{1} & \cdots & \binom{2K-1}{1} \\ \binom{1}{0} & \binom{3}{0} & \binom{5}{0} & \cdots & \binom{2K-1}{0} \end{pmatrix}$$

$$\binom{2j-1}{1} = 2j-1$$

$$2\det \begin{pmatrix} \binom{1}{K-1} & \binom{3}{K-1} & \binom{5}{K-1} & \cdots & \binom{2K-1}{K-1} \\ \binom{1}{K-2} & \binom{3}{K-2} & \binom{5}{K-2} & \cdots & \binom{2K-1}{K-2} \\ \binom{1}{K-3} & \binom{3}{K-3} & \binom{5}{K-3} & \cdots & \binom{2K-1}{K-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{1}{2} & \binom{3}{2} & \binom{5}{2} & \cdots & \binom{2K-1}{2} \\ 1 & 2 & 3 & \cdots & K \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$2\det\begin{pmatrix} \binom{1}{K-1} & \binom{3}{K-1} & \binom{5}{K-1} & \cdots & \binom{2K-1}{K-1} \\ \binom{1}{K-2} & \binom{3}{K-2} & \binom{5}{K-2} & \cdots & \binom{2K-1}{K-2} \\ \binom{1}{K-3} & \binom{3}{K-3} & \binom{5}{K-3} & \cdots & \binom{2K-1}{K-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{1}{2} & \binom{3}{2} & \binom{5}{2} & \cdots & \binom{2K-1}{2} \\ 1 & 2 & 3 & \cdots & K \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$\binom{2j-1}{2} = 2j^2 - 3j + 1$$

$$\prod_{m=0}^{K-1} \frac{2^m}{m!} \det\begin{pmatrix} 1 & 2^{K-1} & \cdots & K^{K-1} \\ 1 & 2^{K-2} & \cdots & K^{K-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^2 & \cdots & K^2 \\ 1 & 2 & \cdots & K \\ 1 & 1 & \cdots & 1 \end{pmatrix} = (-1)^{\frac{K(K-1)}{2}} 2^{\binom{K}{2}}$$