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The supercooled Stefan problem

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Outline

History of Stefan problems

- **Stefan 1889–1891**: **free boundary problems** for the **heat equation**.
- Physical models of **ice formation**; **evaporation** & **condensation**.
- Dormant until **Brillouin '31**, **Rubinshtein**: $\approx$**2500 papers** by **'67**.
- **Kamenomostkaja '61**: definitive solution.
- Today: **supercooled Stefan problem**.
- **Sherman '70**: presence of **blow-ups**.
- For some $T < \infty$: **boundary speed** $\to \infty$.

Mathematical formulation

- **Supercooled Stefan problem (1D, one phase)**:

$$\partial_t u = \frac{1}{2}\partial_{xx} u \quad \text{on} \quad \{(t,x) \in [0,\infty)^2 : x \geq \Lambda_t\},$$

$$\Lambda_t' = C\partial_x u(t,\Lambda_t), \quad t \geq 0,$$

$$u(0,x) = f(x), \quad x \geq 0 \quad \text{and} \quad u(t,\Lambda_t) = 0, \quad t \geq 0,$$

  where $f \geq 0$, $C \geq 0$.

- **Blow-up**: for some $T < \infty$, $\lim_{t\uparrow T} \Lambda_t' = \infty$.
- **Classical solution** on $[0, T)$.

Where is probability?

- **Probabilistic** problem: find a non-decreasing function Λ such that

$$\overline{Y}_t = \overline{Y}_0 + B_t - \Lambda_t, \quad t \le \overline{\tau}, \qquad \overline{Y}_t = \overline{Y}_{\overline{\tau}}, \quad t > \overline{\tau},$$

$$\Lambda_t = C\,\mathbb{P}(\overline{\tau} \le t).$$

- If Λ' exists on $[0, T)$, densities $p(t, \cdot)$ of $\overline{Y}_t$ solve

$$\partial_t p = \frac{1}{2}\partial_{xx} p + \Lambda'_t \partial_x p, \quad p(0, \cdot) = f, \quad p(\cdot, 0) = 0,$$
$$\Lambda'_t = \frac{C}{2}\partial_x p(t, 0), \quad t \in [0, T).$$

$\Longrightarrow u(t, x) := p(t, x - \Lambda_t)$ solves **supercooled Stefan** problem.

- Can look for **global** solutions of both problems!

Additional motivation

Setting 1: neural networks

- Neorons in a part of the brain, e.g. 10^6 in the human hippocampus.
- When the membrane potential of a neuron reaches a critical level ("spike"), the neuron fires.
- This may lead to a spike in surrounding neurons, etc.
- Potentially: macroscopic number of spikes $\to$ synchronization.

Setting 2: systemic risk

- Banking system with banks borrowing from each other.
- Banks default $\to$ losses to other banks $\to$ more banks default $\to$ etc.

Interacting particle system (IPS)

- N particles with initial locations $Y^1(0), Y^2(0), \ldots, Y^N(0) \in [0, \infty)$.
- Particles move according to **indepedent standard Brownian motions**.
- When a particle hits 0, it is **absorbed**.
- This leads to **immediate downward jumps** by other particles, tuned by $C > 0$.
- If some particles cross 0 due to jumps, these particles are **removed**, jump sizes of remaining particles are **adjusted**, etc.
- When **cascade resolved**: remaining particles **continue as BMs**, etc.

IPS: in formulas

- **Particle locations**: Y^1, Y^2, $\ldots$, Y^N.
- As long as particles on $(0, \infty)$:

$$\mathrm{d}Y_t^i = \mathrm{d}B_t^i, \quad i = 1, 2, \ldots, N,$$

B^1, B^2, $\ldots$, B^N **independent standard BMs**.

- **Hitting times**:

$$\tau^i = \inf\{t > 0 : \ Y_t^i \leq 0\}, \quad i = 1, 2, \ldots, N.$$

- Suppose Y^i hits 0 at time t and is **removed**.

IPS: cascades, in words

- **Shift** the remaining particles by

$$C \log\left(1 - \frac{1}{S_{t-}}\right),$$

  where S_{t-} is the **pre-absorption size of the system**.

- **Note**: factor $\downarrow$ in size S_{t-}, $\uparrow$ in parameter C.

- Update may lead to particles $i_1, i_2, \ldots, i_k$ crossing 0, these are removed, and we **adjust the shift** to

$$C \log\left(1 - \frac{k+1}{S_{t-}}\right).$$

- May cause **more immediate absorptions**, in which case **repeat** procedure etc., until determine all particles to remove at time t.

IPS: cascades, in formulas

- **System size**: $S_t := \sum_{i=1}^{N} \mathbf{1}_{\{\tau^i > t\}}$.
- **Order statistics**: $Y_{t-}^{(1)} \leq Y_{t-}^{(2)} \leq \cdots \leq Y_{t-}^{(S_{t-})}$ of $(Y_{t-}^i : \tau^i \geq t)$.
- **# of particles removed at time** t:

  $D_t := \inf \left\{ k : \; Y_{t-}^{(k)} + C \log \left(1 - \frac{k-1}{S_{t-}} \right) > 0 \right\} - 1.$
- **Particle locations**:

  $Y_t^i := Y_0^i + B_t^i + \sum_{u \leq t} C \log \left(1 - \frac{D_u}{S_{u-}} \right).$

Large system limit: starting point

To construct **global** solutions:

- take $N \to \infty$;
- blow-ups $\leftrightarrow$ macroscopic cascades.

Crucial observation: sum of jumps

$$\sum_{u\le t} C \log\left(1 - \frac{D_u}{S_{u-}}\right) = \sum_{u\le t} C \log\left(\frac{S_u}{S_{u-}}\right) = C \log\left(\frac{1}{N}\sum_{j=1}^{N} \mathbf{1}_{\{\tau^j > t\}}\right).$$

$\Longrightarrow$ functional of the **empirical measure** $\varrho^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{Y^i}$.

$\longrightarrow$ **Interaction of mean-field type** $:\Longleftrightarrow$ **dynamics** of every particle **functional** of the **empirical measure**, **own location** (process) & independent **random input**; **same functional across particles**.

Large system limit: McKeav-Vlasov heuristics

McKean-Vlasov heuristics (cf. **Sznitman '89**):

- **Classical setting**:

  $Y_t^i = Y_0^i + \int_0^t b(Y_s^i, \varrho_s^N)\,\mathrm{d}s + \int_0^t \sigma(Y_s^i, \varrho_s^N)\,\mathrm{d}B_s^i,\ i = 1, 2, \ldots, N.$

- **Guess**: $\varrho^N \stackrel{N\to\infty}{\longrightarrow} \varrho$, **deterministic**.

- $\implies$ for large N, particle locations **well-approximated** by

  $\overline{Y}_t^i = \overline{Y}_0^i + \int_0^t b(\overline{Y}_s^i, \varrho_s)\,\mathrm{d}s + \int_0^t \sigma(\overline{Y}_s^i, \varrho_s)\,\mathrm{d}B_s^i,\ i = 1, 2, \ldots, N.$

- $\implies \varrho = \lim_{N\to\infty} \varrho^N = \lim_{N\to\infty} \overline{\varrho}^N = \mathcal{L}(\overline{Y}^1)$.

- **Conclusion**: in $N \to \infty$ limit, Y^i converge to **unique solution** of

  $\overline{Y}_t = \overline{Y}_0 + \int_0^t b(\overline{Y}_s, \mathcal{L}(\overline{Y}_s))\,\mathrm{d}s + \int_0^t \sigma(\overline{Y}_s, \mathcal{L}(\overline{Y}_s))\,\mathrm{d}B_s.$

Large system limit: our setting

- **McKean-Vlasov heuristics** suggests Y^i converge to **unique sol.** of

  $\overline{Y}_t = \overline{Y}_0 + B_t + \Lambda_t,$

  where

  $\Lambda_t := C \log \mathbb{P}(\overline{\tau} > t), \quad \overline{\tau} := \inf\{t \geq 0 : \overline{Y}_t \leq 0\}.$

- **Problems**: **non-existence**, **non-uniqueness** in $C([0,\infty),\mathbb{R})$.
- $\mathbb{P}(\overline{\tau} > t)$ or $\frac{1}{N}\sum_{j=1}^{N} \mathbf{1}_{\{\tau^j > t\}}$ do not specify **cascade mechanism**.
- $\rightsquigarrow D_t := \inf\{y > 0 : y - F_t(y) > 0\}$

  $$:= \inf\left\{y > 0 : y + C\log\left(1 - \frac{\mathbb{P}(\overline{\tau} \geq t, \overline{Y}_{t-} \in (0,y))}{\mathbb{P}(\overline{\tau} \geq t)}\right) > 0\right\}.$$

- **Specify** $\Lambda_t = \Lambda_{t-} + F_t(D_t)$, rcll.
- Call solutions with **correct cascade mechanism physical solutions**.

A first limit theorem

Theorem (Nadtochiy, S. '17) Suppose $\frac{1}{N}\sum_{i=1}^{N}\delta_{Y_i(0)} \to \nu$; ν has a bounded density f_ν on $[0,\infty)$ vanishing in a neighborhood of 0.

Then:

The sequence $\frac{1}{N}\sum_{i=1}^{N}\delta_{Y^i}$, $N \in \mathbb{N}$ is tight and any limit point is supported on physical solutions $\overline{Y}$ with $\overline{Y}_0 \overset{d}{=} \nu$.

Technical point: Skorokhod M1 topology on rcll paths (key observation of **Delarue, Inglis, Rubenthaler, Tanré '15**).

Analysis of physical solutions: questions

- By the theorem, a **physical solution** $\overline{Y}$ with **rcll paths exists**.
- How do the **jumps** in $\overline{Y}$ arise? $\longleftrightarrow$ **leaps of the solid-liquid frontier**.
- E.g., what can one say about

$$t_\Delta := \inf\{t \geq 0 : \Delta \overline{Y}_t \neq 0\}$$

  and the **particle density** $\mathcal{L}(\overline{Y}_{t_\Delta -})$ **right before** t_Δ?
- **Structure** of **blow-ups**? **Uniqueness** of **physical solutions**?

Main theorem I: regular interval

Theorem (Nadtochiy, S. '17) Suppose $\overline{Y}_0 \overset{d}{=} \nu$ has a density $f_\nu \in W_2^1([0,\infty))$ and $f_\nu(0) = 0$.

Then: there exists $t_{reg} > 0$ such that on $[0, t_{reg})$ all physical solutions are indistinguishable and satisfy

$$\overline{Y}_t = \overline{Y}_0 + B_t + \int_0^t \lambda_s \,\mathrm{d}s, \quad t \in [0, \overline{\tau} \wedge t_{reg}),$$
$$\lambda_t = C\,\partial_t \log \mathbb{P}(\overline{\tau} > t), \quad t \in [0, t_{reg}).$$

Moreover, $t_{reg} = \inf\{t > 0 : \|\lambda\|_{L^2([0,t])} = \infty\}$.

Regular interval: some ideas from proof

- As long as $\dot{\Lambda}_t = \lambda_t \in L^2$, density $p(t, y)$ of $\overline{Y}_t \, \mathbf{1}_{\{\overline{\tau} > t\}}$ solves

$$\partial_t p = -\lambda_t \, \partial_y p + \frac{1}{2} \partial_y^2 p, \quad p(0, y) = f_\nu(y), \quad p(t, 0) = 0.$$

- More precisely: p coincides with $W_2^{1,2}([0, T] \times [0, \infty))$ solution.
- **Fixed-point constraint**:

$$\lambda_t = C \, \partial_t \log \mathbb{P}(\overline{\tau} > t) = C \, \frac{\partial_t \mathbb{P}(\overline{Y}_t > 0)}{\mathbb{P}(\overline{Y}_t > 0)} = C \, \frac{\partial_t \int_0^\infty p(t, y) \, \mathrm{d}y}{\int_0^\infty p(t, y) \, \mathrm{d}y}$$
$$= -\frac{C}{2} \, \frac{\partial_y p(t, 0)}{\int_0^\infty p(t, y) \, \mathrm{d}y}.$$

Regular interval: some ideas from proof, cont.

- **PDE fixed-point problem**: given $\lambda \in L^2([0, T])$, solve

$$\partial_t p = -\lambda_t \, \partial_y p + \frac{1}{2}\partial_y^2 p, \quad p(0, y) = f_\nu(y), \quad p(t, 0) = 0$$

  in $W_2^{1,2}([0, T] \times [0, \infty))$.

- **Want**:

$$-\frac{C}{2} \frac{\partial_y p(t, 0)}{\int_0^\infty p(t, y) \, \mathrm{d}y} = \lambda_t.$$

- Would be nice:

$$\lambda_t \mapsto -\frac{C}{2} \frac{\partial_y p(t, 0)}{\int_0^\infty p(t, y) \, \mathrm{d}y}$$

  is a **contraction** ($\Longrightarrow$ uniqueness of physical solution on $[0, T]$).

Regular interval: some ideas from proof, cont.

Turns out: **contraction property** holds for **truncated fixed-point problem**

$$\partial_t p = -\lambda_t^{M,T}\, \partial_y p + \frac{1}{2}\partial_y^2 p, \quad p(0,y) = f_\nu(y), \quad p(t,0) = 0,$$
$$-\frac{C}{2}\,\frac{\partial_y p(t,0)}{\int_0^\infty p(t,y)\,\mathrm{d}y} = \lambda_t,$$

with

$$\lambda^{M,T} = \lambda\, \mathbf{1}_{\{\|\lambda\|_{L^2([0,T])}\leq M\}} + \lambda\, \frac{M}{\|\lambda\|_{L^2([0,T])}}\, \mathbf{1}_{\{\|\lambda\|_{L^2([0,T])}>M\}},$$

when $T = T(M) > 0$ **small enough**.

Regular interval: some ideas from proof, cont.

- Given λ, $\widetilde{\lambda}$, get p, $\widetilde{p}$, need to control
  $|\partial_y p(t,0) - \partial_y \widetilde{p}(t,0)|$ and $\left| \int_0^\infty p(t,y)\,\mathrm{d}y - \int_0^\infty \widetilde{p}(t,y)\,\mathrm{d}y \right|$.
- Write PDE for $u := p - \widetilde{p}$

$$\partial_t u = \frac{1}{2}\partial_y^2 u - \widetilde{\lambda}^{M,T}\,\partial_y u + (\widetilde{\lambda}^{M,T} - \lambda^{M,T})\,\partial_y p,$$

$$u(0,y) = 0, \quad u(t,0) = 0.$$

- **Two step approach**: **a priori estimate** on $\partial_y u$, $\partial_y p$, then treat PDE as **heat equation with source** to get desired estimates.
- **Short time mixed-norm of heat kernel** small $\Longrightarrow$ **contraction**.

Main theorem II: description of jumps

Theorem (Nadtochiy, S. '17) Consider a physical solution $\overline{Y}$.

Then:

(a) the time of the first jump $t_\Delta := \inf\{t \geq 0 : \ \Delta \overline{Y}_t \neq 0\}$ is given by

$$t_\Delta = \inf \left\{ t \geq 0 : \ \exists\, \eta > 0 \ \text{ s.t. } \ \frac{\mathbb{P}(\overline{\tau} \geq t, \overline{Y}_{t-} \in (0,y))}{\mathbb{P}(\overline{\tau} \geq t)} \geq \frac{y}{C},\ y \in [0, \eta] \right\},$$

and

(b) the size of the jump at t_Δ is

$$\sup \left\{ \eta \geq 0 : \ \frac{\mathbb{P}(\overline{\tau} \geq t_\Delta, \overline{Y}_{t_\Delta -} \in (0,y))}{\mathbb{P}(\overline{\tau} \geq t_\Delta)} \geq \frac{y}{C},\ y \in [0, \eta] \right\}.$$

Description of jumps: some ideas from proof

- Given $t \geq 0$ and $\eta > 0$ such that

$$\frac{\mathbb{P}(\overline{\tau} \geq t,\, \overline{Y}_{t-} \in (0,y))}{\mathbb{P}(\overline{\tau} \geq t)} \geq \frac{y}{C}, \quad y \in [0,\eta],$$

  we claim: $\Delta Y_t \leq -\eta$.

- If not, easy to check:

$$\frac{\mathbb{P}(\overline{\tau} > t,\, \overline{Y}_{t} \in (0,y))}{\mathbb{P}(\overline{\tau} > t)} \geq \frac{y}{C}, \quad y \in [0,\widetilde{\eta}]$$

  for some $\widetilde{\eta} > 0$.

- Will use **hierarchical structure of cascades** to get a **contradiction**.

Description of jumps: some ideas from proof, cont.

- $t = 0$ (wlog). Then, for any $t_m \downarrow 0$:

$$
\begin{aligned}
C \log \mathbb{P}(\overline{\tau} > t_m) &= C \log \mathbb{P}\Big(\overline{Y}_0 + \inf_{s \le t_m} (B_s + \Lambda_s) > 0\Big) \\
&\le C \log \left(1 - \frac{1}{C} \int_0^{\tilde{\eta}} \mathbb{P}\Big(y + \inf_{s \le t_m} (B_s + \Lambda_s) \le 0\Big)\, \mathrm{d}y\right) \\
&\le - \int_0^{\tilde{\eta}} \mathbb{P}\Big(y + \inf_{t_{m+1} \le s \le t_m} B_s + \Lambda_{t_{m+1}} \le 0\Big)\, \mathrm{d}y \\
&\lesssim \mathbb{E}\Big[\inf_{t_{m+1} \le s \le t_m} (B_s - B_{t_m})\Big] + C \log \mathbb{P}(\overline{\tau} > t_{m+1}) \\
&= -\sqrt{\frac{2}{\pi}} \sqrt{t_m - t_{m+1}} + C \log \mathbb{P}(\overline{\tau} > t_{m+1}).
\end{aligned}
$$

- **Iterate** and choose $t_m = \frac{1}{m} \implies$ **contradiction**.

Regularity of physical solutions, uniqueness

- For **uniqueness**, need to understand **all** regimes.
- **Case 1:** $\overline{Y}_{t-}$ has a density $f \in C^1([0,\infty)) \cap C^\omega((0,\infty))$, $f(0) = 0$.

  $\Longrightarrow \dot{\Lambda} = \lambda$ is continuous on $[t, t+\varepsilon)$ for some $\varepsilon > 0$.
- **Case 2:** $\overline{Y}_{t-}$ has a density $f \in C^\omega((0,\infty))$, $f(0+) \in [0, 1/C)$.

  $\Longrightarrow \Lambda$ is $(1/2+\delta)$-Hölder on $[t, t+\varepsilon)$, back to **Case 1** on $(t, t+\varepsilon)$.
- **Case 3:** $\overline{Y}_{t-}$ has a density $f \in C^\omega((0,\infty))$, $f(0+) \geq 1/C$.

  $\Longrightarrow \Lambda_t - \Lambda_{t-} = -\inf\left\{y \geq 0 : \mathbb{P}(\overline{Y}_{t-} \in (0,y] < y/C)\right\}$,

  back to **Case 1** on $(t, t+\varepsilon)$.
- **Uniqueness** follows from this and sandwiching between two **maximal** physical solutions.

THANK YOU
FOR YOUR ATTENTION!