Determinantal point processes and spaces of holomorphic functions

Yanqi Qiu

AMSS, Chinese Academy of Sciences; CNRS

2019 Apr 09

Part I. Lyons-Peres completeness conjecture

joint with Alexander Bufetov and Alexander Shamov

Sets related to Bergman spaces

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk,
$\mathrm{Hol}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} | f \text{ holomorphic}\}$.

- Bergman space: $A^p(\mathbb{D}) := L^p(\mathbb{D}, \mathrm{Leb}) \cap \mathrm{Hol}(\mathbb{D})$.

Sets related to Bergman spaces

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk,
$\mathrm{Hol}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} | f \text{ holomorphic}\}$.

- Bergman space: $A^p(\mathbb{D}) := L^p(\mathbb{D}, \mathrm{Leb}) \cap \mathrm{Hol}(\mathbb{D})$.
- A set $X \subset \mathbb{D}$ is called $A^p(\mathbb{D})$-uniqueness set if ANY $f \in A^p(\mathbb{D})$ is uniquely determined by its restriction $f \restriction_X$

Sets related to Bergman spaces

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk,
$\mathrm{Hol}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} | f \text{ holomorphic}\}$.

- Bergman space: $A^p(\mathbb{D}) := L^p(\mathbb{D}, \mathrm{Leb}) \cap \mathrm{Hol}(\mathbb{D})$.
- A set $X \subset \mathbb{D}$ is called $A^p(\mathbb{D})$-uniqueness set if ANY $f \in A^p(\mathbb{D})$ is uniquely determined by its restriction $f \restriction_X$ ($f \in A^p(\mathbb{D})$ and $f \restriction_X = 0$ implies $f \equiv 0$).

Sets related to Bergman spaces

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk,
$\mathrm{Hol}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} | f \text{ holomorphic}\}$.

- Bergman space: $A^p(\mathbb{D}) := L^p(\mathbb{D}, \mathrm{Leb}) \cap \mathrm{Hol}(\mathbb{D})$.
- A set $X \subset \mathbb{D}$ is called $A^p(\mathbb{D})$-uniqueness set if ANY $f \in A^p(\mathbb{D})$ is uniquely determined by its restriction $f \restriction_X$ ($f \in A^p(\mathbb{D})$ and $f \restriction_X = 0$ implies $f \equiv 0$).
- A set $Y \subset \mathbb{D}$ is called an $A^p(\mathbb{D})$-zero set if $\exists f \in A^p(\mathbb{D}) \setminus \{0\}$ such that $Z(f) = Y$.

Sets related to Bergman spaces

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk,
$\mathrm{Hol}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} | f \text{ holomorphic}\}$.

- Bergman space: $A^p(\mathbb{D}) := L^p(\mathbb{D}, \mathrm{Leb}) \cap \mathrm{Hol}(\mathbb{D})$.
- A set $X \subset \mathbb{D}$ is called $A^p(\mathbb{D})$-uniqueness set if ANY $f \in A^p(\mathbb{D})$ is uniquely determined by its restriction $f \restriction_X$ ($f \in A^p(\mathbb{D})$ and $f \restriction_X = 0$ implies $f \equiv 0$).
- A set $Y \subset \mathbb{D}$ is called an $A^p(\mathbb{D})$-zero set if $\exists f \in A^p(\mathbb{D}) \setminus \{0\}$ such that $Z(f) = Y$.

  Remark: $A^p(\mathbb{D})$-uniqueness set $\Longleftrightarrow$ non-$A^p(\mathbb{D})$-zero set.

Gaussian Analytic Function

Consider the *random series*

$$\mathfrak{F}_{\mathbb{D}}(z) = \sum_{n=0}^{\infty} g_n z^n,$$

g_n are i.i.d. complex Gaussian random variables with expectation 0, variance 1.

Gaussian Analytic Function

Consider the *random series*

$$\mathfrak{F}_{\mathbb{D}}(z) = \sum_{n=0}^{\infty} g_n z^n,$$

g_n are i.i.d. complex Gaussian random variables with expectation 0, variance 1.

Elementary fact:

- Almost surely, $\mathfrak{F}_{\mathbb{D}}(z)$ has radius of convergence 1 and defines a holomorphic function on $\mathbb{D}$.

Gaussian Analytic Function

Consider the *random series*

$$\mathfrak{F}_{\mathbb{D}}(z) = \sum_{n=0}^{\infty} g_n z^n,$$

g_n are i.i.d. complex Gaussian random variables with expectation 0, variance 1.

Elementary fact:

- Almost surely, $\mathfrak{F}_{\mathbb{D}}(z)$ has radius of convergence 1 and defines a holomorphic function on $\mathbb{D}$.

We want to study the random subset of $\mathbb{D}$ by

$$Z(\mathfrak{F}_{\mathbb{D}}) := \{z \in \mathbb{D} : \mathfrak{F}_{\mathbb{D}}(z) = 0\}.$$

Gaussian Analytic Function

Consider the *random series*

$$\mathfrak{F}_{\mathbb{D}}(z) = \sum_{n=0}^{\infty} g_n z^n,$$

g_n are i.i.d. complex Gaussian random variables with expectation 0, variance 1.

Elementary fact:

- Almost surely, $\mathfrak{F}_{\mathbb{D}}(z)$ has radius of convergence 1 and defines a holomorphic function on $\mathbb{D}$.

We want to study the random subset of $\mathbb{D}$ by

$$Z(\mathfrak{F}_{\mathbb{D}}) := \{z \in \mathbb{D} : \mathfrak{F}_{\mathbb{D}}(z) = 0\}.$$

Conjecture (**Lyons-Peres conjecture: particular case)**

Almost surely, $Z(\mathfrak{F}_{\mathbb{D}})$ is an $A^2(\mathbb{D})$-uniqueness set.

$Z(\mathfrak{F}_{\mathbb{D}})$ is an $A^2(\mathbb{D})$-uniqueness set

Theorem (Bufetov - Q.- Shamov)

Almost surely, $Z(\mathfrak{F}_{\mathbb{D}})$ is an $A^2(\mathbb{D})$-uniqueness set.

How we solve this conjecture?

I: determinantal structure.
$A^2(\mathbb{D})$ is a reproducing kernel Hilbert space with reproducing kernel

$$K_{\mathbb{D}}(z,w) = \frac{1}{\pi(1 - z\bar{w})^2}.$$

How we solve this conjecture?

I: determinantal structure.
$A^2(\mathbb{D})$ is a reproducing kernel Hilbert space with reproducing kernel

$$K_{\mathbb{D}}(z,w) = \frac{1}{\pi(1 - z\bar{w})^2}.$$

Theorem (Peres-Virág, 2005)

The random subset $Z(\mathfrak{F}_{\mathbb{D}})$ is a realization of the determinantal point process on $\mathbb{D}$ with correlation kernel given by the Bergman kernel $K_{\mathbb{D}}$.

How this conjecture is solved?

II: resolution of a general conjecture.
A set $\mathscr{X} \subset E$ is called the uniqueness set for a reproducing kernel Hilbert space $\mathscr{H} \subset L^2(E, \mu)$ if any $f \in \mathscr{H}$ vanishing on $\mathscr{X}$ is identically zero.

Theorem (Bufetov - Q.- Shamov, **Lyons-Peres completeness conjecture**)

If a random set $\mathscr{X}$ is a determinantal point process induced by the kernel for a reproducing kernel Hilbert space $\mathscr{H}$, then almost surely, $\mathscr{X}$ is a **uniqueness set** *for $\mathscr{H}$.*

Key Ingredient: conditional measure of DPP

Theorem (Bufetov-Q.-Shamov)

Given any DPP $\mathscr{X}$ on any metric complete separable space E, with self-adjoint kernel and any subset $W \subset E$, the conditional measure

$$\mathcal{L}\left(\mathscr{X}|_W \middle| \mathscr{X}|_{W^c}\right)$$

describes again a new DPP on W. Moreover, the kernel is computed explicitly.

Example of conditional measures of DPP

The Fock projection

$$L^2(\mathbb{C}, e^{-|z|^2} dV(z)) \to L^2_{hol}(\mathbb{C}, e^{-|z|^2} dV(z))$$

induces the DPP process $\mathscr{X} \subset \mathbb{C}$ is the famous Ginibre point process.

Theorem (Bufetov-Q.)

For Ginibre process $\mathscr{X}$, if W is bounded, then the conditional measure

$$\mathcal{L}\left(\mathscr{X}|_W \middle| \mathscr{X}|_{W^c}\right)$$

is an orthogonal polynomial ensemble.

Application of conditional measures of DPP

Let $U \subset \mathbb{C}^d$ be a connected domain.

$$H^\infty(U) := \Big\{\text{bounded hol. functions on } U\Big\}.$$

Theorem (Bufetov-Shilei Fan-Q.)

Suppose that $H^\infty(U)$ contains a non-constant element. Then for the DPP $\mathscr{X} \subset U$ induced by the Bergman projection, if $W \subset U$ is relatively compact, then the conditional measure

$$\mathcal{L}\left(\mathscr{X}|_W \,\Big|\, \mathscr{X}|_{W^c}\right)$$

is measure equivalent to a Poisson point process on U.

Part II. Patterson-Sullivan construction

joint with Alexander Bufetov

Reconstruction problems

Recall almost surely, $Z(\mathfrak{F}_{\mathbb{D}})$ is an $A^2(\mathbb{D})$-uniqueness set.

Reconstruction problems

Recall almost surely, $Z(\mathfrak{F}_{\mathbb{D}})$ is an $A^2(\mathbb{D})$-uniqueness set. That is, fix generic realization (in probability sense) $X = Z(\mathfrak{F}_{\mathbb{D}})$. Then any $f \in A^2(\mathbb{D})$ is uniquely determined by its restriction onto X.

Reconstruction problems

Recall almost surely, $Z(\mathfrak{F}_{\mathbb{D}})$ is an $A^2(\mathbb{D})$-uniqueness set. That is, fix generic realization (in probability sense) $X = Z(\mathfrak{F}_{\mathbb{D}})$. Then any $f \in A^2(\mathbb{D})$ is uniquely determined by its restriction onto X.
Problems:

- How to recover simulataneously and explicitly all functions $f \in A^2(\mathbb{D})$ from its restriction onto a fixed generic realization of $Z(\mathfrak{F}_{\mathbb{D}})$?

Reconstruction problems

Recall almost surely, $Z(\mathfrak{F}_{\mathbb{D}})$ is an $A^2(\mathbb{D})$-uniqueness set. That is, fix generic realization (in probability sense) $X = Z(\mathfrak{F}_{\mathbb{D}})$. Then any $f \in A^2(\mathbb{D})$ is uniquely determined by its restriction onto X.
Problems:

- How to recover simulataneously and explicitly all functions $f \in A^2(\mathbb{D})$ from its restriction onto a fixed generic realization of $Z(\mathfrak{F}_{\mathbb{D}})$?
- How about general random countable subset of $\mathbb{D}$ without accumulation points?
- How about more general Banach space $\mathcal{B}$ of holomorphic or harmonic functions on $\mathbb{D}$?

The reconstruction for a fixed $f \in A^2(\mathbb{D})$ and $z \in \mathbb{D}$

The **Poincaré-Lobachevsky hyperbolic metric** on $\mathbb{D}$ is given by

$$d_{\mathbb{D}}(x, z) := \log \frac{1 + \left| \dfrac{z - x}{1 - \bar{x} z} \right|}{1 - \left| \dfrac{z - x}{1 - \bar{x} z} \right|} \qquad \text{for } x, z \in \mathbb{D}.$$

The reconstruction for a fixed $f \in A^2(\mathbb{D})$ and $z \in \mathbb{D}$

The **Poincaré-Lobachevsky hyperbolic metric** on $\mathbb{D}$ is given by

$$d_{\mathbb{D}}(x, z) := \log \frac{1 + \left|\dfrac{z - x}{1 - \bar{x}z}\right|}{1 - \left|\dfrac{z - x}{1 - \bar{x}z}\right|} \quad \text{for } x, z \in \mathbb{D}.$$

Let $\mu_{\mathbb{D}}$ be the **hyperbolic area** (up to a multiplicative constant)

$$d\mu_{\mathbb{D}} = \frac{dLeb}{(1 - |x|^2)^2}.$$

The reconstruction for a fixed $f \in A^2(\mathbb{D})$ and $z \in \mathbb{D}$

For any $s \in \mathbb{R}$, set $W_s(x) := e^{-s d_{\mathbb{D}}(x,0)}$ (which is **radial**). Then

$$W_s^z(x) := W_s\left(\frac{z-x}{1-\bar{x}z}\right) = e^{-s d_{\mathbb{D}}(x,z)}.$$

The reconstruction for a fixed $f \in A^2(\mathbb{D})$ and $z \in \mathbb{D}$

For any $s \in \mathbb{R}$, set $W_s(x) := e^{-s d_{\mathbb{D}}(x,0)}$ (which is **radial**). Then

$$W_s^z(x) := W_s\left(\frac{z-x}{1-\bar{x}z}\right) = e^{-s d_{\mathbb{D}}(x,z)}.$$

Proposition (Alexander I. Bufetov- Q.)

Fix $f \in A^2(\mathbb{D})$, $z \in \mathbb{D}$, *then* $\exists C > 0$ *such that* $\forall s > 1$, *we have*

$$\mathbb{E}\left(\left|\frac{\sum_{k=0}^{\infty} \sum_{\substack{x \in Z(\mathfrak{F}_{\mathbb{D}}) \\ k \le d_{\mathbb{D}}(z,x) < k+1}} W_s^z(x) f(x)}{\mathbb{E} \sum_{x \in Z(\mathfrak{F}_{\mathbb{D}})} W_s^z(x)} - f(z)\right|^2\right) \le C \cdot (s-1)^2.$$

The reconstruction for a fixed $f \in A^2(\mathbb{D})$ and $z \in \mathbb{D}$

Corollary

Fix $f \in A^2(\mathbb{D})$, $z \in \mathbb{D}$, $(s_n)_{n\geq 1}$ *with* $\sum_{n=1}^{\infty}(s_n - 1)^2 < \infty$ *and* $s_n > 1$. *Then for almost every realization* $X = Z(\mathfrak{F}_{\mathbb{D}})$, *we have*

$$f(z) = \lim_{n\to\infty} \frac{\displaystyle\sum_{k=0}^{\infty} \sum_{\substack{x\in X \\ k\leq d_{\mathbb{D}}(z,x)<k+1}} W_{s_n}^z(x) f(x)}{\displaystyle\sum_{x\in X} W_{s_n}^z(x)}.$$

The reconstruction for a fixed $f \in A^2(\mathbb{D})$ and $z \in \mathbb{D}$

Corollary

Fix $f \in A^2(\mathbb{D})$, $z \in \mathbb{D}$, $(s_n)_{n\geq 1}$ *with* $\sum_{n=1}^{\infty}(s_n - 1)^2 < \infty$ *and* $s_n > 1$. *Then for almost every realization* $X = Z(\mathfrak{F}_{\mathbb{D}})$, *we have*

$$f(z) = \lim_{n\to\infty} \frac{\displaystyle\sum_{k=0}^{\infty} \sum_{\substack{x\in X \\ k \leq d_{\mathbb{D}}(z,x) < k+1}} W_{s_n}^{z}(x) f(x)}{\displaystyle\sum_{x\in X} W_{s_n}^{z}(x)}.$$

Remark: The double summation is needed! In general, for s close to 1, we do not know whether or not we have

$$\sum_{x\in X} e^{-s d_{\mathbb{D}}(z,x)} |f(x)| < \infty.$$

We can do a little bit more

A holomorphic can be replaced by harmonic

B $f : \mathbb{D} \to \mathbb{C}$ can be replaced by Hilbert-space-vector valued function.

We can do a little bit more

A holomorphic can be replaced by harmonic

B $f : \mathbb{D} \to \mathbb{C}$ can be replaced by Hilbert-space-vector valued function.

C **Less trivial generalization** is: $f \in A^2(\mathbb{D})$ can be replaced by

$$f \in \bigcap_{\varepsilon > 0} A^2_\varepsilon(\mathbb{D}) \quad \text{and} \quad \|f\|_{A^2_\varepsilon} = o\left(\frac{1}{\varepsilon}\right),$$

where

$$A^2_\varepsilon(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} \Big| \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\varepsilon dLeb(z) < \infty \right\} \cap Hol(\mathbb{D}).$$

Impossibility of simultaneous reconstruction of $A^2(\mathbb{D})$

Proposition (Bufetov- Q.)

For any $z \in \mathbb{D}$, there exists a **universal** *constant $c_z > 0$, such that for any* **compactly supported** *radial weight W, we have*

$$\mathbb{E}\Bigg(\underbrace{\sup_{f \in A^2(\mathbb{D}) : \|f\| \leq 1} \left| \frac{\sum_{x \in Z(\mathfrak{F}_{\mathbb{D}})} W^z(x) f(x)}{\mathbb{E} \sum_{x \in Z(\mathfrak{F}_{\mathbb{D}})} W^z(x)} - f(z) \right|^2}_{\text{denoted } I_{A^2(\mathbb{D})}(W, z)} \Bigg) \geq c_z.$$

Simultaneous reconstructions: Assumptions on $\mathbb{P}$

Assumption (I) (average conformal invariant) $\exists \lambda > 0$ such that

$$\mathbb{E}_{\mathbb{P}}\Big[\#(X \cap B)\Big] = \lambda \times \text{hyperbolic area of } B.$$

Assumption (II). $\exists C > 0$ such that for any $f : \mathbb{D} \to \mathbb{C}$, continuous compactly supported, we have

$$\mathrm{Var}_{\mathbb{P}}\Big(\sum_{x \in X} f(x)\Big) \leq C \cdot \mathbb{E}_{\mathbb{P}}\Big(\sum_{x \in X} |f(x)|^2\Big).$$

(II) is true in many important cases.

1. Poisson point processes
2. determinantal point processes with Hermitian correlation kernels
3. negatively correlated point processes

Which Banach spaces we are going to reconstruct?

Space1 Weighted Bergman space $A^2(\mathbb{D};\omega)$ with rapidly growing weights: for instance

$$\omega(z) = \frac{1}{(1-|z|^2)\log\left(\frac{4}{1-|z|^2}\right)\left[\log\left(\log\left(\frac{4}{1-|z|^2}\right)\right)\right]^{1+\varepsilon}}$$

Space2 reproducing kernel Hilbert space $\mathscr{H}(K) \subset \mathrm{Harm}(\mathbb{D})$ with a growth condition on $K(z,z)$. For instance,

$$K(z,w) = \sum_{n\in\mathbb{Z}} a_n z^n \bar{w}^n \quad \text{with} \lim_{n\to\infty} \frac{a_{|n|}}{\log(|n|+2)} = 0.$$

Space3 space coming from Poisson integrals: μ any fixed Borel probability measure on $\mathbb{T} = \partial\mathbb{D}$.

$$h^2(\mu) = \{h = P[f\mu] : f \in L^2(\mu)\}. \quad \mu \text{ can be singular!!}$$

Statements of results

Theorem (Bufetov -Q. 2018)

Let $\mathbb{P}$ satisfy (I) and (II). Let $\mathcal{B} \in \{Space1, Space\ 2\}$. Then $\exists (s_n)_{n\geq 1}$ with $s_n \to 1^+$ such that, for $\mathbb{P}$-almost every $X \subset \mathbb{D}$,

- *X is a uniqueness set for $\mathcal{B}$.*
- *for ALL $f \in \mathcal{B}$ and all $z \in \mathbb{D} \cap \mathbb{Q}^2$, the limit equality*

$$f(z) = \lim_{n\to\infty} \frac{\displaystyle\sum_{k=0}^{\infty} \sum_{\substack{x\in X \\ k \leq d_{\mathbb{D}}(z,x) < k+1}} e^{-s_n d_{\mathbb{D}}(z,x)} f(x)}{\displaystyle\sum_{x\in X} e^{-s_n d_{\mathbb{D}}(z,x)}}.$$

Statements of results: continued

Theorem (Bufetov -Q. 2018)

Let $\mathbb{P}$ satisfy (I) and (II). Let $\mathcal{B} = h^2(\mu)$. Then for $\mathbb{P}$-almost every $X \subset \mathbb{D}$,

- *X is a uniqueness set for $\mathcal{B}$.*
- *for ALL $f \in \mathcal{B}$ and ALL $z \in \mathbb{D}$, the limit equality*

$$f(z) = \lim_{s \to 1^+} \frac{\sum_{x \in X} e^{-s d_{\mathbb{D}}(z,x)} f(x)}{\sum_{x \in X} e^{-s d_{\mathbb{D}}(z,x)}}.$$

Statements of results: continued

Theorem (Bufetov -Q. 2018)

Let $\mathbb{P}$ satisfy (I) and (II). Let $\mathcal{B} = h^2(\mu)$. Then for $\mathbb{P}$-almost every $X \subset \mathbb{D}$,

- *X is a uniqueness set for $\mathcal{B}$.*
- *for ALL $f \in \mathcal{B}$ and ALL $z \in \mathbb{D}$, the limit equality*

$$f(z) = \lim_{s \to 1^+} \frac{\sum_{x \in X} e^{-s d_{\mathbb{D}}(z,x)} f(x)}{\sum_{x \in X} e^{-s d_{\mathbb{D}}(z,x)}}.$$

Corollary A: Under the above assumptions, $\mathbb{P}$-a.e. $X \subset \mathbb{D}$,

$$\lim_{s \to 1^+} \frac{\sum_{x \in X} e^{-s d_{\mathbb{D}}(z,x)} \delta_x}{\sum_{x \in X} e^{-s d_{\mathbb{D}}(z,x)}} = \text{harmonic measure with repect to point } z.$$

Relax average conformal invariance

Theorem (Bufetov-Q. 2018)

Let $\beta \geq 2$. Fix $(s_n)_{n\geq 1}$ with $s_n > \beta$ and $\sum_{n=1}^{\infty}(s_n - \beta) < \infty$. Let $\mathbb{P}$ be a point process on $\mathbb{D}$ satisfying Assumption (II) such that

$$\mathbb{E}_{\mathbb{P}}\Big[\#(X \cap B)\Big] \propto \int_B \frac{dLeb}{(1-|x|^2)^{\beta+1}}.$$

Then $\mathbb{P}$-almost any $X \subset \mathbb{D}$, the limit equality

$$f(z) = \lim_{n\to\infty} \frac{\displaystyle\sum_{k=0}^{\infty} \sum_{\substack{x\in X \\ k \leq d_{\mathbb{D}}(z,x) < k+1}} e^{-s_n d_{\mathbb{D}}(z,x)} \left(\frac{|1-x\bar{z}|^2}{1-|z|^2}\right)^{\beta-1} f(x)}{\displaystyle\sum_{x\in X} e^{-s_n d_{\mathbb{D}}(z,x)} \left(\frac{|1-x\bar{z}|^2}{1-|z|^2}\right)^{\beta-1}}$$

holds simultaneously for all $f \in A^2(\mathbb{D})$ and all $z \in \mathbb{D} \cap \mathbb{Q}^2$.

Some problems

- Construct deterministic subsets $X \subset \mathbb{D}$ satisfying the reconstruction properties.
- Give sufficient conditions or even geometric criteria for subsets $X \subset \mathbb{D}$ satisfying the reconstruction properties.

More general situations

Our method works in the following situations as well:

- Real hyperbolic spaces: the unit ball in $\mathbb{R}^n$, equipped with the Poincaré metric, (we should consider the invariant harmonic functions)
- Complex hyperbolic spaces: the unit ball in $\mathbb{C}^n$, equipped with the Bergman metric.
- Quaternion hyperbolic spaces.
- Any locally finite infinite connected graph.

Thank you !

Appendix

- The complex hyperbolic spaces: Let $\mathbb{D}_d = \{z \in \mathbb{C}^d : |z| < 1\}$ equipped with the Riemannian metric (called the Bergman metric on $\mathbb{D}_d$) as follows

$$ds_B^2 := 4\frac{|dz_1|^2 + \cdots + |dz_d|^2}{1 - |z|^2} + 4\frac{|z_1 dz_1 + \cdots + z_d dz_d|^2}{(1 - |z|^2)^2}.$$

Bergman Laplacian $\widetilde{\Delta}$ is given by the formula

$$\widetilde{\Delta} = (1 - |z|^2) \sum_{i,j} (\delta_{ij} - z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$

- The real hyperbolic spaces: Let $m \geq 2$ be a positive integer and let $\mathbb{B}_m \subset \mathbb{R}^m$, equipped with the Poincaré metric

$$ds_h^2 = 4\frac{dx_1^2 + \cdots + dx_m^2}{(1-|x|^2)^2}.$$

The hyperbolic Laplacian Δ_h on $\mathbb{B}_m$ is:

$$\Delta_h = (1-|x|^2)^2 \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} + 2(m-2)(1-|x|^2) \sum_{i=1}^{m} x_i \frac{\partial}{\partial x_i}.$$