KP theory, planar bicolored networks in the disk and rational degenerations of M-curves

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CIRM Luminy, April 9, 2019

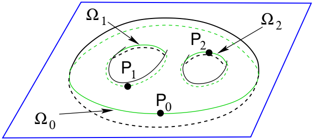
Goal: Connect totally non–negative Grassmannians to M–curves through finite–gap KP theory

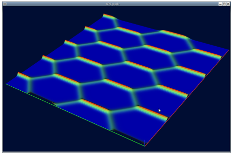
KP-II equation $\quad (-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$

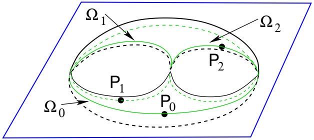
Two relevant classes of solutions:

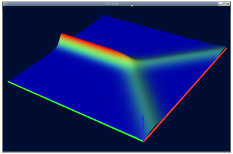
- Real regular multiline KP solitons which are in natural correspondence with totally non–negative Grassmannians [Chakravarthy-Kodama; Kodama-Williams];
- Real regular finite–gap KP solutions parametrized by degree g real regular non–special divisors on genus g M-curves [Dubrovin-Natanzon]

Novikov: relevant to check whether real regular soliton solutions may be obtained from real regular finite–gap solutions



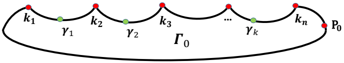






- Soliton data: $(\mathcal{K}, [A])$, with $\mathcal{K} = \{\kappa_1 < \cdots \kappa_n\}$, A real $k \times n$ matrix
- $\tau(x,y,t) = Wr_x(f^{(1)}, \dots f^{(k)})$, where $f^{(i)} = \sum_{j=1}^n A^i_j \exp(\kappa_j x + \kappa_j^2 y + \kappa_j^3 t)$
- $u(x,y,t) = 2\partial_x^2 \log(\tau)$ is regular for real (x,y,t) iff all maximal minors of A are non–negative [Kodama Williams-2013]

Soliton data: $(\mathcal{K}, [A])$ $\mapsto$ Sato algebraic geometric data: Γ_0 rational curve, marked points $P_0, \kappa_1, \dots, \kappa_n$, k-point real non–special divisor $\mathcal{D}_S^{(k)} = \{\kappa_1 < \gamma_1 < \cdots < \gamma_k \le \kappa_n\}$ [Malanyuk 1991]:



Incompleteness of Sato algebraic–geometric data: k divisor points vs $k(n-k)$–dimensional Grassmannian

Idea: use finite–gap theory for degenerate solutions (ex. solitons) on reducible curves! [Krichever 1986]

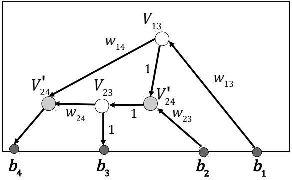
The algebraic curve

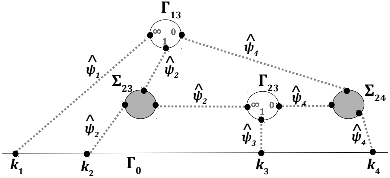
[Postnikov 2006]: Parametrization via planar bicolored networks in the disk of positroid cells (= Gelfand-Serganova stratum + positivity) of totally non–negative Grassmannians

In arXiv:1801.00208: fix soliton data $(\mathcal{K}, [A])$, choose a trivalent $\mathcal{G}$ in Postnikov class and construct Γ rational degeneration of $\mathtt{M}$ curve of genus $g = \#\{f\} - 1$:

$\mathcal{G}$	Γ
Boundary of disk	Sato component Γ_0
Boundary vertex b_l	Marked point κ_l on Γ_0
Internal black vertex V'_s	Copy of $\mathbb{CP}^1$ denoted Σ_s
Internal white vertex V_l	Copy of $\mathbb{CP}^1$ denoted Γ_l
Edge e	Double point
Face f	Oval

• In the special case of Le–networks (arXiv:1805.05641) genus is minimal and equal to the dimension of the positroid cell!





The KP divisor for the soliton data $(\mathcal{K}, [A])$ on Γ

Key ideas:

- Associate to each edge e of the directed network $\mathcal{N}$ an edge vector E_e so that Sato constraints are satisfied;
- Use edge vectors to rule the values of the dressed edge wave function at the edges $e \in \mathcal{N}$ (=double points on Γ) $\Longrightarrow$ the Baker-Akhiezer function on Γ automatically takes equal values at double points;
- Use linear relations at vertices to compute the position of the KP divisor and extend wave function to Γ
- Edge vectors are real $\Longrightarrow$ Edge wave function real for real KP times $\Longrightarrow$ KP divisor belongs to the union of the ovals;
- Combinatorial proof that there is one divisor point in each oval.

◇ The j–th component of E_e: $(E_e)_j = \sum_{\mathcal{P}:e\to b_j} (-1)^{\text{wind}(\mathcal{P})+\text{int}(\mathcal{P})} w(\mathcal{P})$.

◇ Explicit expressions for components of edge vectors on any network (modification of Postnikov and Talaska): the edge vector components are rational in weights with subtraction free denominators;

◇ Linear relations at internal vertices analogous to momentum-elicity conservation conditions in the planar limit of $N = 4$–SYM theory (see Arkani-Ahmed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka [2016]).

Reducible plane curve $P_0(\lambda,\mu)=0$, with

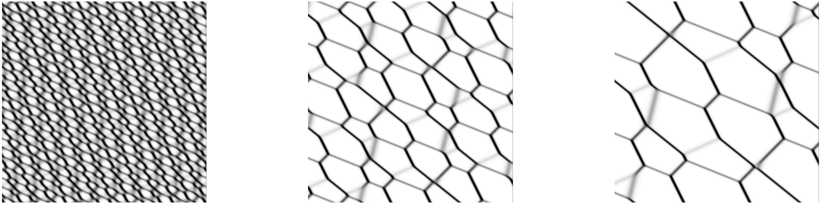
$$P_0(\lambda,\mu)=\mu\cdot(\mu-(\lambda-\kappa_1))\cdot(\mu+(\lambda-\kappa_2))\cdot(\mu-(\lambda-\kappa_3))\cdot(\mu+(\lambda-\kappa_4)).$$

Genus 4 $\mathtt{M}$–curve after desingularization:

$$\Gamma(\varepsilon)\ :\qquad P(\lambda,\mu)=P_0(\lambda,\mu)+\varepsilon(\beta^2-\mu^2)=0,\qquad 0<\varepsilon\ll 1,$$

where

$$\beta=\frac{\kappa_4-\kappa_1}{4}+\frac{1}{4}\max\left\{\kappa_2-\kappa_1,\kappa_3-\kappa_2,\kappa_4-\kappa_3\right\}.$$



$$\kappa_1=-1.5,\quad \kappa_2=-0.75,\quad \kappa_3=0.5,\quad \kappa_4=2.$$

Level plots for the KP-II finite gap solutions for $\epsilon=10^{-2}$ [left], $\epsilon=10^{-10}$ [center] and $\epsilon=10^{-18}$ [right]. The horizontal axis is $-60\le x\le 60$, the vertical axis is $0\le y\le 120$, $t=0$. The white color corresponds to lowest values of u, the dark color corresponds to the highest values of u.

Frobenius manifold as Orbit space of Extended Jacobi Groups

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SISSA, Trieste

April 09, 2019

Frobenius Manifolds

Definition (Frobenius manifold)

A Frobenius structure on M is the data (M, $\bullet$, $<,>$, e, E) satisfying:

1. $\eta := <,>$ is a flat pseudo-Riemannian metric;
2. $\bullet$ is C-linear, associative, commutative product on $T_m M$ which depends smoothly on m;
3. e is the unity vector field for the product and $\nabla e = 0$;
4. $\nabla_w c(x, y, z)$ is symmetric, where $c(x, y, z) := < x \bullet y, z >$;
5. A linear vector field $E \in \Gamma(M)$ must be fixed on M, i.e. $\nabla\nabla E = 0$ such that:

$$L_E <,> = (2 - d) <,>, \; L_E \bullet = \bullet \; L_E e = e$$

Frobenius Manifolds as Ω/W

Theorem (Dubrovin Conjecture, Hertling 1999)

Any irreducible semisimple polynomial Frobenius manifold with positive invariant degrees is isomorphic to the orbit space of a finite Coxeter group.

Main Point

Differential geometry of the orbit spaces of reflection groups and of their extensions $\mapsto$ Frobenius manifolds.

Similar constructions works when W is Extended affine Weyl Group [Dubrovin, Zhang 1998] and for Jacobi groups [Bertola 1999].

Problem Setting

$M_{1,1} \cong \mathbb{C}^3/\hat{A}_1$
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Mixed of Extended Affine Weyl Group + Jacobi Group?

$$M_{1,0,0} \cong \mathbb{C}^4/W$$

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$$M_{1,0,0} \cong \mathbb{C}^4/W$$

Generalization

$$M_{1,n,0} \cong \mathbb{C}^{n+3}/W$$

Thank you!

Moments of Moments

Emma Bailey

Joint work with Jon Keating

arXiv:1807.06605

Take $A \in \mathrm{CUE}_N$, an $N \times N$ **unitary** matrix. Then define

$$P_N(A, \theta) = \det(I - Ae^{-i\theta}).$$

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Then there are two spaces to average over:

- the unit circle in the complex plane,
- $U(N)$ with respect to the Haar measure.

Moments of Moments

$\mathrm{MoM}_{\mathbf{N}}(\boldsymbol{k}, \beta)$

Set

$$\mathrm{MoM}_N(k, \beta) := \mathbb{E}_{A \in \mathrm{U}(N)} \left(\left(\frac{1}{2\pi} \int_0^{2\pi} |P_N(A, \theta)|^{2\beta} d\theta \right)^k \right).$$

Moments of Moments

$\mathrm{MoM_N}(k, \beta)$

Set

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Conjecture (Fyodorov & Keating)

As $N \to \infty$,

$$\mathrm{MoM}_N(k,\beta) \sim \begin{cases} \gamma_{k,\beta} N^{k\beta^2} & k < 1/\beta^2 \\ \rho_{k,\beta} N^{k^2\beta^2-k+1} & k \geq 1/\beta^2, \end{cases}$$

for some coefficients $\gamma_{k,\beta}, \rho_{k,\beta}$.

Results

Consider the case when $k, \beta \in \mathbb{N}$.

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Theorem [B.-Keating (2018)]

Let $k, \beta \in \mathbb{N}$. Then $\mathrm{MoM}_N(k,\beta)$ is a polynomial in N.

Theorem [B.-Keating (2018)]

Let $k, \beta \in \mathbb{N}$. Then with $\rho_{k,\beta}$ an *explicit* function of k and β,

$$\mathrm{MoM}_N(k,\beta) = \rho_{k,\beta} N^{k^2\beta^2 - k + 1} + O(N^{k^2\beta^2 - k}).$$

Example

$$\mathrm{MoM}_N(2,3) = \frac{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)(N+7)(N+8)(N+9)(N+10)(N+11)}{17221913277310241549444418895872000000000}$$
$$\times \Big(12308743625763N^{24} + 1772459082109872N^{23} + 121902830804059138N^{22} +$$
$$+5328802119564663432N^{21} + 166214570195622478453N^{20} + 3937056259812505643352N^{19}$$
$$+73583663800226157619008N^{18} + 1113109355823972261429312N^{17} + 13869840005250869763713293N^{16}$$
$$+144126954435929329947378912N^{15} + 1259786144898207172443272698N^{14}$$
$$+9315726913410827893883025672N^{13} + 58475127984013141340467825323N^{12}$$
$$+311978271286536355427593012632N^{11} + 1413794106539529439589778645028N^{10}$$
$$+5427439874579682729570383266992N^{9} + 17564370687865211818995713096848N^{8}$$
$$+47561382824003032731805262975232N^{7} + 106610927256886475209611301000128N^{6}$$
$$+194861499503272627170466392014592N^{5} + 284303877221735683573377603640320N^{4}$$
$$+320989495108428049992898521600000N^{3} + 266974288159876385845370793984000N^{2}$$
$$+148918006780282798012340305920000N + 43144523802785397500411904000000\Big)$$

Thank you

Speed of Convergence in the Gaussian Distribution for Laguerre Ensembles Under Double Scaling

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Problem statement

Consider **Laguerre Unitary Ensemble**:

$$M = U^* \mathrm{diag}\{\Lambda_1, \ldots, \Lambda_n\} U, \tag{1}$$

where U is distributed uniformly on the unitary group $U(n)$

The random variables $\Lambda_1, \ldots, \Lambda_n$ have the joint probability density

$$P_{n,m}(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_{n,m}} \prod_j \mathbb{1}[\lambda_j > 0] \lambda_j^{\alpha} e^{-4m\lambda_j} \prod_{j<k} (\lambda_k - \lambda_j)^2, \tag{2}$$

where $\alpha > -1$, $m \in \mathbb{N}$, and $Z_{n,m}$ is the partition function.

Let $f(M)$ be a real-valued function defined on the spectrum of M.

Our goal is to **study the characteristic function**

$$\mathbb{E}_{n,m}\left[e^{ih\,\mathrm{Tr}\, f(M)}\right] = \int e^{ih \sum f(\lambda_j)} P_{n,m}(\lambda_1, \ldots, \lambda_n)\, d\lambda_1 \cdot \ldots \cdot d\lambda_n \tag{3}$$

of the linear statistic $\mathrm{Tr}\, f(M)$ in a double-scaling limit

as $n = m \to \infty$.

Main results

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be locally Hölder continuous such that it admits the analytic continuation to some neighborhood of $[0, 1]$.

Theorem (Convergence to the Gaussian law)

$$\operatorname{Tr} f(M) - n\varkappa[f] \xrightarrow{d} N(\mu[f], K[f]), \quad n = m \to \infty. \tag{4}$$

The linear functionals $\varkappa[f]$, $\mu[f]$, and the quadratic functional $K[f]$ are given with the explicit formulas.

Theorem (Speed of convergence)

Let $f(x)$ also satisfy $f(x) = O(e^{Ax})$, $A > 0$, as $x \to +\infty$. Define the cumulative distribution functions $F_n(x)$ and $F(x)$ corresponding to $\operatorname{Tr} f(M) - n\varkappa[f] - \mu[f]$ and to $N(0, K[f])$, respectively. Then

$$\sup |F_n(x) - F(x)| = O(1/n), \quad n = m \to \infty. \tag{5}$$

The proof of Theorems is based on the Riemann–Hilbert analysis similar to Charlier&Gharakhloo (2019). However, unlike them, we are interested in **complex exponents**. In such a case the corresponding Hankel determinants and/or the weight of the corresponding orthogonal polynomials can be zero. Also we need the exponents that grow with n.
To succeed we adopt the approach from Deift, Its&Krasovsky (2014) and use the deformation of

$$w(x) = x^{\alpha} e^{-4nx} e^{ihf(x)}. \tag{6}$$

into

$$\tilde{w}_{l,t}(x) = x^{\alpha} e^{-4nx} \left(1 - t + te^{ih\mathbb{1}[l<n^{\gamma}+1]f(x)}\right) e^{ih(l-1)\mathbb{1}[l<n^{\gamma}+1]f(x)}, \tag{7}$$

We choose $\varepsilon > 0$ small enough so that

$$1 - t + te^{ih\mathbb{1}[l<n^{\gamma}+1]f(x)} \neq 0, \gamma \in [0,1], \tag{8}$$

for all $t \in [0,1]$, x in the neighborhood of $[0,1]$, h such that $|h| < \varepsilon$, and for all n, l.

From Gumbel to Tracy–Widom II, via integer partitions

Dan Betea

University of Bonn

based on joint work with J. Bouttier (Math. Phys. Anal. Geom. 2019, **arXiv:1807.09022 [math-ph]**)

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Partitions

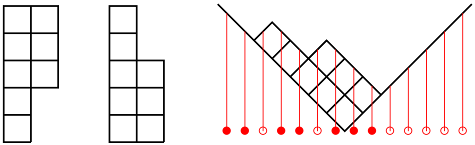


Figure: Partition (Young diagram) $\lambda = (2, 2, 2, 1, 1)$ (Frobenius coordinates $(1, 0|4, 1)$) in English, French and Russian notation, with associated Maya diagram (particle-hole representation). Size $|\lambda| = 8$, length $\ell(\lambda) = 5$.

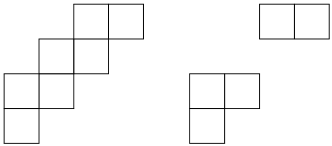


Figure: Skew partitions (Young diagrams) $(4, 3, 2, 1)/(2, 1)$ (but also $(5, 4, 3, 2, 1)/(5, 2, 1), \ldots$) and $(4, 4, 2, 1)/(2, 2)$ (but also $(6, 4, 4, 2, 1)/(6, 2, 2), \ldots$)

Counting tableaux

A standard Young tableau (SYT) is a filling of a (possibly skew) Young diagram with numbers $1, 2, \ldots$ strictly increasing down columns and rows.

$$\begin{ytableau} 1 & 3 & 5 & 6 \\ 2 & 4 & 9 \\ 7 \\ 8 \end{ytableau} \qquad \begin{ytableau} \none & \none & 1 & 7 \\ \none & 3 & 4 \\ 2 & 5 \\ 6 \end{ytableau}$$

$$\dim \lambda := \text{number of SYTs of shape } \lambda$$

and similarly for $\dim \lambda/\mu$.

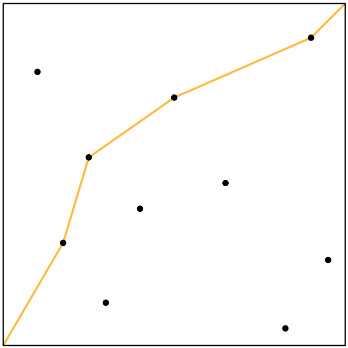
Measures on partitions

There are two natural measures on *all* partitions: poissonized Plancherel vs. (grand canonical) uniform

$$Prob(\lambda) = e^{-\epsilon^2}\epsilon^{2|\lambda|}\frac{(\dim\lambda)^2}{(|\lambda|!)^2} \qquad \text{vs.} \qquad Prob(\lambda) = u^{|\lambda|}\prod_{i\geq 1}(1-u^i)$$

with $\epsilon \geq 0$ and $1 > u \geq 0$ parameters.

Ulam's problem and Hammersley last passage percolation



Quantity of interest: L = longest up-right path from $(0,0)$ to $(1,1)$ (= 4 here).
Schensted's theorem yields that, in distribution,

$$L = \lambda_1$$

with λ coming from the poissonized Plancherel measure.

The Baik–Deift–Johansson theorem and Tracy–Widom

Theorem (BaiDeiJoh 1999)

If λ is distributed as poissonized Plancherel, we have:

$$\lim_{\epsilon\to\infty} Prob\left(\frac{\lambda_1 - 2\epsilon}{\epsilon^{1/3}} \leq s\right) = F_{\mathrm{TW}}(x) := \det(1 - Ai_2)_{(s,\infty)}$$

with

$$Ai_2(x,y) := \int_0^\infty Ai(x+s)Ai(y+s)ds.$$

and Ai the Airy function (solution of $y'' = xy$ decaying at ∞).

F_{TW} is the Tracy–Widom GUE distribution. It is by (original) construction the extreme distribution of the largest eigenvalue of a random hermitian matrix with iid standard Gaussian entries as the size of the matrix goes to infinity.

The Erdős–Lehner theorem and Gumbel

Theorem (ErdLeh 1941)

For the uniform measure $Prob(\lambda) \propto u^{|\lambda|}$ we have:

$$\lim_{u \to 1^-} Prob\left(\lambda_1 < -\frac{\log(1-u)}{\log u} + \frac{\xi}{|\log u|}\right) = e^{-e^{-\xi}}.$$

The finite temperature Plancherel measure

On pairs of partitions $\mu \subset \lambda \supset \mu$ consider the measure

$$Prob(\mu, \lambda) \propto u^{|\mu|} \cdot \frac{\epsilon^{2(|\lambda|-|\mu|)} \dim^2(\lambda/\mu)}{(|\lambda/\mu|!)^2}$$

with $u = e^{-\beta}$, $\beta =$ inverse temperature.

- $u = 0$ yields the poissonized Plancherel measure
- $\epsilon = 0$ yields the (grand canonical) uniform measure

The finite temperature Plancherel measure II

Theorem (B/Bouttier 2019)

Let $M = \frac{\epsilon}{1-u} \to \infty$ *and* $u = \exp(-\alpha M^{-1/3}) \to 1$. *Then*

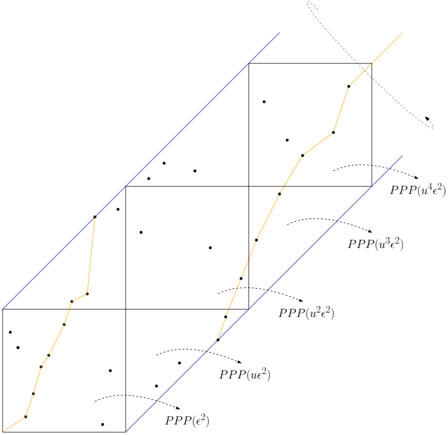
$$\lim_{M\to\infty} Prob\left(\frac{\lambda_1 - 2M}{M^{1/3}} \leq s\right) = F^{\alpha}(x) := \det(1 - Ai^{\alpha})_{(s,\infty)}$$

with

$$Ai^{\alpha}(x,y) := \int_{-\infty}^{\infty} \frac{e^{\alpha s}}{1+e^{\alpha s}} \cdot Ai(x+s)Ai(y+s)ds.$$

the finite temperature Airy kernel.

What is in a part?



With L the longest up-right path in this cylindric geometry, in distribution, Schensted's theorem states that

$$\lambda_1 = L + \kappa_1$$

where κ is a uniform partition $Prob(\kappa) \propto u^{|\kappa|}$ independent of everything else.

A word on the finite temperature Airy kernel Ai^{α}

- introduced by Johansson (Joh07)
- also appearing as the KPZ crossover kernel: SasSpo10 and AmiCorQua11; in random directed polymers BorCorFer11; cylindric OU processes LeDMajSch15
- interpolates between the Airy kernel and a diagonal exponential kernel:

$$\lim_{\alpha\to\infty} Ai^{\alpha}(x,y) = Ai_2(x,y),$$

$$\lim_{\alpha\to 0+} \frac{1}{\alpha} Ai^{\alpha}\left(\frac{x}{\alpha} - \frac{1}{2\alpha}\log(4\pi\alpha^3), \frac{y}{\alpha} - \frac{1}{2\alpha}\log(4\pi\alpha^3)\right) = e^{-x}\delta_{x,y}$$

- with $F^{\alpha}(s), F_{\mathrm{TW}}(s)$, and $G(s)$ the Fredholm determinants on (s,∞) of Ai^{α}, Ai_2 and $e^{-x}\delta_{x,y}$, (Joh07)

$$\lim_{\alpha\to\infty} F^{\alpha}(s) = F_{\mathrm{TW}}(s),$$

$$\lim_{\alpha\to 0+} F^{\alpha}\left(\frac{s}{\alpha} - \frac{1}{2\alpha}\log(4\pi\alpha^3)\right) = G(s) = e^{-e^{-s}}$$

Direct limit to Tracy–Widom

Theorem (B/Bouttier 2019)

Let $u \to 1$ *and* $\epsilon \to \infty$ *in such a way that* $\epsilon(1-u)^2 \to \infty$*. Then we have*

$$Prob\left(\frac{\lambda_1 - 2M}{M^{1/3}} \leq s\right) \to F_{\mathrm{TW}}(s), \qquad M := \frac{\epsilon}{1-u}.$$

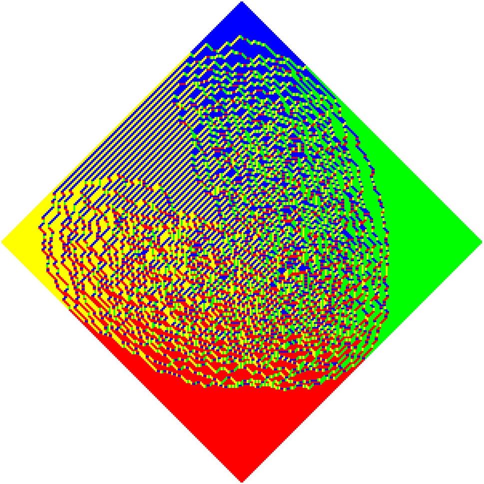
Direct limit to Gumbel

Theorem (B/Bouttier 2019)

Set $u = e^{-r}$ and assume that $r \to 0+$ and $\epsilon r^2 \to 0+$ (with ϵ possibly remaining finite). Then:

$$Prob\left(r\lambda_1 - \ln \frac{I_0(2\epsilon + \epsilon r)}{r} \leq s \right) \to e^{-e^{-s}}$$

where $I_0(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \cos \phi} d\phi$ is the modified Bessel function of the first kind and order zero.



Thank you!

Transition between characters of classical groups, decomposition of Gelfand-Tsetlin patterns, and last passage percolation

(joint work with Nikos Zygouras)

Elia Bisi

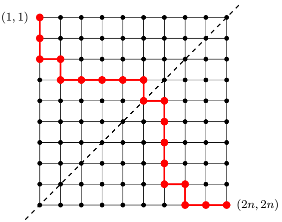
University College Dublin

Integrability and Randomness in Mathematical Physics and Geometry
Marseille, 9 April 2019

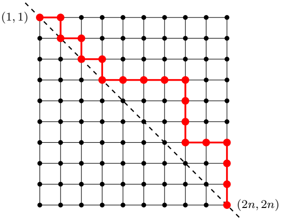
Last passage percolation (LPP)

$$L(2n,2n) := \max_{\pi \in \Pi_{2n,2n}} \sum_{(i,j)\in\pi} W_{i,j}$$

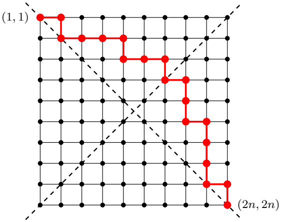
- $\Pi_{2n,2n}$ is the set of directed paths in $\{1,\ldots,2n\}^2$ starting from $(1,1)$ and ending at $(2n,2n)$;
- $\{W_{i,j}\}_{1\le i,j\le 2n}$ is a field of independent geometric random variables with various symmetries



Antidiagonal symmetry



Diagonal symmetry



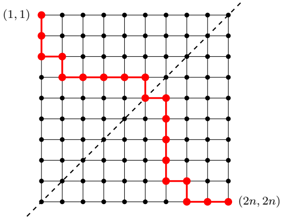
Double symmetry

Character identities and LPP

$$\mathbb{P}\left(L_\beta^{\boxslash}(2n,2n) \le 2u\right) \propto \sum_{\mu \subseteq (2u)^{(2n)}} \beta^{\sum_{i=1}^{2n}(\mu_i \bmod 2)} \cdot s_\mu^{(2n)}(p_1,\dots,p_{2n})$$

$$= \left[\prod_{i=1}^{2n} p_i\right]^u s^{\mathrm{CB}}_{u^{(2n)}}(p_1,\dots,p_{2n};\beta)$$

$$= \left[\prod_{i=1}^{2n} p_i\right]^u \sum_{\lambda \subseteq u^{(n)}} s^{\mathrm{CB}}_\lambda(p_1,\dots,p_n;\beta) \cdot s^{\mathrm{CB}}_\lambda(p_{n+1},\dots,p_{2n};\beta)$$



- $s_\mu^{(2n)}$ is a classical Schur polynomial;
- s_λ^{CB} is a Schur polynomial that interpolates between symplectic and odd orthogonal characters.

Duality between determinants and Pfaffians

Baik-Rains' formulas and ours show a *duality* between

- Pfaffians and determinants, for finite N.
- Fredholm Pfaffian and Fredholm determinantal expressions of the limiting distribution functions, as $N \to \infty$.

E.g., we obtain:

- Sasamoto's Fredholm determinant for the GOE Tracy-Widom distribution in the case of *antidiagonal symmetry*:

$$F_1(s) = \det(I - B_s)$$

- Ferrari-Spohn's Fredholm determinant for the GSE Tracy-Widom distribution in the case of *diagonal symmetry*:

$$F_4(s) = \frac{1}{2}\left[\det(I - B_{\sqrt{2}s}) + \det(I + B_{\sqrt{2}s})\right]$$

with the kernel being $B_s(x,y) := \mathrm{Ai}(x+y+s)$ on $L^2([0,\infty))$.

Painlevé II τ-function as a Fredholm determinant

Harini Desiraju

SISSA, Trieste

Introduction

- **Question:** Can the τ-function of Painlevé II be expressed as a Fredholm determinant?

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$$\frac{d^2}{ds^2} \ln \tau[s] = -q^2(s) \tag{2}$$

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$$q(s) \approx \kappa Ai(s); \quad \kappa \in \mathbb{C}; s \to \infty \tag{3}$$

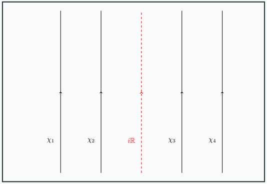
  - It is a known result that the τ-function in this case is the determinant of the Airy Kernel.

$$\tau[s] = \det[\mathbb{I} - \kappa^2 K_{Ai}]|_{[s,\infty)]} \tag{4}$$

General Painlevé II using IIKS construction

- The Riemann Hilbert problem of Painlevé II, after some transformations, can be reduced to the following RHP on $i\mathbb{R}$

$$\Gamma_+(z) = \Gamma_-(z)J(z); \quad \Gamma(z) = 1 + \mathcal{O}(z^{-1}) \quad \text{as} \quad z \to \infty \tag{5}$$



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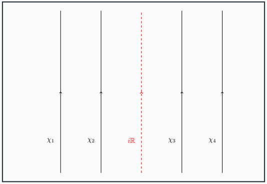

- Using χ_i, the jump function is $J(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} = \mathbb{1} - 2\pi i f(z) g^T(z)$

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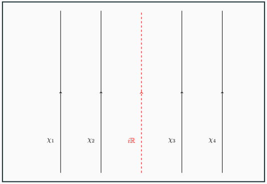

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- with

$$f(z) = \begin{bmatrix} \chi_2(z) + \frac{(b(z)-1)}{a(z)}\chi_4(z) \\ \frac{(1+c(z)-a(z))}{a(z)}\chi_1(z) + (a(z)-1)\chi_3(z) \end{bmatrix}; g(z) = \frac{1}{2\pi i}\begin{bmatrix} \chi_1(z) + \chi_3(z) \\ \chi_2(z) + \chi_4(z) \end{bmatrix}$$

$a(z), b(z), c(z), d(z)$ are given in terms of parabolic cylinder functions.

Results

- The integrable kernel on $L^2(i\mathbb{R})$ is given by

$$K(z,w) = \frac{f^T(z)g(w)}{2\pi i(z-w)} \tag{6}$$

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$$\tau[s] = \det(\mathbb{1} - K) \tag{7}$$

- $\tau[s]$ is related to the JMU τ-function as

$$\partial_s \ln \tau[s] = \partial_s \ln \tau_{JMU} + \left[\frac{2i\nu}{3} + \frac{\nu^2}{s}\right] + A(\nu) \tag{8}$$

where $\nu = -\frac{1}{2\pi i}\ln(1 - s_1 s_3)$ and s_1, s_3 are Stokes' parameters and s is the PII parameter and $A(\nu)$ is a non-vanishing depending only on ν.

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Extreme gap problems in random matrix theory

Renjie Feng

BIMCR, Peking University

Previous results I: smallest gaps for CUE

Let $e^{i\theta_1}, \cdots, e^{i\theta_n}$ be n eigenvalues of CUE, consider

$$\chi_n = \sum_{i=1}^{n} \delta_{(n^{4/3}(\theta_{i+1}-\theta_i),\theta_i)}.$$

Theorem (Vinson, Soshnikov, Ben Arous-Bourgade)

χ_n tends to a Poisson process χ with intensity

$$\mathbb{E}\chi(A \times I) = \left(\frac{1}{24\pi}\int_A u^2 du\right)\left(\int_I \frac{du}{2\pi}\right).$$

The kth smallest gap has limiting density

$$\frac{3}{(k-1)!} x^{3k-1} e^{-x^3}.$$

Previous results II: smallest gaps for GUE

For GUE

$$\chi_n = \sum_{i=1}^{n} \delta_{(n^{\frac{4}{3}}(\lambda_{i+1}-\lambda_i),\lambda_i)} 1_{|\lambda_i|<2-\eta}$$

Theorem (Ben Arous-Bourgade, AOP 2013)

χ_n tends to a Poisson process χ with intensity

$$\mathbb{E}\chi(A \times I) = (\frac{1}{48\pi^2}\int_A u^2 du)(\int_I (4-x^2)^2 dx),$$

where $A \subset \mathbb{R}_+$ and $I \subset (-2+\eta, 2-\eta)$.

The kth smallest gap has the limiting density $\frac{3}{(k-1)!}x^{3k-1}e^{-x^3}$, same as CUE.

New results I: smallest gaps for CβE

When β is an positive integer, consider

$$\chi_n = \sum_{i=1}^{n} \delta_{\left(n^{\frac{\beta+2}{\beta+1}}(\theta_{i+1}-\theta_i),\theta_i\right)}$$

Theorem [F.-Wei]

χ_n tends to a Poisson point process χ with intensity

$$\mathbb{E}\chi(A \times I) = \frac{A_\beta |I|}{2\pi} \int_A u^\beta du,$$

where $A_\beta = (2\pi)^{-1} \frac{(\beta/2)^\beta (\Gamma(\beta/2+1))^3}{\Gamma(3\beta/2+1)\Gamma(\beta+1)}$. For COE, CUE and CSE,

$$A_1 = \frac{1}{24},\ A_2 = \frac{1}{24\pi},\ A_4 = \frac{1}{270\pi}.$$

New results II: smallest gaps for GOE

For GOE

$$\chi^{(n)} = \sum_{i=1}^{n-1} \delta_{n^{3/2}(\lambda_{(i+1)}-\lambda_{(i)})}$$

Theorem [F.-Tian-Wei]

$\chi^{(n)}$ converges to a Poisson point process χ with intensity

$$\mathbb{E}\chi(A) = \frac{1}{4}\int_A u du.$$

the limiting density of the kth smallest gap is

$$\frac{2}{(k-1)!} x^{2k-1} e^{-x^2},$$

same as COE.

Conjecture: CβE and GβE share the same smallest gaps.

Previous III: order of largest gaps

For CUE and interior of GUE, m_k is the kth largest gap,

Theorem (Ben Arous-Bourgade, AOP 2013)

For any $p > 0$ and $l_n = n^{o(1)}$, one has

$$m_{l_n} \times \frac{n}{\sqrt{32 \ln n}} \xrightarrow{L^p} 1.$$

New results III: fluctuation of largest gaps

Theorem (F.-Wei)

Let's denote m_k as the k-th largest gap of CUE, and

$$\tau_k^n = (2\ln n)^{\frac{1}{2}}(nm_k - (32\ln n)^{\frac{1}{2}})/4 - (3/8)\ln(2\ln n),$$

then $\{\tau_k^n\}$ tends to a Poisson process and τ_k^n has the limit of the Gumbel distribution,

$$\frac{e^{k(c_1-x)}}{(k-1)!}e^{-e^{c_1-x}}.$$

Here, $c_1 = \frac{1}{12}\ln 2 + 3\zeta'(-1) + \ln\frac{\pi}{2}$.

New results III: fluctuation of largest gaps

Theorem (F.-Wei)

Let's denote m_k^* as the k-th largest gap of GUE, $S(I) = \inf_I \sqrt{4 - x^2}$ and

$$\tau_k^* = (2\ln n)^{\frac{1}{2}}(nS(I)m_k^* - (32\ln n)^{\frac{1}{2}})/4 + (5/8)\ln(2\ln n),$$

$\{\tau_k^*\}$ tends to a Poisson process and has the limit of the Gumbel distribution,

$$\frac{e^{k(c_2 - x)}}{(k-1)!} e^{-e^{c_2 - x}}.$$

Here, $c_2 = \frac{1}{12}\ln 2 + 3\zeta'(-1) + M_0(I)$ depending on I, where
$M_0(I) = (3/2)\ln(4 - a^2) - \ln(4|a|)$ if $a + b < 0$,
$M_0(I) = (3/2)\ln(4 - b^2) - \ln(4|b|)$ if $a + b > 0$,
$M_0(I) = (3/2)\ln(4 - a^2) - \ln(2|a|)$ if $a + b = 0$.

Extreme gaps IV: universality of extreme gaps

Recently, our results are generalized for Hermitian/symmetric Wigner matrices with mild assumptions.

- P. Bourgade, Extreme gaps between eigenvalues of Wigner matrices, arXiv:1812.10376.
- B. Landon, P. Lopatto, J. Marcinek, Comparison theorem for some extremal eigenvalue statistics, arXiv:1812.10022.

References

- Large gaps of CUE and GUE, arXiv:1807.02149.
- Small gaps of circular beta-ensemble, arXiv:1806.01555
- Small gaps of GOE, arXiv:1901.01567.

MATRIX MODELS FOR CLASSICAL GROUPS AND TOEPLITZ+HANKEL MINORS WITH APPLICATIONS TO CHERN-SIMONS THEORY AND FERMIONIC MODELS

David García-García

Joint work with Miguel Tierz (arXiv:1901.08922)

MINORS OF TOEPLITZ+HANKEL MATRICES

$$\frac{1}{N!}\int_{[0,2\pi]^N} |\Delta(e^{i\theta})|^2 \prod_{k=1}^{N} f(e^{i\theta_k})\frac{d\theta_k}{2\pi}$$

MINORS OF TOEPLITZ+HANKEL MATRICES

$$\int_{U(N)} f(M)dM =$$

$$\frac{1}{N!}\int_{[0,2\pi]^N} |\Delta(e^{i\theta})|^2 \prod_{k=1}^{N} f(e^{i\theta_k})\frac{d\theta_k}{2\pi} =$$

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$$\det_{N\times N}\begin{pmatrix} d_0 & d_{-1} & d_{-2} & d_{-3} & d_{-4} & d_{-5} & \cdots \\ d_1 & d_0 & d_{-1} & d_{-2} & d_{-3} & d_{-4} & \cdots \\ d_2 & d_1 & d_0 & d_{-1} & d_{-2} & d_{-3} & \cdots \\ d_3 & d_2 & d_1 & d_0 & d_{-1} & d_{-2} & \cdots \\ d_4 & d_3 & d_2 & d_1 & d_0 & d_{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

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MINORS OF TOEPLITZ+HANKEL MATRICES

$$\int_{U(N)} s_\lambda(M^{-1}) s_\mu(M) f(M) dM =$$

$$\frac{1}{N!}\int_{[0,2\pi]^N} s_\lambda(e^{-i\theta}) s_\mu(e^{i\theta}) |\Delta(e^{i\theta})|^2 \prod_{k=1}^{N} f(e^{i\theta_k}) \frac{d\theta_k}{2\pi} =$$

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MINORS OF TOEPLITZ+HANKEL MATRICES

$$\int_{G(N)} f(M)dM = \qquad (G(N) = Sp(2N), O(2N), O(2N+1))$$

$$\frac{1}{N!}\int_{[0,2\pi]^N} |\Delta_{G(N)}(e^{i\theta})|^2 \prod_{k=1}^{N} f(e^{i\theta_k})\frac{d\theta_k}{2\pi} =$$

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$$\int_{G(N)} \chi^{\lambda}_{G(N)}(M^{-1})\chi^{\mu}_{G(N)}(M)f(M)dM = \qquad (G(N) = Sp(2N), O(2N), O(2N+1))$$

$$\frac{1}{N!}\int_{[0,2\pi]^N} \chi^{\lambda}_{G(N)}(e^{-i\theta})\chi^{\mu}_{G(N)}(e^{i\theta})|\Delta_{G(N)}(e^{i\theta})|^2 \prod_{k=1}^{N} f(e^{i\theta_k})\frac{d\theta_k}{2\pi} =$$

$$\det_{N\times N}\begin{pmatrix} d_1 - d_3 & d_1 - d_5 & d_3 - d_7 & d_4 - d_8 & \cdots \\ & & & & \\ d_3 - d_5 & d_1 - d_7 & d_1 - d_9 & d_2 - d_{10} & \cdots \\ d_4 - d_6 & d_2 - d_8 & d_0 - d_{10} & d_1 - d_{11} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

SOME RESULTS AND APPLICATIONS

- Factorizations

$$\int_{U(2N)} f(U)dU = \int_{O(2N+1)} f(U)dU \int_{O(2N+1)} f(-U)dU,$$
$$\int_{U(2N+1)} f(U)dU = \int_{Sp(2N)} f(U)dU \int_{O(2N+2)} f(U)dU.$$

- Expansions in terms of Toeplitz minors

$$\det\left(d_{j-k} - d_{j+k}\right)_{j,k=1}^{N} = \frac{1}{2^N} \sum_{\lambda,\mu \in R(N)} (-1)^{(|\lambda|+|\mu|)/2} D_N^{\lambda,\mu}(f).$$

- Chern-Simons theory

$$\int_{G(N)} \Theta(U)dU \qquad \text{Partition function}$$

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$$\int_{G(N)} \chi^{\mu}_{G(N)}(U)\Theta(U)dU \qquad \text{Wilson loop}$$

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- Chern-Simons theory

$$\int_{G(N)} \chi_{G(N)}^{\lambda}(U^{-1})\chi_{G(N)}^{\mu}(U)\Theta(U)dU \qquad \text{Hopf link}$$

Thank you!

Semigroups for One-Dimensional Schrödinger Operators with Multiplicative White Noise[1]

Pierre Yves Gaudreau Lamarre

Princeton University

[1] Based on a paper of the same name; arXiv:1902.05047.

Let ξ be a Gaussian white noise on $\mathbb{R}^d$, and $V : \mathbb{R}^d \to \mathbb{R}$ be a deterministic function.

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Consider the random Schrödinger operator

$$\hat{H} := \left(-\tfrac{1}{2}\Delta + V\right) + \xi.$$

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Consider the random Schrödinger operator

$$\hat{H} := \left(-\tfrac{1}{2}\Delta + V\right) + \xi.$$

Problem. Develop a semigroup theory for $\hat{H}$, i.e.,

$$\left\{ \mathrm{e}^{-t\hat{H}} : t > 0 \right\}$$

1. SPDEs: $u(t,x) := \mathrm{e}^{-t\hat{H}} u_0(x)$ solves

$$\partial_t u = \left(\tfrac{1}{2}\Delta - V\right) u + \xi u, \qquad u(0,x) = u_0(x).$$

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2. Spectral Analysis of SPDEs:

$$\mathrm{e}^{-t\hat{H}}u_0 = \sum_{k=1}^{\infty} \mathrm{e}^{-t\lambda_k(\hat{H})}\langle\psi_k(\hat{H}), u_0\rangle\psi_k(\hat{H}).$$

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$$\partial_t u = \left(\tfrac{1}{2}\Delta - V\right)u + \xi u, \qquad u(0,x) = u_0(x).$$

2. Spectral Analysis of SPDEs:

$$\mathrm{e}^{-t\hat{H}}u_0 = \sum_{k=1}^{\infty} \mathrm{e}^{-t\lambda_k(\hat{H})}\langle \psi_k(\hat{H}), u_0\rangle \psi_k(\hat{H}).$$

3. Feynman-Kac formula:

$$\begin{aligned}&\mathrm{e}^{-t\hat{H}}f(x)\\ &= \mathbf{E}^x\left[\exp\left(-\int_0^t V\big(B(s)\big) + \xi\big(B(s)\big)\mathrm{d}s\right) f\big(B(t)\big)\right].\end{aligned}$$

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It is thus nontrivial to define

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2. Discuss applications in random matrix theory and SPDEs with multiplicative white noise.

3. Discuss partial results in higher dimensions, and connection to regularity structures/paracontrolled calculus/renormalization of SPDEs.

The longest increasing subsequence problem for correlated random variables

J. Ricardo G. Mendonça

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Escola de Artes, Ciências e Humanidades, Universidade de São Paulo, SP, Brasil

L'Intégrabilité et l'Aléatoire en Physique Mathématique et en Géométrie
CIRM, Marseille Luminy, France, 8–12 April 2019

The longest increasing subsequence problem

LIS problem
To find an increasing subsequence of maximum length of a finite sequence of n elements taken from a partially ordered set

$$(a_1, a_2, \ldots, a_n) \quad \Rightarrow \quad LIS = (a_{i_1}, a_{i_2}, \ldots a_{i_k})$$

such that $a_{i_1} \leqslant a_{i_2} \leqslant \cdots \leqslant a_{i_k}$ with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$

Applications

- Bioinformatics — gene sequence alignment
- Computational linguistics — querying, string matching, diff
- Statistical process control — trend marker

To find one representative LIS of a sequence is an $O(n \log n)$ task

LIS problem for random permutations

How does the length L_n of the LIS of random permutations grow with n?

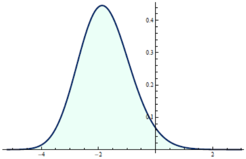
(S. Ulam, ∼1960)

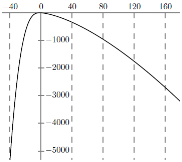
Example

$\sigma = (2\,4\,3\,5\,1\,7\,6\,9\,8) \;\Rightarrow\; LIS = \left\{(2\,3\,5\,6\,8),\ (2\,4\,5\,7\,9),\ \cdots\right\}, L_n = 5$

Solution took nearly 40 years to complete (Baik, Deift & Johansson, 1999)

$$L_n \sim 2\sqrt{n} + n^{1/6}\chi_2$$





with $\chi_2 \sim TW_2$, the distribution for the fluctuations of the largest eigenvalue of a random GUE matrix (Tracy & Widom, 1993)

The LIS of random walks

A random walk (RW) of length n is the r. v.

$$S_n = X_1 + X_2 + \cdots + X_n$$

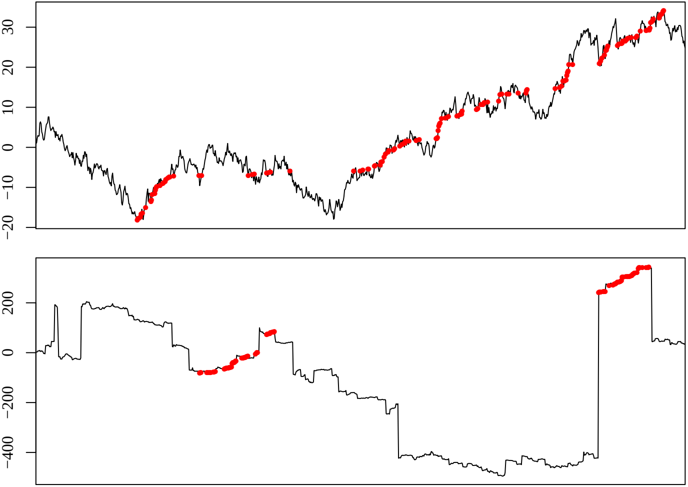
with X_k i.i.d. according to some to some zero-mean, symmetric p.d.f.

$\mathcal{S}_n = (S_1, S_2, \dots, S_n)$ is a sequence of ***correlated*** random variables

How does the length of the LIS of $\mathcal{S}_n$ scales with n and the law of increments?

Surprisingly, this problem has been posed only recently in the literature (Angel, Balkay & Peres, 2014; Pemantle & Peres, 2016)

What we are looking for



Rigorous results on the LIS of random walks

LIS of RW with finite variance (Angel, Balkay & Peres, 2014)

Let $S_n = \sum_{i=1}^{n} X_i$ be a RW on $\mathbb{R}$ with i.i.d. X_i such that $\mathbb{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = 1$. Then for all $\varepsilon > 0$ and large enough n,

$$c\sqrt{n} \leqslant \mathbb{E}(L_n) \leqslant n^{\frac{1}{2}+\varepsilon}.$$

LIS of RW with infinite variance (Pemantle & Peres, 2016)

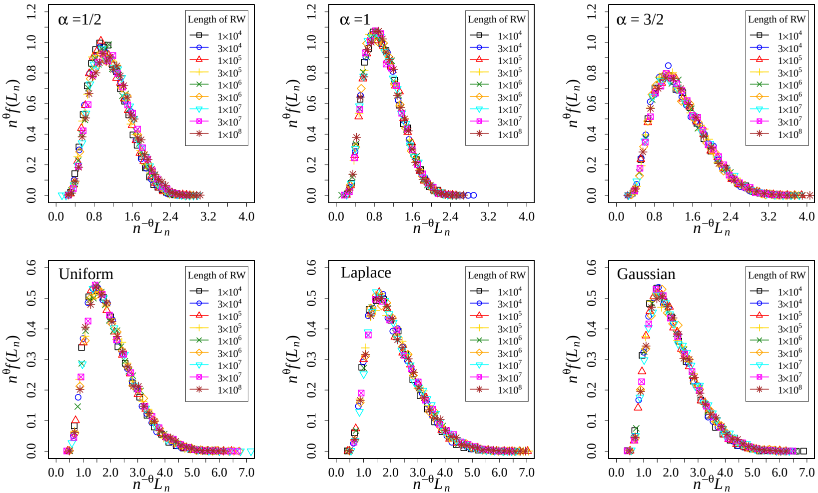
If the steps X_i are i.i.d. according to a symmetric α-stable law with a sufficiently small index $\alpha \leqslant 1$, then

$$n^{\beta_0 - o(1)} \leqslant \mathbb{E}(L_n) \leqslant n^{\beta_1 + o(1)},$$

with $\beta_0 = 0.690093\cdots$ and $\beta_1 = 0.814835\cdots$ (not sharp).

Numerical experiments

Numerical evidence suggests that the p.d.f. $f(L_n) = n^{-\theta} g(n^{-\theta} L_n)$.



Correction to scaling

Conjectural asymptotics (JR, 2017)

The length L_n of the LIS of random walks with step lengths of finite variance scales with n like

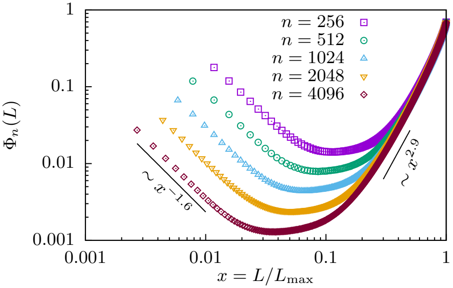
$$L_n \sim \frac{1}{e}\sqrt{n}\ln n + \frac{1}{2}\sqrt{n} + \text{lower order terms}$$

Recent data (Börjes, Schawe & Hartmann, 2019) seem to confirm this scaling over several orders of magnitude

Large deviation function

The empirical large deviation rate function $\Phi_n(L)$ associated with the distribution of L_n ($n \gg 1$) observes

$$f(L_n > L) \asymp \exp(-n\Phi_n(L)) \sim \begin{cases} L^{-1.6} & \text{(maybe } L^{-3/2}\text{?) in the left tail} \\ L^{2.9} & \text{(maybe } L^{3}\text{?) in the right tail} \end{cases}$$



The distribution $f(u)$ (or $g(u)$) remains unknown

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- O. Angel, R. Balka & Y. Peres, Increasing subsequences of random walks, *Math. Proc. Cambridge Phil. Soc.* **163** (1), 173–185 (2017)
- J. R. G. Mendonça, Empirical scaling of the length of the longest increasing subsequences of random walks, *J. Phys. A: Math. Theor.* **50** (8), 08LT02 (2017)
- J. Börjes, H. Schawe & A. K. Hartmann, Large deviations of the length of the longest increasing subsequence of random permutations and random walks, *Phys. Rev. E* **99** (4), 042104 (2019)
- J. R. G. Mendonça, Leading asymptotic behavior of the length of the longest increasing subsequences of heavy-tailed random walks, in preparation (2019)

Merci beaucoup!

Planar orthogonal polynomials with logarithmic singularities in the external potential

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Luminy, France

April 9th, 2019

Planar Orthogonal Polynomials

Let $p_n(z)$ be the monic polynomial of degree n satisfying the orthogonality condition:

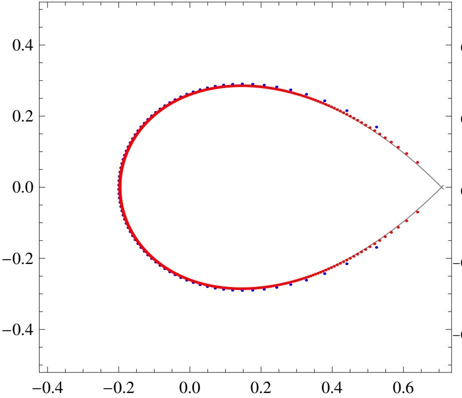
$$\int_{\mathbb{C}} p_n(z)\,\overline{p_m(z)}\,\mathrm{e}^{-NQ(z)}\,\mathrm{d}A(z) = h_n\delta_{nm}, \quad n, m \geq 0,$$

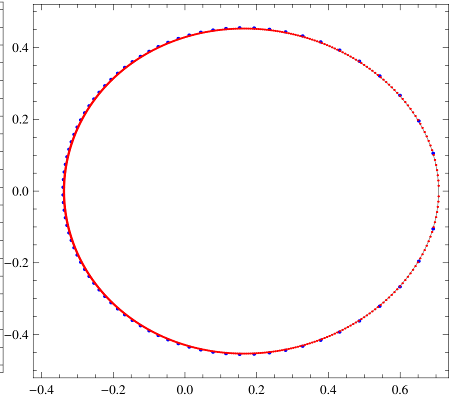
where the external potential is given by

$$Q(z) = |z|^2 + 2\sum_{j=1}^{\nu} \frac{c_j}{N} \log \frac{1}{|z - a_j|},$$

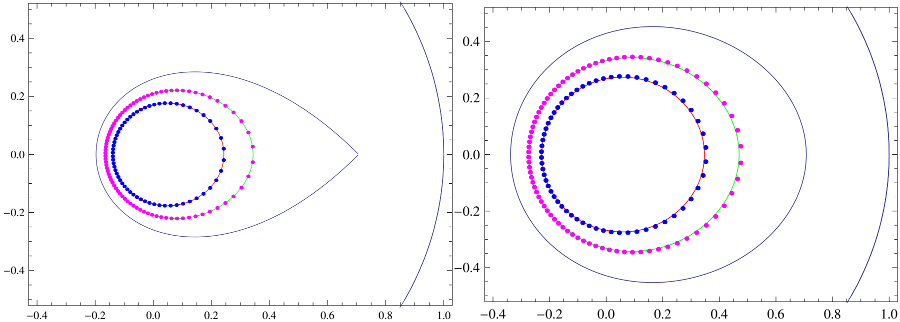
where $\{c_1, \cdots, c_\nu\}$ are positive integers and $\{a_1, \cdots, a_\nu\}$ are distinct points in $\mathbb{C}$.

For $\nu = 1$, the zeros of orthogonal polynomials for $c = 1$. The left is for $a > 1$ and the right is for $a < 1$.

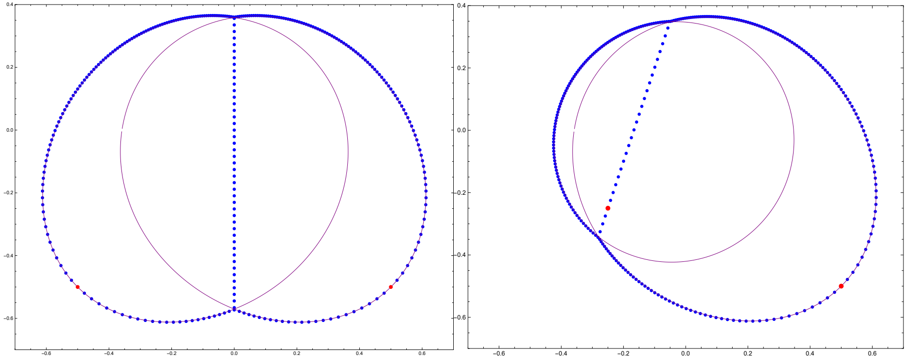




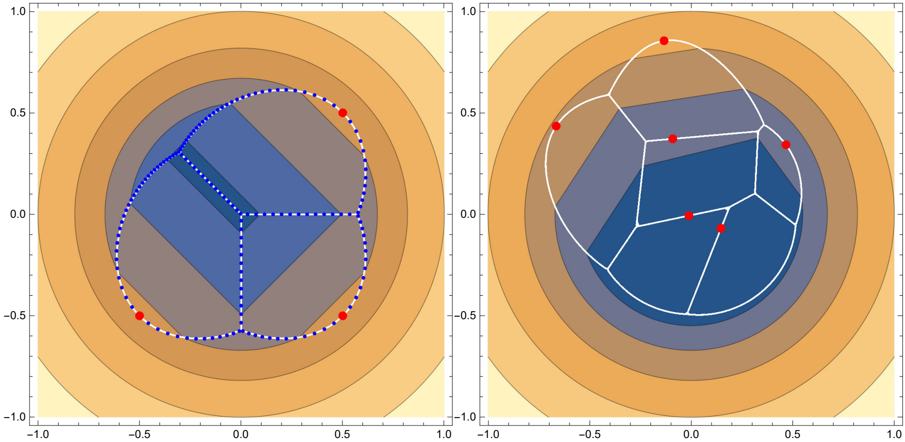
The zeros of orthogonal polynomials for $c = e^{-\eta n}$, where $\eta = 0.4$ (blue) and $\eta = 0.2$ (magenta).



The limiting locus (Purple lines).



The limiting locus for $\nu = 3$ and $\nu = 6$.



Thank You!

A class of unbounded solutions of the Korteweg-de Vries equation

A. A. Minakov
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joint work with B. A. Dubrovin
arxiv: 1901.07470

Integrability and Randomness in Mathematical Physics and Geometry
Marseille, France
April 8 – 12, 2019

Korteweg-de Vries equation

$$u_t(x,t) + u(x,t)u_x(x,t) + \frac{1}{12}u_{xxx}(x,t) = 0, \quad x \in \mathbb{R}, t \geq 0.$$

Lax pair representation

$$\varphi_{xx} + 2u\varphi = \lambda\varphi,$$
$$\varphi_t = \frac{u_x}{6}\varphi - \frac{\lambda + u}{3}\varphi_x.$$

Question: Can this be applied to solve an initial value problem (ivp) with an initial function $u(x,0) = u_0(x)$?

Answer: As a rule, if $u_0(x) \to u_+(x,t_0)$ as $x \to +\infty$, and $u_0(x) \to u_-(x,t_0)$ as $x \to -\infty$, where $u_\pm(x,t)$ are exact solutions of the KdV with known solutions of the Lax pair, then the answer is: yes.

Goal: to solve ivp for KdV with $u_0(x)$ from a class of unbounded as $x \to \pm\infty$ functions.

Known examples of exact solutions of KdV:

- $u_\pm(x,t) \equiv 0$. The corresponding continuous spectrum is two folded $(-\infty, 0]$.

- $u_\pm(x,t) \equiv c_\pm$, where $c_\pm$ are constants. The continuous spectrum is partially one or two folded.

- $u_\pm(x,t)$ are the so-called *finite gap* (quasi periodic) solutions of KdV, who bear their name after the form of the spectrum (B. Dubrovin, S. Novikov, P. Lax, A. Its, V. Matveev, V. Marchenko, B. Levitan, H. Knörer, E. Trubowitz). The solutions of the Lax pair are the Baker-Akhiezer functions, which are meromorphic functions on the corresponding Riemann surface. The typical spectrum has the following shape:

- $u_\pm(x,t) = U(x,t)$, where the $U(x,t) \sim \sqrt[3]{-x/6}$ as $x \to \pm\infty$ is some particular function, defined through a Riemann-Hilbert problem. The corresponding spectrum is one folded real line $\mathbb{R}$.

Scheme of integration of the initial value problem:

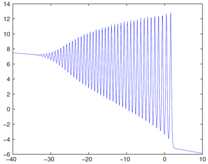
Usual scheme for integrating the ivp for KdV consists of two steps:

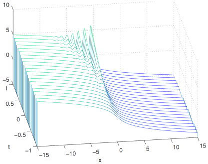
- **Forward scattering transform:** Given $u_0(x)$, construct the solutions of the Lax pair at the time $t=0$, and construct the associated spectral functions, and then
- **Inverse scattering transform:** Given the spectral functions, plug in the evolution in time t and reconstruct the solution $u(x,t)$ of the ivp.

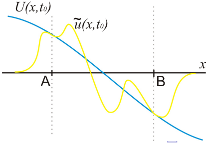
This is known in the case of an initial function which is
• a perturbation of zero (Gardner Green Kruskal Miura),
• periodic initial function (V. Marchenko, B. Levitan),
• step-like perturbations of finite-gap functions (E. Khruslov, I. Egorova, G. Teschl),
• rapidly vanishing at positive half-axis, arbitrary on the left one (A. Rybkin).

Our goal here is to develop such a theory for $u_0(x)$, which is a perturbation of $U(x,t_0)$.

$U(x,t=4)$, from T. Grava, A. Kapaev, C. Klein, '15





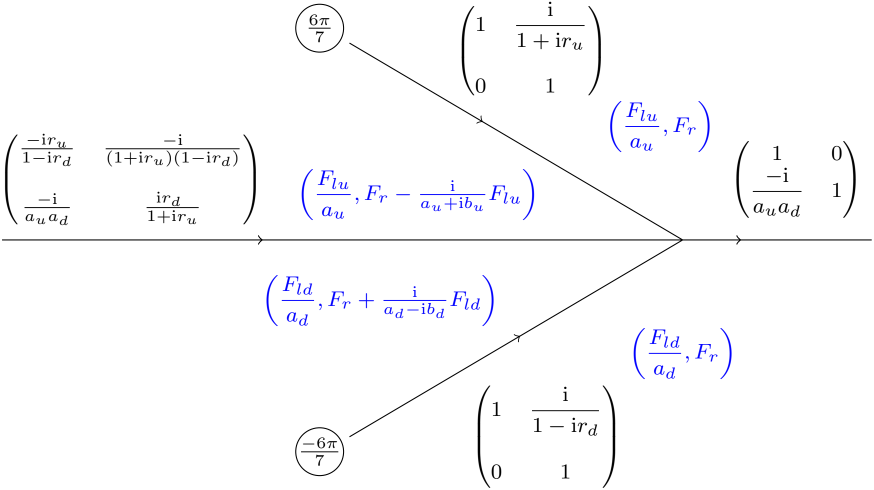


Main features of analysis

- the Jost solutions of the Lax operator are not similar:
  left solution $f_-(x;\lambda) \in \mathcal{H}(\mathbb{C}\setminus\mathbb{R})$ is discontinuous across $\lambda \in \mathbb{R}$,
  right solution $f_+(x;\lambda) \in \mathcal{H}(\mathbb{C}\setminus\mathbb{R})$ is an entire function;
- as a consequence, there is only one (scattering) relation between $f_\pm$;
- only one spectral function, $a(\lambda)$ is determined through that (scattering) relation,

$$f_+(x;\lambda) = \mathrm{i}a(\lambda - \mathrm{i}0) f_-(x;\lambda + \mathrm{i}0) - \mathrm{i}a(\lambda + \mathrm{i}0) f_-(x;\lambda - \mathrm{i}0);$$

- another spectral function, $b(\lambda)$, is determined through asymptotics as $x \to +\infty$ of $f_-(x;\lambda)$;
- both $a(\lambda), b(\lambda)$ are $\in \mathcal{H}(\mathbb{C}\setminus\mathbb{R})$ and discontinuous across $\lambda \in \mathbb{R}$;
- to reconstruct solution $u(x,t)$ of KdV from $a(\lambda)$, $b(\lambda)$, one needs to define a piece-wise meromorphic matrix-valued function in the complex plane, using as entries linear combinations of f_-, f_+;
- we use compactness of perturbation in order to construct the above matrix;
- poles of the conjugation problem solved by the above matrix are caused not by zeros of $a(\lambda)$ ($a(\lambda) \neq 0$ everywhere), but by zeros of $a(\lambda) + ib(\lambda)$ in the upper half-plane.
- instead of $|R|^2 + |T|^2 = 1$, or $|r|^2 \equiv \frac{|b|^2}{|a|^2} = 1 - \frac{1}{|a|^2}$,we have
  $-\mathrm{i}\,(r(\lambda + \mathrm{i}0) - r(\lambda - \mathrm{i}0)) = 1 - \frac{1}{|a(\lambda)|^2}, \lambda \in \mathbb{R}.$



$\frac{6\pi}{7}$
$\begin{pmatrix} 1 & \frac{\mathrm{i}}{1+\mathrm{i}r_u} \\ 0 & 1 \end{pmatrix}$
$\left(\frac{F_{lu}}{a_u}, F_r\right)$
$\begin{pmatrix} \frac{-\mathrm{i}r_u}{1-\mathrm{i}r_d} & \frac{-\mathrm{i}}{(1+\mathrm{i}r_u)(1-\mathrm{i}r_d)} \\ \frac{-\mathrm{i}}{a_u a_d} & \frac{\mathrm{i}r_d}{1+\mathrm{i}r_u} \end{pmatrix}$
$\left(\frac{F_{lu}}{a_u}, F_r - \frac{\mathrm{i}}{a_u+\mathrm{i}b_u}F_{lu}\right)$
$\begin{pmatrix} 1 & 0 \\ \frac{-\mathrm{i}}{a_u a_d} & 1 \end{pmatrix}$
$\left(\frac{F_{ld}}{a_d}, F_r + \frac{\mathrm{i}}{a_d-\mathrm{i}b_d}F_{ld}\right)$
$\left(\frac{F_{ld}}{a_d}, F_r\right)$
$\begin{pmatrix} 1 & \frac{\mathrm{i}}{1-\mathrm{i}r_d} \\ 0 & 1 \end{pmatrix}$
$\frac{-6\pi}{7}$

Universality of the conditional measure of the Bessel Process

Joint work with Marco Stevens

L.D. Molag

KU Leuven

March 12, 2019, Marseille

- Loosely speaking, a point process is **rigid** when the position of the points *outside* a certain interval I, a.s. determine the number of points *inside* that interval.

- When a point process is indeed rigid, one can consider the induced finite point process on I, called the **conditional measure**.
- It is a recent find by Subhroshekhar Ghosh that the sine process is rigid. It is a more recent find by Alexander Bufetov that also the Bessel and Airy process are rigid, and in particular that the conditional measures of these three processes a.s. are orthogonal polynomial ensembles.

- Bufetov posed the question, what would happen when we let I grow to cover the whole space (i.e. $R \to \infty$ in the figure)? Would we end up with the original point process?
- This question was answered affirmatively by Arno Kuijlaars and Erwin Miña-Díaz for the sine process.
  That is: the correlation kernel of the conditional measure on $[-R, R]$ a.s. converges back to the sine kernel as $R \to \infty$.
- Marco Stevens and I proved that this also holds for the Bessel process.
  That is: the correlation kernel of the conditional measure on $[0, R]$ a.s. converges back to the Bessel kernel.

Our approach is as follows:

1 Find the behavior of the (increasingly ordered) points (p_n) in a typical configuration $X = \{p_1, p_2, \ldots\}$ of the Bessel process.
2 Approximate the weight of the OP ensemble by a convenient weight.
3 Relate the approximating OP ensemble to a Riemann-Hilbert problem and solve it using the Deift-Zhou steepest decent method.
4 Use a technique by Lubinsky to find the limit of the correlation kernel corresponding to the original OP weight.

Matrix models and isomonodromic tau functions

Integrability and Randomness in Mathematical Physics
CIRM, Luminy, 8-12 April 2019

Giulio Ruzza (SISSA, Trieste)
joint work with Marco Bertola (SISSA, Trieste / Concordia University, Montreal)

Matrix models & multipoint correlators

Matrix integrals $Z = Z(t_1, t_2, ...)$ as generating functions of algebro geometric - combinatorial objects: (connected) multipoint correlators

$$\left.\frac{\partial^s \log Z}{\partial t_{\ell_1} \cdots \partial t_{\ell_s}}\right|_{t_*=0}.$$

Examples:

- GUE: $Z_n = \dfrac{\int_{\mathcal{H}_n} e^{\mathrm{tr} \sum_{\ell\geq 3} t_\ell M^\ell} e^{-\mathrm{tr}\, \frac{M^2}{2}} \mathrm{d}M}{\int_{\mathcal{H}_n} e^{-\mathrm{tr}\, \frac{M^2}{2}} \mathrm{d}M}$ (*ribbon graphs* [Bessis, Itzykson & Zuber, 1980])
- LUE: $Z_n(m) = \dfrac{\int_{\mathcal{H}_n^+} e^{\mathrm{tr} \sum_{\ell\geq 1} t_\ell M^\ell} e^{-\mathrm{tr}\, M} \det^{m-n} M \mathrm{d}M}{\int_{\mathcal{H}_n^+} e^{-\mathrm{tr}\, M} \det^{m-n} M \mathrm{d}M}$ (*monotone Hurwitz numbers* [Cunden, Dahlqvist & O' Connell, 2018])
- Kontsevich matrix integral:

$$Z_n(\Lambda) = \frac{\int_{\mathcal{H}_n} e^{\mathrm{tr}(\frac{M^3}{3} - \Lambda M^2)} \mathrm{d}M}{\int_{\mathcal{H}_n} e^{\mathrm{tr}(-\Lambda M^2)} \mathrm{d}M}$$ (*psi-classes intersection numbers on* $\overline{\mathcal{M}_{g,s}}$ [Kontsevich, 1992])

and various generalizations (*generalized Kontsevich models*).

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(*psi-classes intersection numbers on* $\overline{\mathcal{M}_{g,s}}$ [Kontsevich, 1992])

and various generalizations (*generalized Kontsevich models*).

Results: formulæ for multipoint correlators

Q: How to effectively compute these numbers?

Recently, formulæ of the kind

$$\sum_{\ell_1,\ldots,\ell_s} \frac{1}{x_1^{\ell_1+1}\cdots x_s^{\ell_s+1}} \frac{\partial^s \log Z}{\partial t_{\ell_1}\cdots\partial t_{\ell_s}}\Big|_{t_*=0} = -\sum_{\sigma\in\mathfrak{S}_s/\mathfrak{C}_s} \frac{\mathrm{tr}\Big(R(x_{\sigma(1)})\cdots R(x_{\sigma(s)})\Big)}{(x_{\sigma(1)}-x_{\sigma(2)})\cdots(x_{\sigma(s)}-x_{\sigma(1)})} - \frac{\delta_{s,2}}{(x_1-x_2)^2}.$$

have been found, for the generating functions of connected correlators (*matrix resolvent approach*, *topological recursion*, **identification with isomonodromic tau functions**).

More formulæ in the poster!

Bertola, Dubrovin & Yang, 2016 - Dubrovin & Yang, 2017 - R & Bertola, 2018 - R & Bertola, 2019 - ...,

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More formulæ in the poster!

Bertola, Dubrovin & Yang, 2016 - Dubrovin & Yang, 2017 - R & Bertola, 2018 - R & Bertola, 2019 - ...,

Main technical tool: isomonodromic tau functions

Isomonodromic system: monodromy-preserving deformations of a rational connection on $\mathbb{P}^1$. Independent deformation parameters are called **isomonodromic times** and are the argument of the **isomonodromic tau function** (Sato, Jimbo–Miwa–Ueno), which plays the role of a generalized determinant (Malgrange–Miwa).

Isomonodromic approach: identification of matrix integrals (and their limits) with suitable isomonodromic tau functions.

Applications. Let the matrix integral $Z(\mathbf{t})$ (depending on parameters $\mathbf{t} = (t_1, t_2, \ldots)$) be identified with the tau function of the isomonodromic system

$$\partial_x \Psi(x,\mathbf{t}) = A(x,\mathbf{t}) \cdot \Psi(x,\mathbf{t}), \;\; \partial_{t_\ell} \Psi(x,\mathbf{t}) = \Omega_\ell(x,\mathbf{t}) \cdot \Psi(x,\mathbf{t}), \;\; \Psi(x,\mathbf{t}) = Y(x,\mathbf{t}) \cdot e^{\Theta(x,\mathbf{t})}, \;\; Y(x,\mathbf{t}) \sim \mathbf{1} + \mathcal{O}(x^{-1}).$$

(For simplicity, the only pole of A is at $x = \infty$.)

- Limits of the matrix integral admit rigorous analytic interpretation in this setting (the limit itself is in turn interpreted as an isomonodromic tau function).
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$$\partial_{t_\ell} \log Z(\mathbf{t}) = -\operatorname*{res}_{x=\infty} \operatorname{tr}(Y^{-1} \cdot \partial_x Y \cdot \partial_{t_j} \Theta)$$
allows to write the non-recursive formulæ of previous last slide.
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Thank you for your time!

On insertion of a point charge in the random normal matrix model

Joint work with Yacin Ameur and Nam-Gyu Kang

Seong-Mi Seo (KIAS)

April 09 2019

Random normal matrix model with a logarithmic singularity

Consider n point charges on $\mathbb{C}$ influenced by an external potential V_n where

$$V_n(\zeta) = Q(\zeta) - \frac{2c}{n}\log|\zeta|, \quad c > -1.$$

- Energy of a configuration $(\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n$:

$$H_n(\zeta_1, \cdots, \zeta_n) = \sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^{n} V_n(\zeta_j).$$

- The Boltzmann-Gibbs distribution at inverse temperature $\beta = 1$:

$$\frac{1}{Z_n} e^{-H_n(\zeta_1, \cdots, \zeta_n)}, \quad \zeta_1, \cdots, \zeta_n \in \mathbb{C}.$$

With a sufficient growth condition on the external potential, the eigenvalues (particles) condensate on a compact set S, called the droplet, as n tends to ∞.

Effects of inserting a point charge

- Microscopic properties of eigenvalues at the singularity in the bulk.
- Difference between the one point functions with and without insertion: balayage operation.
- Gaussian convergence of the logarithmic potential.

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Bulk universality for dominant radial potentials

Consider the external potential

$$V_n(\zeta) = Q_r(\zeta) + \mathrm{Re} \sum_{j=1}^{d} t_j \zeta^j - \frac{2c}{n} \log |\zeta|,$$

where Q_r is a radially symmetric function such that $Q_r = |\zeta|^{2\lambda} + O(|\zeta|^{2\lambda+\epsilon})$ near 0 for $\lambda > 0$. Assume that $0 \in \mathrm{Int}\, S$. Then the limiting correlation kernel of the rescaled system

$$z_j = n^{1/2\lambda} \zeta_j$$

can be described in terms of the two-parametric Mittag-Leffler function which depends on λ and c.

Non-Hermitian ensembles and Painlevé critical asymptotics

Alfredo Deaño[†] and Nick Simm[*]

[†]Department of Mathematics, University of Kent, UK
[*]Department of Mathematics, University of Sussex, UK

CIRM conference: Integrability and Randomness in Mathematical Physics, April 2019

The model

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Consider the normal matrix model

$$\mathcal{Z}_N(t) = \int_{\mathbb{C}^N} \prod_{1 \le k < j \le N} |z_k - z_j|^2 \prod_{j=1}^{N} e^{-NV_t(z_j)} \, d^2 z_j$$

where (Balogh, Merzi '13, ..., Bertola, Rebelo, Grava '18)

$$V_t(z) = |z|^{2s} - t(z^s + \overline{z}^s), \qquad t \in \mathbb{R}, \quad s \in \mathbb{N}.$$

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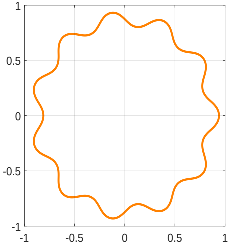
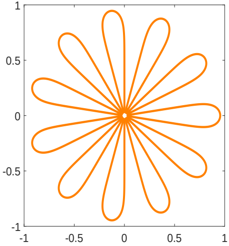
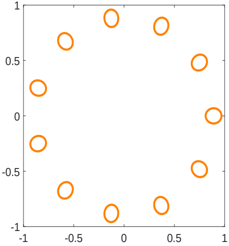
The equilibrium measure for this potential:







The figures are $t < t_{\rm c}$, $t = t_{\rm c}$ and $t > t_{\rm c}$, $d = 11$ and $t_{\rm c} = 1/\sqrt{s}$.

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- **Painlevé integrability:** the partition function $\mathcal{Z}_N(t)$ can be represented *exactly* in terms of a Painlevé V τ-function.
- **Double scaling limit $N \to \infty$:** scaling near $t \sim t_{\mathrm{c}}$ we calculate an asymptotic expansion in terms of a Painlevé IV τ-function.
- **Moments of characteristic polynomials of non-Hermitian matrices:** the results are intimately related to averages, say of Ginibre type. (e.g. Akemann and Vernizzi '02, Fyodorov and Khoruzhenko '06, Forrester and Rains '08):

$$\mathcal{Z}_N(t) = \prod_{l=0}^{s-1} \mathbb{E}_{\mathrm{Gin}}\left(|\det(A - t\sqrt{s})|^{-\gamma_l}\right)$$

but now the exponents may not be even integers (*fractional moments*).

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Thank you for listening. See our poster for more detail and please ask us any questions.