Transfer matrix approach to 1d random band matrices

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Local statistics, localization and delocalization

One of the key physical parameter of models is the localization length, which describes the typical length scale of the eigenvectors of random matrices. The system is called delocalized if the localization length ℓ is comparable with the matrix size, and it is called localized otherwise.

- Localized eigenvectors: lack of transport (insulators), and Poisson local spectral statistics (typically strong disorder)
- Delocalization: diffusion (electric conductors), and GUE/GOE local statistics (typically weak disorder).

The questions of the order of the localization length are closely related to the universality conjecture of the bulk local regime of the random matrix theory.

From the RMT point of view, the main objects of the local regime are k-point correlation functions R_k $(k = 1, 2, \ldots)$, which can be defined by the equalities:

$$\mathbb{E}\left\{\sum_{j_1\neq\ldots\neq j_k}\varphi_k(\lambda_{j_1}^{(N)},\ldots,\lambda_{j_k}^{(N)})\right\}$$
$$= \int_{\mathbb{R}^k}\varphi_k(\lambda_1^{(N)},\ldots,\lambda_k^{(N)})R_k(\lambda_1^{(N)},\ldots,\lambda_k^{(N)})d\lambda_1^{(N)}\ldots d\lambda_k^{(N)},$$

where $\varphi_k : \mathbb{R}^k \to \mathbb{C}$ is bounded, continuous and symmetric in its arguments.

Universality conjecture in the bulk of the spectrum (hermitian case, deloc.eg.s.) (Wigner – Dyson):

$$(N\rho(E))^{-k}R_k\big(\{E\ +\ \xi_j/N\rho(E)\}\big)\quad\xrightarrow{N\to\infty}\quad\det\left\{\frac{\sin\pi(\xi_i-\xi_j)}{\pi(\xi_i-\xi_j)}\right\}_{i,j=1}^{k}.$$

- Wigner matrices, β-ensembles with $\beta = 1, 2$, sample covariance matrices, etc.: delocalization, GUE/GOE local spectral statistics
- Anderson model (Random Schrödinger operators):

$$H_{RS} = -\triangle + V,$$

where $\triangle$ is the discrete Laplacian in lattice box $\Lambda = [1, n]^d \cap \mathbb{Z}^d$, V is a random potential (i.e. a diagonal matrix with i.i.d. entries).
In $d = 1$: narrow band matrix with i.i.d. diagonal

$$H_{RS} = \begin{pmatrix} V_1 & 1 & 0 & 0 & \dots & 0 \\ 1 & V_2 & 1 & 0 & \dots & 0 \\ 0 & 1 & V_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & V_{n-1} & 1 \\ 0 & \dots & 0 & 0 & 1 & V_n \end{pmatrix}.$$

Localization, Poisson local spectral statistics (Fröhlich, Spencer, Aizenman, Molchanov, ...)

Random band matrices

Can be defined in any dimension, but we will speak about $d = 1$.

Entries are independent (up to the symmetry) but not identically distributed.

$$H = \{H_{jk}\}_{j,k=1}^{N}, \quad H = H^*, \quad \mathbb{E}\{H_{jk}\} = 0.$$

Variance is given by some function J (even, compact support or rapid decay)

$$\mathbb{E}\{|H_{jk}|^2\} = W^{-1}\, J\big(|j - k|/W\big)$$

Main parameter: band width $W \in [1; N]$.

1d case

$$
H = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

$W = O(1)$ [$\sim$ random Schrödinger] $\longleftrightarrow$ $W = N$ [Wigner matrices]

We consider the following two models:

- Random band matrices: specific covariance

$$J_{ij} = \left(-W^2\Delta + 1\right)_{ij}^{-1} \approx C_1 W^{-1} \exp\{-C_2|i-j|/W\}$$

- Block band matrices

  Only 3 block diagonals are non zero.

$$H = \begin{pmatrix} A_1 & B_1 & 0 & 0 & 0 & \dots & 0 \\ B_1^* & A_2 & B_2 & 0 & 0 & \dots & 0 \\ 0 & B_2^* & A_3 & B_3 & 0 & \dots & 0 \\ . & . & B_3^* & . & . & . & . \\ . & . & . & . & . & A_{n-1} & B_{n-1} \\ 0 & . & . & . & 0 & B_{n-1}^* & A_n \end{pmatrix}$$

  A_j – independent $W \times W$ GUE-matrices with entry's variance $(1-2\alpha)/W, \quad \alpha < \frac{1}{4}$

  B_j -independent $W \times W$ Ginibre matrices with entry's variance α/W

Anderson transition in random band matrices

Varying W, we can see the transition:

Conjecture (in the bulk of the spectrum):

$d = 1:$	$\ell \sim W^2$	$W \gg \sqrt{N}$	Delocalization, GUE statistics
		$W \ll \sqrt{N}$	Localization, Poisson statistics

Partial results ($d = 1$):

- Schenker (2009): $\ell \leq W^8$ localization techniques; improved to W^7;
- Erdős, Yau, Yin (2011): $\ell \geq W$ – RM methods;
- Erdős, Knowles (2011): $\ell \gg W^{7/6}$ (in a weak sense);
- Erdős, Knowles, Yau, Yin (2012): $\ell \gg W^{5/4}$ (in a weak sense, not uniform in N);
- Bourgade, Erdős, Yau, Yin (2016): gap universality for $W \sim N$;
- Bourgade, Yau, Yin (2018): $W \gg N^{3/4}$ (quantum unique ergodicity);

Another method, which allows to work with random operators with non-trivial spatial structures, is supersymmetry techniques (SUSY), which based on the representation of the determinant as an integral over the Grassmann (anticommuting) variables.

The method allows to obtain an integral representation for the main spectral characteristic (such as density of states, second correlation functions, or the average of an elements of the resolvent) as the averages of certain observables in some SUSY statistical mechanics models (so-called dual representation in terms of SUSY). This is basically an algebraic step, and usually can be done by the standard algebraic manipulations. The real mathematical challenge is a rigour analysis of the obtained integral representation.

"Generalised" correlation functions

$$\mathcal{R}_1(z_1, z_1') := \mathbb{E}\Big\{ \frac{\det(H - z_1')}{\det(H - z_1)} \Big\}$$

$$\mathcal{R}_2(z_1, z_1'; z_2, z_2') := \mathbb{E}\Big\{ \frac{\det(H - z_1')\det(H - z_2'))}{\det(H - z_1)\det(H - z_2))} \Big\}$$

We study these functions for $z_{1,2} = E + \xi_{1,2}/\rho(E)N$, $z'_{1,2} = E + \xi'_{1,2}/\rho(E)N$, $E \in (-2, 2)$.

Link with the spectral correlation functions:

$$E\{\mathrm{Tr}(H - z_1)^{-1}\mathrm{Tr}(H - z_2)^{-1}\} = \frac{d^2}{dz_1' dz_2'} \mathcal{R}(z_1, z_1'; z_2, z_2')\Big|_{z_1' = z_1, z_2' = z_2}$$

Correlation function of the characteristic polynomials:

$$\mathcal{R}_0(\lambda_1, \lambda_2) = \mathbb{E}\Big\{ \det(H - \lambda_1)\det(H - \lambda_2) \Big\}, \quad \lambda_{1,2} = E \pm \xi/\rho(E)N.$$

Integral representation for characteristic polynomials

$$\mathcal{R}_0(\lambda_1,\lambda_2) = \mathrm{C_N} \int\limits_{\mathcal{H}_2^{\mathrm{N}}} \exp\Big\{ -\frac{1}{2}\sum_{\mathrm{j,k}} \mathrm{J}^{-1}_{\mathrm{jk}} \mathrm{Tr}\, \mathrm{X_j X_k} \Big\} \prod_{\mathrm{j}} \det\big(\mathrm{X_j} - \mathrm{i}\Lambda/2\big) d\overline{\mathrm{X}},$$

where $\{\mathrm{X_j}\}$ are hermitian 2×2 matrices, $\Lambda = \mathrm{diag}\{\lambda_1,\lambda_2\}$,and $\hat{\xi} = \mathrm{diag}\{\xi,-\xi\}$.

For the density of states or the second correlation function $\mathrm{X_j}$ will be super-matrices

$$\mathrm{X}_{\mathrm{j}}^{1} = \begin{pmatrix} \mathrm{a_j} & \rho_{\mathrm{j}} \\ \tau_{\mathrm{j}} & \mathrm{b_j} \end{pmatrix}, \quad \mathrm{X}_{\mathrm{j}}^{2} = \begin{pmatrix} \mathrm{A_j} & \bar{\rho}_{\mathrm{j}} \\ \bar{\tau}_{\mathrm{j}} & \mathrm{B_j} \end{pmatrix}$$

with real variables $\mathrm{a_j}$, $\mathrm{b_j}$ and Grassmann variables ρ_{j}, τ_{j}, or hermitian $\mathrm{A_j}$, hyperbolic $\mathrm{B_j}$ and Grassmann 2×2 matrices $\bar{\rho}_{\mathrm{j}}$, $\bar{\tau}_{\mathrm{j}}$.

The formulas can be obtain in any dimension and for any J, although the specific $J = \left(-W^2\Delta + 1\right)^{-1}$ gives a nearest neighbour model. In particular, it becomes accessible for transfer matrix approach.

For the specific covariance $(-W^2\triangle + 1)^{-1}$:

$$\mathcal{R}_0(\lambda_1, \lambda_2) = C_N \int\limits_{\mathcal{H}_2^N} \exp\Big\{ -\frac{W^2}{2} \sum_{j=2}^{N} \mathrm{Tr}\,(X_j - X_{j-1})^2 \Big\} \times$$

$$\exp\Big\{ -\frac{1}{2} \sum_{j=1}^{N} \mathrm{Tr}\,\Big(X_j + \frac{iE \cdot I}{2} + \frac{i\hat{\xi}}{2N\rho(\lambda_0)}\Big)^2 \Big\} \prod_{j=1}^{N} \det\,(X_j - iE \cdot I/2) d\overline{X},$$

The idea of the transfer operator approach is very simple and natural. Let $\mathcal{K}(X, Y)$ be the matrix kernel of the compact integral operator in $\oplus_{i=1}^{p} L_2[X, d\mu(X)]$. Then

$$\int g(X_1)\mathcal{K}(X_1, X_2)\dots\mathcal{K}(X_{n-1}, X_n)f(X_n)\prod d\mu(X_i) = (\mathcal{K}^{n-1}f, \bar{g})$$

$$= \sum_{j=0}^{\infty} \lambda_j^{n-1}(\mathcal{K})c_j, \quad \text{with} \quad c_j = (f, \psi_j)(g, \tilde{\psi}_j).$$

Here $\{\lambda_j(\mathcal{K})\}_{j=0}^{\infty}$ are the eigenvalues of $\mathcal{K}$ ($|\lambda_0| \geq |\lambda_1| \geq \dots$), ψ_j are corresponding eigenvectors, and $\tilde{\psi}_j$ are the eigenvectors of $\mathcal{K}^*$. Hence, to study the correlation function, it suffices to study the integral operator with a kernel $\mathcal{K}(X, Y)$.

For characteristic polynomials with $J = (-W^2\Delta + 1)^{-1}$:

$$\mathcal{K}_\xi(X, Y) = \frac{W^4}{2\pi^2} \mathcal{F}_\xi(X) \exp\Big\{ -\frac{W^2}{2} \mathrm{Tr}\,(X - Y)^2 \Big\} \mathcal{F}_\xi(Y),$$

where $\mathcal{F}_\xi(X)$ is the operator of multiplication by

$$\mathcal{F}_\xi(X) = \mathcal{F}(X) \cdot \exp\Big\{ -\frac{i}{2n\rho(E)} \mathrm{Tr}\, X\hat{\xi} \Big\}$$

with

$$\mathcal{F}(X) = \exp\Big\{ -\frac{1}{4} \mathrm{Tr}\, \Big(X + \frac{i\Lambda_0}{2}\Big)^2 + \frac{1}{2} \mathrm{Tr}\, \log\big(X - i\Lambda_0/2\big) - C_+ \Big\}$$

and some specific C_+

Saddle-points: $X_j = \pi\rho(E) \cdot U_j^* L U_j$, $\hat{A}_j = \mathrm{diag}\{1, -1\}$, $X_j = \pm\pi\rho(E) \cdot I_2$

The main difficulties:

1. the transfer operator is not self-adjoint, and thus the perturbation theory is not easily applied in a rigorous way;
2. the transfer operator has a complicated structure including a part that acts on unitary and hyperbolic groups, hence we need to work with corresponding special functions;
3. the kernel of the transfer operator for the density of states and for the second correlation function contains not only only the complex, but also some Grassmann variables. Therefore, for the density of states $\mathcal{K}_1$ is a 2×2 matrix kernel, containing the Jordan cell, and for the second correlation function $\mathcal{K}_2$ is a $2^8 \times 2^8$ matrix kernel, containing 4×4 Jordan cell in the main block.
   Using the symmetry of the problem, $\mathcal{K}_2$ could be replaced by 70×70 matrix kernel, but it is still very complicated.

Results for the characteristic polynomials:

Let $D_2 = \mathcal{R}_0(E, E)$, $\bar{\mathcal{R}}_0(E, \xi) = D_2^{-1} \cdot \mathcal{R}_0\Big(E + \hat{\xi}/2N\rho(E)\Big)$.

$$\lim_{n\to\infty} \bar{\mathcal{R}}_0(E, \xi) = \begin{cases} \dfrac{\sin \pi\xi}{\pi\xi}, & W \geq N^{1/2+\theta}; \\ (e^{-C_* t_* \Delta_U - i\xi\hat{\nu}} \cdot 1, 1), & N = C_* W^2 \\ 1, & 1 \ll W \leq \sqrt{\dfrac{N}{C_* \log N}}, \end{cases}$$

where $t_* = (2\pi\rho(E))^2$,

$$\Delta_U = -\frac{d}{dx} x(1-x) \frac{d}{dx}, \quad \nu(U) = \pi(1-2x), \quad x = |U_{12}|^2.$$

Delocalization part: S., 2013 – saddle-point analysis; (the case of orthogonal symmetry is also done, S., 2015)

Localization part: M. Shcherbina, S., 2016 – transfer matrix approach.

Near the crossover: S., 2018

SUSY results for the density of states:

Let $g(z) = N^{-1}\mathbb{E}\{\mathrm{Tr}\,(H-z)^{-1}\}$, g_{sc} is a Stieltjes transform of semi-circle.

- Disertori, Pinson, Spencer, 2002: The smoothness and the local semicircle for averaged density for RBM in 3d, i.e.

  $$|g(z) - g_{sc}(z)| \le C/W^2$$

  uniformly in $\mathrm{Im}\, z$, $W \ge W_0$.
- Disertori, Lager, 2016: the same in 2d.
- M. Shcherbina, S., 2016: local semicircle for averaged density for RBM in 1d (with an arrow W^{-1}).

First and second results use the cluster expansion, the third one uses the supersymmetric transfer matrices.

Sigma-model $\mathcal{R}_2^{(\sigma)}$

The model can be obtained by some scaling limit ($\alpha = \beta/\mathrm{W}$, $\mathrm{W} \to \infty$, β, n-fixed) from the expression for $\mathcal{R}_2$.

The crossover is expected for $\beta \sim \mathrm{n}$. First result is a rigorous derivation of sigma-model approximation:

$$\mathcal{R}_2^{(\sigma)} = \int \exp\left\{\frac{\beta}{4}\sum \mathrm{Str}\, \mathrm{Q_j Q_{j+1}} + \frac{\varepsilon + \mathrm{i}\xi}{4\mathrm{n}} \sum \mathrm{Str}\, \mathrm{Q_j}\Lambda\right\} \prod \mathrm{dQ_j}$$

Here $\mathrm{Q_j}$ is a 4×4 super matrix of the block form:

$$\mathrm{Q_j} = \begin{pmatrix} \mathrm{U_j^*} & 0 \\ 0 & \mathrm{S_j^{-1}} \end{pmatrix} \begin{pmatrix} (\mathrm{I} + 2\hat{\rho}_\mathrm{j}\hat{\tau}_\mathrm{j})\mathrm{L} & 2\hat{\tau}_\mathrm{j} \\ 2\hat{\rho}_\mathrm{j} & -(\mathrm{I} - 2\hat{\rho}_\mathrm{j}\hat{\tau}_\mathrm{j})\mathrm{L} \end{pmatrix} \begin{pmatrix} \mathrm{U_j} & 0 \\ 0 & \mathrm{S_j} \end{pmatrix},$$

$$\mathrm{dQ} = \prod \mathrm{dQ_j}, \quad \mathrm{dQ_j} = (1 - 2\rho_{\mathrm{j}1}\tau_{\mathrm{j}1}\rho_{\mathrm{j}2}\tau_{\mathrm{j}2})\, d\rho_{\mathrm{j}1} d\tau_{\mathrm{j}1}\, d\rho_{\mathrm{j}2} d\tau_{\mathrm{j}2}\, \mathrm{dU_j}\, \mathrm{dS_j}$$

with

$$\hat{\rho}_\mathrm{j} = \mathrm{diag}\{\rho_{\mathrm{j}1}, \rho_{\mathrm{j}2}\}, \quad \hat{\tau}_\mathrm{j} = \mathrm{diag}\{\tau_{\mathrm{j}1}, \rho_{\mathrm{j}2}\}, \quad \mathrm{L} = \mathrm{diag}\{1, -1\}.$$

Here $\{\mathrm{U_j}\}$ are unitary matrices, $\{\mathrm{S_j}\}$ are hyperbolic matrices, $\mathrm{Q_j^2} = \mathrm{I}$.

Result for $\mathcal{R}_2^{(\sigma)}$ [M. Shcherbina, S., 2018]

In the dimension $\mathrm{d} = 1$ the behavior of the sigma-model approximation $\mathcal{R}_2^{(\sigma)}$ of the second order correlation function, as $\beta \gg \mathrm{n}$, in the bulk of the spectrum coincides with those for the GUE. More precisely, if $\Lambda = [1, \mathrm{n}] \cap \mathbb{Z}$ and $\mathrm{H_N}$, $\mathrm{N} = \mathrm{Wn}$ are block RBM with $\mathrm{J} = 1/\mathrm{W} + \beta\Delta/\mathrm{W}^2$, then for any $|\mathrm{E}| < \sqrt{2}$

$$(\mathrm{N}\rho(\mathrm{E}))^{-2}\mathcal{R}_2\left(\mathrm{E} + \frac{\xi_1}{\rho(\mathrm{E})\,\mathrm{N}}, \mathrm{E} + \frac{\xi_2}{\rho(\mathrm{E})\,\mathrm{N}}\right) \longrightarrow 1 - \frac{\sin^2(\pi(\xi_1 - \xi_2))}{\pi^2(\xi_1 - \xi_2)^2},$$

in the limit first $\mathrm{W} \to \infty$, and then $\beta, \mathrm{n} \to \infty$, $\beta \geq \mathrm{Cn}\log^2 \mathrm{n}$.

"Right" limit: $\beta = \alpha \mathrm{W}$, α is fixed, $\mathrm{W}, \mathrm{n} \to \infty$, $\mathrm{W} \gg \mathrm{n}$.

Resolvent version of the transfer operator approach

$$(\mathcal{K}^{n-1}f, \bar{g}) = -\frac{1}{2\pi i}\oint_L z^{n-1}(\mathcal{G}(z)f, \bar{g})dz, \quad \mathcal{G}(z) = (\mathcal{K} - z)^{-1}$$

where L is any closed contour which contains all eigenvalues of $\mathcal{K}$.

Set

$$\lambda_* = \lambda_0(\mathcal{K}), \quad (\lambda_* \sim 1),$$

then it suffices to choose L as $L_0 = \{z : |z| = |\lambda_*|(1 + O(n^{-1}))\}$.
We choose $L = L_1 \cup L_2$ where $L_2 = \{z : |z| = |\lambda_*|(1 - \log^2 n/n)\}$, and L_1 is some special contour, containing all eigenvalues between L_0 and L_2. Then

$$(\mathcal{K}^{n-1}f, \bar{g}) = -\frac{1}{2\pi i}\oint_{L_1} z^{n-1}(\mathcal{G}(z)f, \bar{g})dz \\ - \frac{1}{2\pi i}\oint_{|z|=|\lambda_*|(1-\log^2 n/n)} z^{n-1}(\mathcal{G}(z)f, \bar{g})dz$$

The second integral is small comparing with $|\lambda_*|^{n-1}$, since

$$|z|^{n-1} \le |\lambda_*|^{n-1} \cdot e^{-\log^2 n}$$

Definition of asymptotically equivalent operators $(n, W \to \infty)$

$$\mathcal{A} \sim \mathcal{B} \quad \Leftrightarrow \oint_{L_1} z^{n-1}((\mathcal{A}-z)^{-1}f, \bar{g})dz = \oint_{L_1} z^{n-1}((\mathcal{B}-z)^{-1}f, \bar{g})dz + o(1)$$

for certain L_1

Mechanism of the crossover for $\mathcal{R}_0$

Key technical step

$$\mathcal{K}_\xi \sim \mathcal{K}_{*\xi} \otimes \mathcal{A},$$

$$\mathcal{K}_{*\xi}(\mathrm{U}_1, \mathrm{U}_2) = e^{-i\xi\nu(\mathrm{U}_1)/N} \mathrm{K}_{*0}(\mathrm{U}_1 \mathrm{U}_2^*) e^{-i\xi\nu(\mathrm{U}_2)/N}, \quad \mathcal{K}_{*0} : \mathrm{L}_2(\mathring{\mathrm{U}}(2)) \to \mathrm{L}_2(\mathring{\mathrm{U}}(2)),$$

$$\mathcal{A}(x_1, x_2, y_1, y_2) = \mathrm{A}_1(x_1, x_2)\mathrm{A}_2(y_1, y_2), \qquad \mathrm{L}_2(\mathbb{R}^2) \to \mathrm{L}_2(\mathbb{R}^2).$$

Here $\xi_1 = -\xi_2 = \xi$, and $\nu(\mathrm{U}) = \pi(1 - 2|\mathrm{U}_{12}|^2)$

Then

$$\mathcal{R}_0 = (\mathcal{K}_{*\xi}^N \otimes \mathcal{A}^N f, \bar{g})(1 + o(1)) = (\mathcal{K}_{*\xi}^N \cdot 1, 1)(\mathcal{A}^N f_1, \bar{g}_1)(1 + o(1)).$$

Here we used that both f, g asymptotically can be replaced by $1 \otimes f_1(x, y)$.
After normalization we get:

$$\mathrm{D}_2^{-1}\mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{(\mathcal{K}_{*\xi}^N \cdot 1, 1)}{(\mathcal{K}_{*0}^N \cdot 1, 1)}(1 + o(1))$$

Spectral analysis of $\mathcal{K}_{*\xi}$

A good news is that $\mathcal{K}_{*0}$ with a kernel

$$\mathcal{K}_{*0} = \mathrm{t}_* \mathrm{W}^2 \mathrm{e}^{-\mathrm{t}_* \mathrm{W}^2 |(\mathrm{U}_1 \mathrm{U}_2^*)_{12}|^2}$$

is a self-adjoint "difference" operator. It is known that his eigenfunctions are Legendre polynomials P_j. Moreover, it is easy to check that corresponding eigenvalues have the form:

$$\lambda_\mathrm{j} = 1 - \mathrm{t}_* \mathrm{j}(\mathrm{j}+1)/\mathrm{W}^2 + \mathrm{O}((\mathrm{j}(\mathrm{j}+1)/\mathrm{W}^2)^2), \quad \mathrm{j} = 0, 1 \ldots.$$

Besides,

$$\mathcal{K}_{*\xi} = \mathcal{K}_{*0} - 2\mathrm{i}\xi\hat{\nu}/\mathrm{N} + \mathrm{O}(\mathrm{N}^{-2})$$

where $\hat{\nu}$ is the operator of multiplication by ν. Thus the eigenvalues of $\mathcal{K}_{*\xi}$ are in the N^{-1}-neighbourhood of λ_j.

Mechanism of the Poisson behavior for $W^2 \ll N$

For $W^{-2} \gg N^{-1}$ (the spectral gap is much less then the perturbation norm)

$$\lambda_0(\mathcal{K}_{*\xi}) = 1 - 2N^{-1} i\xi(\nu \cdot 1, 1) + o(N^{-1}),$$
$$|\lambda_1(\mathcal{K}_{*\xi})| \le 1 - O(W^{-2}) \quad \Rightarrow \quad |\lambda_j(\mathcal{K}_{*\xi})|^N \to 0, \ (j = 1, 2, \dots).$$

Since

$$(\nu \cdot 1, 1) = 0,$$

we obtain that

$$\lambda_0(\mathcal{K}_{*\xi}) = 1 + o(N^{-1}),$$

and

$$D_2^{-1}\mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{\lambda_0^N(\mathcal{K}_{*\xi})}{\lambda_0^N(\mathcal{K}_{*0})}(1 + o(1)) \to 1$$

The relation corresponds to the Poisson local statistics.

Mechanism of the GUE behavior for $W^2 \gg N$

In the regime $W^{-2} \ll N^{-1}$ we have $\mathcal{K}_{*0}^{N} \to I$ in the strong vector topology, hence one can prove that

$$\mathcal{K}_{*\xi} \sim 1 + O(W^{-2}) - N^{-1} 2i\xi\nu \Rightarrow (\mathcal{K}_{*\xi}^{N} \cdot 1, 1) \to (e^{-2i\xi\hat{\nu}} \cdot 1, 1)$$

and

$$D_2^{-1}\mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{(e^{-2i\xi t^*\hat{\nu}} \cdot 1, 1)}{(1,1)}(1 + o(1)) \to \frac{\sin(2\pi\xi)}{2\pi\xi}.$$

The expression for $D_2^{-1}\mathcal{R}_0$ coincides with that for GUE.

In the regime $W^{-2} = C_* N^{-1}$ observe that $\mathcal{K}_{*\xi}$ is reduced by the subspace $\mathcal{E}_0$ of the functions depending only on $|U_{12}|^2$.
Recall also that the Laplace operator on $\mathring{U}(2)$ is reduced by $\mathcal{E}_0$ and have the form

$$\Delta_U = -\frac{d}{dx} x(1-x) \frac{d}{dx}, \quad x = |U_{12}|^2.$$

Besides, the eigenvectors of Δ_U and $\mathcal{K}_{*0}$ coincide (they are Legendre's polynomials P_j) and corresponding eigenvalues of Δ_U are

$$\lambda_j^* = j(j+1).$$

Hence we can write $\mathcal{K}_{*\xi}$ as

$$\mathcal{K}_{*\xi} \sim 1 - N^{-1}(C_* t_* \Delta_U + 2i\xi\nu) + o(N^{-1}) \Rightarrow (\mathcal{K}_{*\xi}^N \cdot 1, 1) \to (e^{-C\Delta_U - 2i\xi\hat{\nu}} \cdot 1, 1)$$

Other SUSY results for the full model:

- S., 2014: Gaussian case, three diagonal block band matrices with $J = \frac{\alpha}{W}\triangle + \frac{1}{W}$. If $W \sim N$, then

$$\frac{1}{(N\rho(\lambda_0))^2}\, R_2\big(\lambda_0 + x/N\rho(\lambda_0), \lambda_0 + y/N\rho(\lambda_0)\big) \overset{N\to\infty}{\longrightarrow} 1 - \frac{\sin^2(\pi(x-y))}{\pi^2(x-y)^2}$$

in any dimension.

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  in any dimension.

- Erdős, Bao, 2015: Combining this techniques with Green's function comparison strategy (Erdős-Yau), they proved

$$\ell \geq W^{7/6}$$

  in a strong sense for the block band matrices with more or less general element's distribution (subexponential tails, four Gaussian moments).