The Speed of a Second Class Particle in the ASEP

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April 11, 2019

Joint work with Promit Ghosal and Ethan Zell

CIRM
Integrability and Randomness in Mathematical Physics and Geometry

Outline

Section 1

The Model and The Result

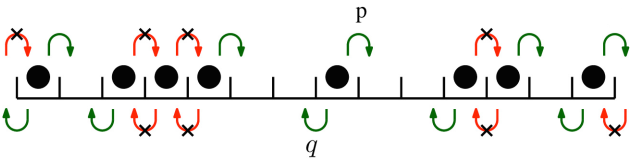


Figure: ASEP on a line.

A jumping rates asymmetry $p \neq q$

S particles only move up to one position to the left or right at each instance

E particles may not occupy the same position

P the model is a Markov process

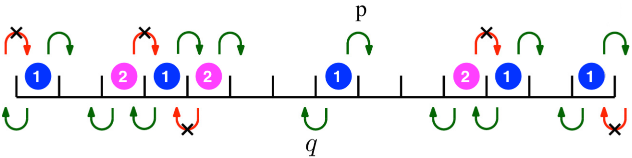


Figure: ASEP with Second class particles.

Step Initial Conditions

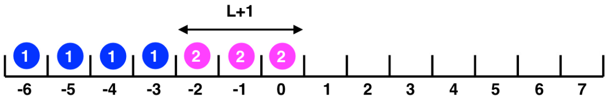


Figure: Step Initial Conditions

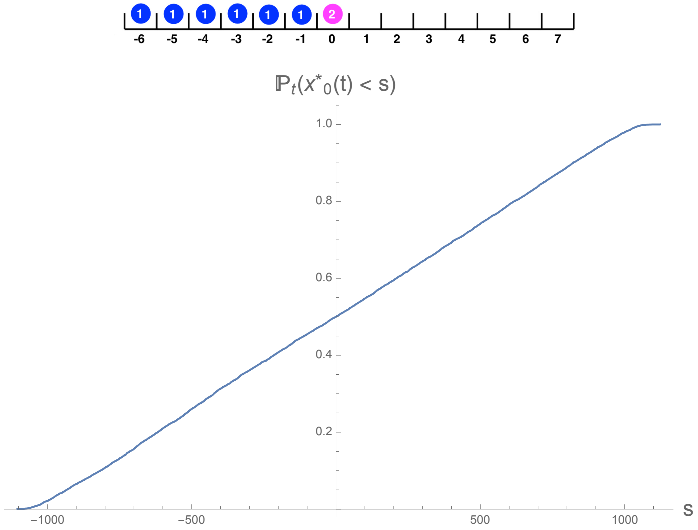


Figure: $t = 1,000$ and $10,000$ trails

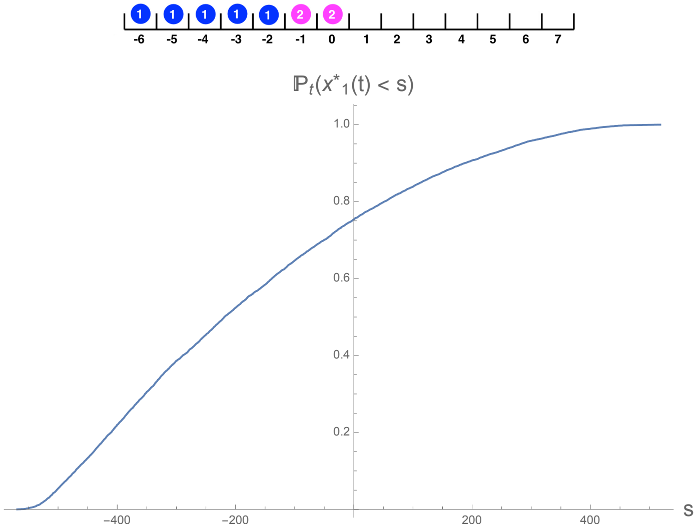
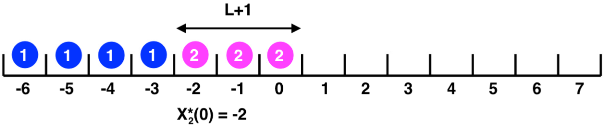


Figure: $t = 500$ and $10,000$ trails

The Result



Theorem (Ghosal, S., Zell (2019))

Given the asymmetry parameter $\gamma := p - q$ *for* $p \in (\frac{1}{2}, 1]$ *and* $q = 1 - p$*, we have*

$$\frac{x_L^*(t)}{t} \xrightarrow{d} U_L, \quad as\ t \to \infty \tag{1}$$

with U_L*, a random variable supported on* $[-\gamma, \gamma]$*, and*

$$\mathbb{P}(U_L \geq s) = \left(\frac{1 - \gamma^{-1} s}{2}\right)^{L+1}, \qquad \forall s \in [-\gamma, \gamma]. \tag{2}$$

Section 2

Background

Introduce the occupation variable

$$\eta(x,t)=\begin{cases}1, & x \text{ is occupied at time } t\\ 0, & x \text{ is not occupied at time } t\end{cases} \tag{3}$$

for $x \in \mathbb{Z}$ and $t \in \mathbb{R}_{\geq 0}$. Then, for the scaling limit with

$$\tau = \epsilon t, \quad \chi = \epsilon x, \text{ and } \quad \epsilon \to 0, \tag{4}$$

we have the inviscid Burgers equation

$$\frac{\partial u(\tau,\chi)}{\partial \tau} + (p-q)\frac{\partial[u(\tau,\chi)(1-u(\tau,\chi))]}{\partial \chi} = 0. \tag{5}$$

for $u(\chi,\tau) = \mathbb{E}(\eta(\tau,\chi))$.

A characteristic line $\chi(\tau)$ is defined so that

$$u\big(\chi(\tau), \tau\big) = \text{ constant}, \tag{6}$$

In fact, the characteristic line are straight lines

$$\chi(\tau) = v\tau + \chi(0) \tag{7}$$

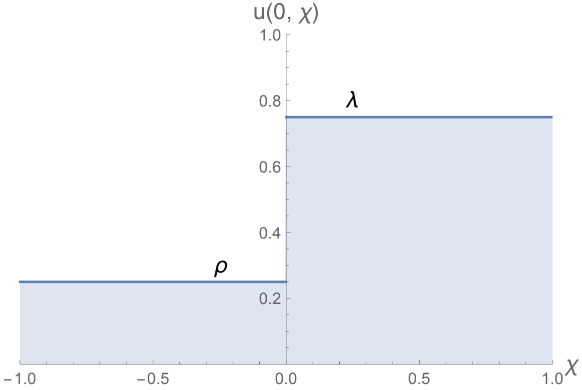
with

$$v = 1 - 2\eta\big(\chi(0), 0\big). \tag{8}$$

Take the (partial) step initial conditions, or *shock initial conditions*,

$$u(\chi,0)=\begin{cases}\rho, & \chi<0\\ \lambda, & \chi>0\end{cases} \tag{9}$$

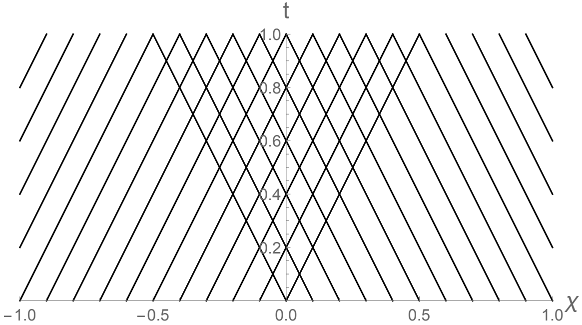
with $\rho<\lambda$.



Then, the speed of the characteristic line is given by

$$v = 1 - 2\eta(\chi(0), 0) = \begin{cases} 1 - 2\rho, & \chi < 0 \\ 1 - 2\lambda, & \chi > 0 \end{cases} \tag{10}$$

with $\rho < \lambda$.



In the pre-scaling regime, the *shock initial conditions* are given by independent occupation variables $\{\eta(0,x)\}_{x\in\mathbb{Z}}$ with

$$\mathbb{P}(\eta(x,0)=1)=\begin{cases}\rho, & x>0\\ \lambda, & x<0\end{cases}\quad. \tag{11}$$

with $\rho<\lambda$. Moreover, we introduce a second class particle at the origin

$$\mathbb{P}(\eta(x,0)=2)=1, \tag{12}$$

and denote the location of the second class particle by $X(t)$.

A Second Class Particle in the Shock

Theorem (Ferrari (1991))

For the ASEP with shock initial conditions, the second class particle stays at the shock of the Burgers equation:

$$\lim_{\epsilon\to 0} \mathbb{P}\left[\eta(\epsilon^{-1}\tau, \epsilon^{-1}(\chi + X(\epsilon^{-1}\tau))) = 1\right] = \begin{cases} \rho, & \chi < 0 \\ \lambda, & \chi > 0 \end{cases} . \tag{13}$$

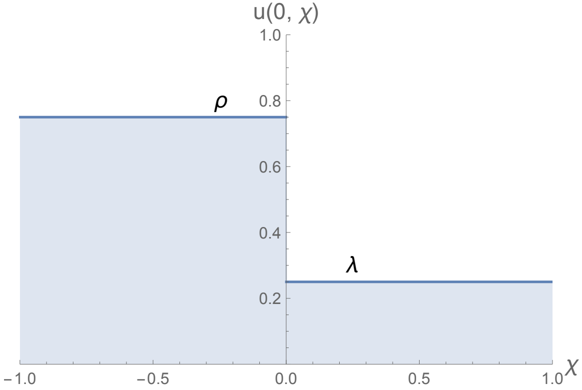
Moreover, the speed of the second class particle convergence almost surely

$$\frac{X(t)}{t} \to (p-q)(1-\lambda-\rho). \tag{14}$$

Take the (partial) step initial conditions, or *rarefaction initial conditions*,

$$u(\chi,0)=\begin{cases}\rho, & \chi<0\\ \lambda, & \chi>0\end{cases} \tag{15}$$

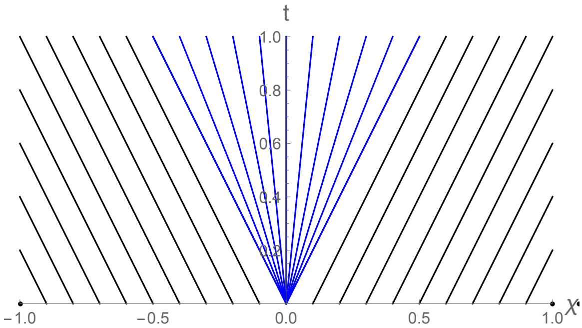
with $\rho>\lambda$.



Then, the speed of the characteristic line is given by

$$v = 1 - 2\eta(\chi(0), 0) = \begin{cases} 1 - 2\rho, & \chi < 0 \\ 1 - 2\lambda, & \chi > 0 \end{cases} \tag{16}$$

with $\rho > \lambda$.



In the pre-scaling regime, the *rarefaction initial conditions* are given by independent occupation variables $\{\eta(0,x)\}_{x\in\mathbb{Z}}$ with

$$\mathbb{P}(\eta(x,0)=1)=\begin{cases}\rho, & x>0\\ \lambda, & x<0\end{cases}. \tag{17}$$

with $\rho>\lambda$. Moreover, we introduce a second class particle at the origin

$$\mathbb{P}(\eta(x,0)=2)=1, \tag{18}$$

and denote the location of the second class particle by $X(t)$.

Theorem (Ferrari, Kipnis (1995), Ferrari, Goncalves, Martin (2009))

For the ASEP with rarefaction initial conditions, the speed of the second class particle convergence in distribution

$$\frac{X(t)}{t} \to \mathcal{U}_p \tag{19}$$

with $\mathcal{U}_p$, a uniformly distributed random variable on the interval $[-(p-q),(p-q)]$.

A Second Class Particle in the Rarefaction

Theorem (Mountford and Guiol (2005))

For the TASEP with rarefaction initial conditions, the speed of the second class particle convergence almost surely

$$\frac{X(t)}{t} \to \mathcal{U} \tag{20}$$

with $\mathcal{U}$*, a uniformly distributed random variable on the interval* $[-1, 1]$*.*

For TASEP with arbitrary rarefaction initial data,

$$\rho = \liminf_{n\to\infty} \sum_{x=-n}^{-1} \eta(x) > \limsup_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n} \eta(x) = \lambda, \tag{21}$$

Cator and Pimentel gave the law of the speed of the second class particle in 2013. For instance, if we take the initial data

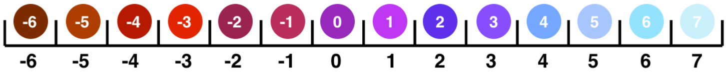
$$\eta(x) = \begin{cases} 0, & x \geq 1 \\ 1, & x \leq 0, x \neq -L \\ 2, & x = -L \end{cases} \tag{22}$$

with $L \geq 0$, then

$$\frac{X(t)}{t} \xrightarrow{d} U_L, \quad \text{as } t \to \infty \tag{23}$$

with U_L, a random variable supported on $[-1, 1]$, and

$$\mathbb{P}(U_L \geq s) = \left(\frac{1-s}{2}\right)^{L+1}, \qquad \forall s \in [-1, 1]. \tag{24}$$

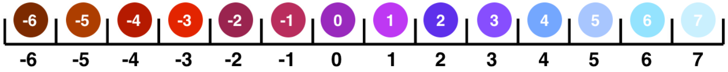


In 2011, Amir, Angel and Valko consider the TASEP with all different types of classes (colors). In particular, let

$$\begin{aligned} \eta(t,n) &:= \text{color of particle at location } n \\ X(t,n) &:= \text{location of particle colored } n, \end{aligned} \tag{25}$$

with the stationary initial condition

$$\eta(0,n) = X(0,t) = n. \tag{26}$$



Theorem (Amir, Angel and Valko (2011))

In the TASEP with stationary initial condition $\eta(0, n) = n$, the speed of every particle converges almost surely:

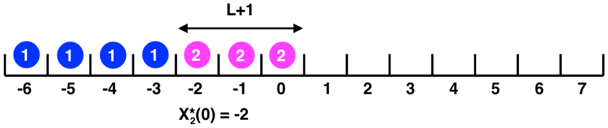
$$\frac{X(t,n) - n}{t} \to U_n, \quad \text{as } t \to \infty \tag{27}$$

with $\{U_n\}_{n\in\mathbb{Z}}$ a family of random variables, each uniform on $[-1, 1]$.

Definition

The process $\{U_n\}_{n\in\mathbb{Z}}$ is called the TASEP speed process.

Proof



Theorem (Ghosal, S., Zell (2019))

Given the asymmetry parameter $\gamma := p - q$ *for* $p \in (\frac{1}{2}, 1]$ *and* $q = 1 - p$, *we have*

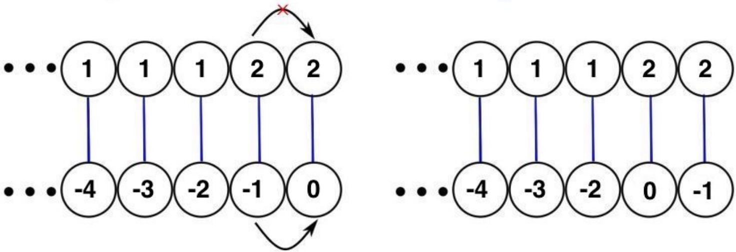
$$\frac{x_L^*(t)}{t} \xrightarrow{d} U_L, \quad \text{as } t \to \infty \tag{28}$$

with U_L, *a random variable supported on* $[-\gamma, \gamma]$, *and*

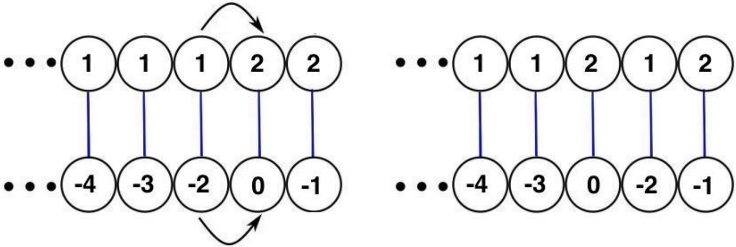
$$\mathbb{P}(U_L \geq s) = \left(\frac{1 - \gamma^{-1}s}{2}\right)^{L+1}, \qquad \forall s \in [-\gamma, \gamma]. \tag{29}$$

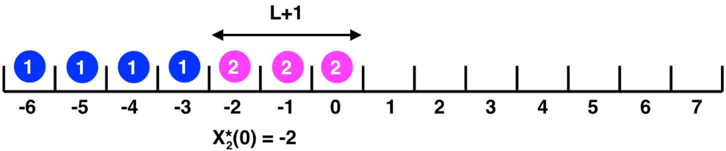
Strategy

- Coupling between second class ASEP and multi-colored ASEP.
- Symmetry between multi-colored and colorblind ASEP.
- Asymptotic analysis for block probabilities.


1 1 1 2 2
-4 -3 -2 -1 0
1 1 1 2 2
-4 -3 -2 0 -1

Coupling



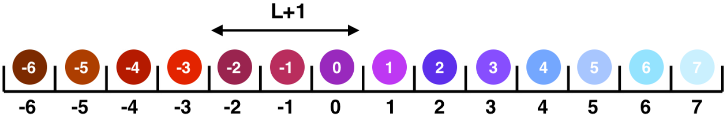


Lemma (Ghosal, S. , Zell (2019))

Let $x_L^*(t)$ be the position of the last second class particle in ASEP with $L+1$ total second class particles and step initial conditions, and let $X(t,n)$ be the location of the n^{th} colored particle in the multi-colored ASEP with stationary initial conditions $X(0,n)=n$. Then, there exist a coupling between the two systems so that

$$x_L^*(t) = \min\{X(t,n) : -L \leq n \leq 0\} \tag{30}$$

for all $t \geq 0$.



Theorem (Amir, Angel and Valko (2011), Borodin and Wheeler (2018))

For ASEP with stationary initial conditions $\eta(0, n) = n$, the process $\{\eta(t, n)\}_{n\in\mathbb{Z}}$ has the same distribution as $\{X(t, n)\}_{n\in\mathbb{Z}}$ for any $t > 0$.

In particular,

$$\begin{aligned} &\mathbb{P}^{mASEP}(X(t,0) > s, X(t,-1) > s, \ldots, X(t,-L) > s) \\ &= \mathbb{P}^{mASEP}(\eta(t,0) > s, \eta(t,-1) > s, \ldots, \eta(t,-L) > s) \\ &= \mathbb{P}^{mASEP}(\eta(t,s+L) > 0, \eta(t,s+L-1) > 0, \ldots, \eta(t,s) > 0) \\ &= \mathbb{P}^{ASEP}(\eta(t,s+L) = 0, \eta(t,s+L-1) = 0, \ldots, \eta(t,s) = 0) \\ &= \mathbb{P}^{ASEP}(\text{ gap of length } L+1 \text{ starting at } s). \end{aligned} \tag{31}$$

Both configurations $\{\eta(t,n)\}_{n\in\mathbb{Z}}$ and $\{X(t,n)\}_{n\in\mathbb{Z}}$ may be considered to be *random* permutations of $\mathbb{Z}$. Thus, we introduce random permutations

$$\pi_n Y = \begin{cases} \tau_n Y, & y_n < y_{n+1} \\ \tau_n Y, & y_n > y_{n+1} \text{ with prob } p \\ Y, & y_n > y_{n+1} \text{ with prob } 1-p \end{cases} \tag{32}$$

with $Y = (y_n)_{n\in\mathbb{Z}}$ and τ_n acting by transposition of the entries (y_n, y_{n+1}).

Lemma (Amir, Angel and Valko (2011))

The operators $\{\pi_n\}_{n\in\mathbb{Z}}$ satisfy the relations

$$\begin{aligned} \pi_i^2 &= pI + (1-p)\pi_i, \\ \pi_i\pi_j &= \pi_j\pi_i \quad |i-j| > 1, \\ \pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} \end{aligned} \tag{33}$$

Lemma (Amir, Angel and Valko (2011))

Fix a sequence $i_1, \dots, i_r$. *Then*

$$\pi_{i_r} \cdots \pi_{i_1} \cdot id \stackrel{d}{=} (\pi_{i_1} \cdots \pi_{i_r} \cdot id)^{-1}. \tag{34}$$

For instance, consider a simple example

$$\{0,1,2\} \xrightarrow{\pi_1} \{0,2,1\} \xrightarrow{\pi_1} \begin{cases} \{0,1,2\}, & p \\ \{0,2,1\}, & 1-p \end{cases} \xrightarrow{\pi_0} \begin{cases} \{1,0,2\}, & p \\ \{2,0,1\}, & 1-p \end{cases}$$

$$\{0,1,2\} \xrightarrow{\pi_0} \{1,0,2\} \xrightarrow{\pi_1} \{1,2,0\} \xrightarrow{\pi_1} \begin{cases} \{1,0,2\}, & p \\ \{1,2,0\}, & 1-p \end{cases} \xrightarrow{\text{inv}} \begin{cases} \{1,0,2\}, & p \\ \{2,0,1\}, & 1-p \end{cases} .$$

Theorem (Tracy and Widom (2018))

For the (colorblind) ASEP with step initial conditions, let $\mathcal{P}_{L,\mathrm{step}}(x,m,t)$ *be the probability of the event*

$$x_m^{\mathrm{asep}}(t) = x, \quad x_{m+2}^{\mathrm{asep}}(t) = x+1, \quad \ldots \quad , \quad x_{m+L}^{\mathrm{asep}}(t) = x+L. \tag{35}$$

Additionally, we set $m = \sigma t$ *for some* $\sigma \in (0,1)$ *and introduce the parameters*

$$c_1 = 1 - 2\sqrt{\sigma} \quad \text{and} \quad c_2 = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}. \tag{36}$$

Then, for $x = c_1 t - c_2 \zeta t^{1/3}$, *one has*

$$\mathcal{P}_{L,\mathrm{step}}\left(x, m, t/\gamma\right) = c_2^{-1}\sigma^{(L-1)/2} F'_{\mathrm{GUE}}(\zeta) t^{-\frac{1}{3}} + o(t^{-\frac{1}{3}}) \tag{37}$$

with F'_{GUE}, *the derivative of the Tracy-Widom GUE distribution.*

Recall,

$$\mathbb{P}(x_L^*(t) > st) = \mathbb{P}(\text{ block of length } L+1 \text{ starting at } st). \tag{38}$$

We expect

$$st = c_1(\gamma t) - c_2\zeta(\gamma t)^{1/3}, \quad \Rightarrow \quad \sigma \approx \left(\frac{1-\gamma^{-1}s}{2}\right)^2. \tag{39}$$

Then,

$$\begin{aligned}\mathbb{P}(\text{ block of length } L+1 \text{ starting at } st) &\approx \sum_{m=\sigma(\gamma t)-\zeta(\gamma t)^{1/3}}^{\sigma(\gamma t)+\zeta(\gamma t)^{1/3}} \mathcal{P}_{L,\mathrm{step}}\Big(st, m, t\Big) \\ &\approx \sigma^{\frac{L+1}{2}} \int_{-\infty}^{\infty} F'_{\mathrm{GUE}}(\xi) d\xi. \\ &= \left(\frac{1-\gamma^{-1}s}{2}\right)^{L+1}. \end{aligned} \tag{40}$$

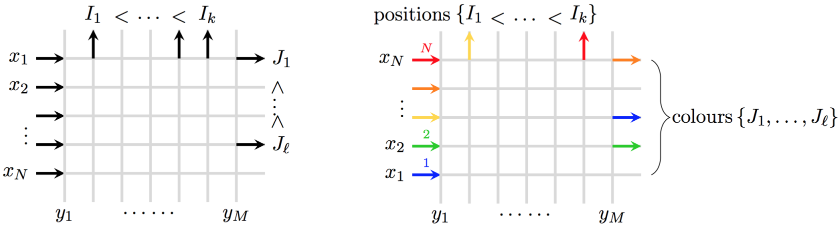


Figure: Image from Borodin and Wheeler

Thank you for your attention!