Moments of Random Matrices and Hypergeometric Orthogonal Polynomials

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Integrability and Randonmness in
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Outline

Moments of Random Matrices

Moments & Hypergeometric OP's

Wronskians & Hypergeometric OP's

Moments for $\beta = 1$ and $\beta = 4$

Conclusions

Moments of Random Matrices

- j,p.d.f. of the eigenvalues at the classical RMT Ensembles

$$\frac{1}{C_{n,\beta}} \prod_{j=1}^{n} w_\beta(x_j)\chi_I(x_j) \prod_{1\le j<k\le n} |x_k - x_j|^\beta dx_1 \cdots dx_n$$

$\beta = 1, 2, 4$, $I = \mathbb{R}$, $I = \mathbb{R}_+$ and $I = [0, 1]$

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$\beta = 1, 2, 4$, $I = \mathbb{R}$, $I = \mathbb{R}_+$ and $I = [0, 1]$

- The weights are

$$w_\beta(x) = \begin{cases} e^{-(\beta/2)x^2} & \text{Hermite} \\ x^{(\beta/2)(m-n+1)-1}\, e^{-(\beta/2)x} & \text{Laguerre} \\ (1-x)^{\frac{\beta}{2}(m_1-n+1)-1}\, x^{\frac{\beta}{2}(m_2-n+1)-1} & \text{Jacobi} \end{cases}$$

Moments of Random Matrices

- GUE ensemble

$$P_n(x_1, \ldots, x_n) = C_n \prod_{j=1}^{n} \exp\left(-x_j^2\right) \prod_{1 \le j < k \le n} |x_k - x_j|^2$$

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- The k-point correlation function

$$R_k(x_1, \dots, x_k) = \det_{k \times k} \left[K_n(x_i, x_j)\right]_{i,j=1}^{k}$$

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- The kernel is expressed in terms of Hermite polymonials,

$$K_n(x,y) = e^{-\left(x^2+y^2\right)/2} \sum_{k=0}^{n-1} \frac{H_k(x)H_k(y)}{\sqrt{\pi}2^k k!}$$

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$$

$$\int_{-\infty}^{\infty} H_k(x)H_j(x)e^{-x^2}dx = \sqrt{\pi}2^k k!\delta_{jk}$$

Moments of Random Matrices

- We define the *eigenvalue density* $\rho_n^{(\beta)}(x)$

$$\rho_n^{(\beta)}(x) = R_1(x) = \mathbb{E}\left(\sum_{j=1}^{n} \delta(x - x_j)\right)$$

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  - Others...

Moments of Random Matrices

- If X_n is a GUE matrix then

$$Q_k^{\mathbb{C}}(n) = \mathbb{E}\,\mathrm{Tr}\, X_n^{2k} = n^{k+1} \sum_{g=0}^{[k/2]} \frac{\epsilon_g(k)}{n^{2g}}.$$

$\epsilon_g(k)$ is the number of maps of genus g with k edges.

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- Then $\tau = (1/n)\,\mathrm{Tr}\,X_n^{-1}$ is the Wigner delay time
- The CGF $H_n(t)$ satisfies Painlevé III (FM & Simm, 2013)

$$\begin{aligned}(zH_n'')^2 &= 4H_n\big((H_n')^2 - H_n'\big) - \big(4z(H_n')^2 \\ &\quad -(4z + (b-2n)^2)H_n' - 2n(b-2n)\big)H_n' + n^2.\end{aligned}$$

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$$\mathcal{C}_\nu = \frac{1}{(2n^2)^{\nu-1}} \sum_{g \geq 0} \frac{c_g(\nu)}{n^g}$$

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- Now take the asymptotics expansion of the moments

$$M_k^{(\beta)}(n) = \sum_{g=0}^{\infty} \kappa_g^{(\beta)}(k) n^{-g}, \quad \beta = 1, 2.$$

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with $(\mu_1, \ldots, \mu_k) \in \mathbb{N}^k$.

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  with $(\mu_1, \ldots, \mu_k) \in \mathbb{N}^k$.
- The $c_g(\mu_1, \ldots, \mu_k)$ are Hurwitz numbers (Cunden, Dahlqvist & O'Connell 2018)

Moments & Hypergeometric OP's

- If X_n is a GUE matrix then $\mathbb{E}\,\mathrm{Tr}\,X_n^{2k}$ is a polynomial in n

$$\mathbb{E}\,\mathrm{Tr}\,X_n^8 = 14n^5 + 70n^3 + 21n. \qquad (*)$$

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- Can we say something about it as a function of k?

$$(k+2)Q_{k+1}^{\mathbb{C}}(n) = 2n(2k+1)Q_k^{\mathbb{C}}(n) + k(2k+1)(2k-1)Q_{k-1}^{\mathbb{C}}(n),$$

(Harer and Zagier, 1986)

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- It turns out that

$$\frac{1}{(2k-1)!!}\mathbb{E}\,\mathrm{Tr}\,X_4^{2k} = \frac{4}{3}k^3 + 4k^2 + \frac{20}{3}k + 4.$$

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This is a Meixner polynomial!

Moments & Hypergeometric OP's

- Meixner polynomials have the representation

$$M_n(x;\gamma,c) = {}_2F_1\left(\begin{matrix}-n, -x \\ \gamma\end{matrix}; 1 - \frac{1}{c}\right)$$

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- They obey the orthogonality relation

$$\sum_{x=0}^{\infty} \frac{(\gamma)_x}{x!} c^x M_n(x;\gamma,c) M_m(x;\gamma,c)$$

$$= \frac{c^{-n} n!}{(\gamma)_x (1-c)^{\gamma}} \delta_{mn}, \quad \gamma > 0, \quad 0 < c < 1$$

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- They obey the recurrence relation

$$(c-1)xM_n(x;\gamma,c) = c(n+\gamma)M_{n+1}(x;\gamma,c) - \left[n+(n+\gamma)c\right]M_n(x;\gamma,c) + nM_{n-1}(x;\gamma,c)$$

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- We have

$$\frac{Q_k^{\mathbb{C}}(m,n)}{\Gamma(k+\alpha+1)} = \frac{mn}{\Gamma(2+\alpha)}\,{}_3F_2\left(\begin{matrix}1-n, 1-k, 2+k\\ 2, 2+\alpha\end{matrix}; 1\right)$$
$$= mn(2+\alpha)_{k-1} R_{n-1}((k-1)(k+2);\ 1, 1, -2-\alpha).$$

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- The polynomial

$$R_n(\lambda(x);\gamma,\delta,N) = {}_3F_2\left(\begin{matrix}-n, -x, x+\gamma+\delta+1\\ \gamma+1, -N\end{matrix}; 1\right),$$

$\lambda(x) = x(x+\gamma+\delta+1)$, is a dual-Hahn polynomial.

Moments & Hypergeometric OP's

- Let's consider again

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- Now take $k \in \mathbb{C}$ and set $k = -\frac{1}{2} + ix$, $x \in \mathbb{R}$

$$\frac{Q_{-\frac{1}{2}+ix}^{\mathbb{C}}(m,n)}{\Gamma(-\frac{1}{2}+ix+\alpha+1)} =$$

$$\frac{mn}{\Gamma(2+\alpha)}\,{}_3F_2\left(\begin{matrix}1-n, \frac{3}{2}+ix, \frac{3}{2}-ix\\ 2, 2+\alpha\end{matrix};1\right)$$

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$$\frac{1}{\Gamma(n)\,\Gamma(m)} S_{n-1}\left(x^2; \frac{3}{2}, \frac{1}{2}, \alpha+\frac{1}{2}\right)$$

Moments & Hypergeometric OP's

- The polynomials

$$S_n(x^2; a, b, c) = (a+b)_n(a+c)_n \, {}_3F_2\left(\begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix}; 1\right)$$

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  are continuous dual Hahn polynomials.
- They obey the orthogonality relation

$$\frac{1}{2\pi}\int_{\mathbb{R}_+}\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 \times$$
$$\times S_m(x^2;\ a, b, c) S_n(x^2;\ a, b, c) dx$$
$$= \Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c) n!\ \delta_{mn}.$$

Moments & Hypergeometric OP's

We compute $\rho_n^{(2)}(x)$ using orthogonal polynomials. For the GUE

$$\int_{-\infty}^{\infty} H_k(x) H_j(x) e^{-x^2} dx = \sqrt{\pi} 2^k k! \delta_{jk}$$

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$$\mathbb{E}\,\mathrm{Tr}\, X_n^k = \int_{-\infty}^{\infty} x^{2k} \rho_n^{(2)}(x) dx$$

are Hypergeometric OP's in k of degree $n-1$.

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This is a statement on the Mellin transform

$$\mathcal{M}[\rho_n(x); s] = \int_0^{\infty} \rho_n^{(2)}(x) x^{s-1} dx.$$

Moments & Hypergeometric OP's

- They admit a representation in terms of hypergeometric functions

$$
{}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{z^j}{j!}
$$

$$
(q)_n = q(q+1)(q+2)\cdots(q+n-1), \qquad (q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}
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- Hermite polynomials are hypergeometric OP (of first type)

$$ H_n(x) = (2x)^{2n} {}_2F_0\left(\begin{matrix} -n/2, & -(n-1)/2 \\ & \end{matrix}; -\frac{1}{x^2}\right) $$

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They satisfy the second order ODE

$$ y''(x) - 2xy'(x) + 2ny(x) = 0 $$

Moments & Hypergeometric OP's

1. First Type: Solutions of continuous second order ODE's. (Classical Orthogonal Polynomials: Hermite, Laguerre and Jacobi, etc.)

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3. Third Type: Solutions of second order discrete difference equations with complex coefficients. (Meixner-Pollaczek, continuous Hahn, continuous dual Hahn and others.)

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4. Askey Scheme of Hypergeometric OP's.

Moments & Hypergeometric OP's

- Define

$$\zeta_{X_n}(s) = \operatorname{Tr}|X_n|^{-s} = \sum_{j=1}^{n} \frac{1}{|\lambda_j|^s}, \quad X_n \in \{\mathrm{GUE}, \mathrm{LUE}, \mathrm{JUE}\},$$

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- and

$$\xi_n(s) := \begin{cases} \dfrac{2^{2s}}{\Gamma(1/2 - 2s)}\, \mathbb{E}\, \zeta_{X_n}(4s) \quad \text{if } X_n \in \mathrm{GUE}\ , \\[2ex] \dfrac{1}{\Gamma(1+\alpha-s)}\, \mathbb{E}\, \zeta_{X_n}(s) \quad \text{if } X_n \in \mathrm{LUE}\ , \\[2ex] \dfrac{\Gamma(1+\alpha_1+\alpha_2+2n-s)}{\Gamma(1+\alpha_2-s)} \\ \times \mathbb{E}\left(\zeta_{X_n}(s) - \zeta_{X_n}(s-1)\right) \quad \text{if } X_n \in \mathrm{JUE}\ , \end{cases}$$

Moments & Hypergeometric OP's

Theorem (Cunden,FM,O'Connell and Simm, 2019)

For all n, $\xi_n(s)$ is a hypergeometric orthogonal polynomial:

$$\xi_n(s) = \begin{cases} \dfrac{i^{1-n}}{\sqrt{\pi}}\, P^{(1)}_{n-1}\left(2ix;\ \pi/2\right) & X_n \in \mathrm{GUE} \\[2ex] \dfrac{1}{\Gamma(n)\Gamma(\alpha+n)}\, S_{n-1}\left((ix)^2;\ \dfrac{3}{2}, \dfrac{1}{2}, \alpha+\dfrac{1}{2}\right) & X_n \in \mathrm{LUE} \\[2ex] \dfrac{\Gamma(\alpha_1+\alpha_2+n+1)}{\Gamma(n)\Gamma(\alpha_2+n)}(-1)^{n-1}(\alpha_1+n) & X_n \in \mathrm{JUE} \\ \times\, W_{n-1}\left((ix)^2;\ \dfrac{3}{2}, \dfrac{1}{2}, \alpha_2+\dfrac{1}{2}, \dfrac{1}{2}-\alpha_1-\alpha_2-2n\right), & \end{cases}$$

where $x = 1/2 - s$. In particular, $\xi_n(s)$ satisfies the functional equation $\xi_n(s) = \xi_n(1-s)$, and all its zeros lie on the critical line $\mathrm{Re}(s) = 1/2$.

Moments & Hypergeometric OP's

These polynomials have the hypergeometric representations:

$$P_n^{(\lambda)}(x;\phi) = (2\lambda)_n \frac{e^{in\phi}}{n!} \times {}_2F_1\left(\begin{matrix}-n, \lambda + ix \\ 2\lambda\end{matrix}; 1 - e^{2i\phi}\right)$$ Meixner-Pollaczek

$$S_n(x^2; a, b, c) = (a+b)_n(a+c)_n \times {}_3F_2\left(\begin{matrix}-n, a+ix, a-ix \\ a+b, a+c\end{matrix}; 1\right)$$ continuous dual Hahn

$$W_n(x^2; a, b, c, d) = (a+b)_n(a+c)_n(a+d)_n \times {}_4F_3\left(\begin{matrix}-n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d\end{matrix}; 1\right)$$ Wilson

Moments & Hypergeometric OP's

Matrix ens.	**Correlation func.** (classical OP's)	**Moments** (hypergeo. OP's)
GUE	Hermite	Meixner-Pollaczek
LUE	Laguerre	continuous dual Hahn
JUE	Jacobi	Wilson

Moments & Hypergeometric OP's

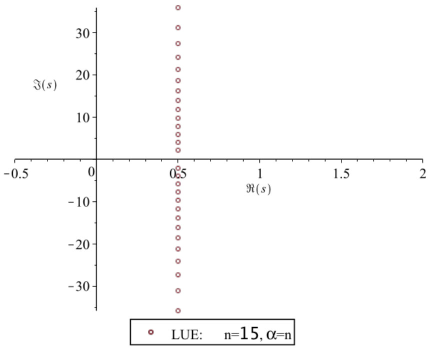
They obey the orthogonality relations

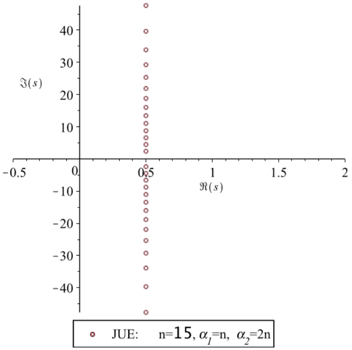
$$\frac{1}{2\pi i} \int_{\frac{1}{2}+i\mathbb{R}_+} \xi_m(s)\overline{\xi_n}(s)w(s)ds = h_m \, \delta_{mn}$$

where

$$w(s) = \begin{cases} \left|2\sqrt{\pi}\Gamma(2s)\right|^2 & \text{if } X_n \in \text{GUE} \\ \left|\frac{\Gamma(s)\Gamma(s+1)\Gamma(s+\alpha)}{\Gamma(2s-1)}\right|^2 & \text{if } X_n \in \text{LUE} \\ \left|\frac{\Gamma(s)\Gamma(s+1)\Gamma(s+\alpha_2)}{\Gamma(s+\alpha_1+\alpha_2+2n)\Gamma(2s-1)}\right|^2 & \text{if } X_n \in \text{JUE} \,, \end{cases}$$

Moments & Hypergeometric OP's





The zeros of $\xi(s)$ for the LUE and JUE.

The limit $n \to \infty$

- Take the equilibrium measure for the LUE

$$\rho_\infty(x) = \frac{1}{2\pi x}\sqrt{(x_+ - x)(x - x_-)}\, 1_{x\in(x_-,x_+)}$$

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The functional equation $\xi_\infty(s) = \xi_\infty(1-s)$ holds, and the zeros of the $\zeta_\infty(s)$ all lie on the critical line $\mathrm{Re}(s) = 1/2$.

Wronskians & Hypergepometric OP's

- Set

$$\phi_n(x) = (2^n n!\sqrt{\pi})^{-1/2} H_n(x) e^{-\frac{x^2}{2}}$$
$$\phi_n^*(s) = \int_0^\infty x^{s-1}\phi_n(x)dx$$

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- The density of state is

$$\rho_n^{(2)}(x) = \sum_{j=0}^{n-1} \phi_j^2(x) = \frac{k_{n-1}}{k_n}\left[\phi_n(x)\phi'_{n-1}(x) - \phi'_n(x)\phi_{n-1}(x)\right]$$
$$= \frac{k_{n-1}}{k_n}\mathrm{Wr}(\phi_{n-1}(x), \phi_n(x)).$$

Wronskians & Hypergeometric OP's

- Let $\phi_n(x)$ be Hermite wavefunctions and set

$$\begin{aligned}\omega_{n,\ell}(x) &= \phi_n(x)\phi_{n+\ell}(x)\\ W_{n,\ell}(x) &= \mathrm{Wr}(\phi_n(x), \phi_{n+\ell}(x))\end{aligned}$$

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  1. The Mellin transform of the products is

$$\omega^*_{n,\ell}(s) = i^n 2^{\frac{\ell}{2}-s}\sqrt{\frac{n!}{(n+\ell)!}}\frac{\Gamma(s)}{\Gamma\left(\frac{s-\ell+1}{2}\right)}P_n^{\left(\frac{\ell+1}{2}\right)}\left(-\frac{is}{2};\frac{\pi}{2}\right)$$

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  2. The Mellin transform of the Wronskians is

$$W^*_{n,\ell}(s-1) = \frac{2\ell}{s-1}\omega^*_{n,\ell}(s)$$

$\beta = 1$ and $\beta = 4$

- Set

$$Q_k^{\mathbb{R}}(n) = \mathbb{E}\operatorname{Tr} X_n^{2k} \qquad \text{if } X_n \in \mathrm{GOE}$$

$$Q_k^{\mathbb{H}}(n) = \mathbb{E}\operatorname{Tr} X_n^{2k} \qquad \text{if } X_n \in \mathrm{GSE}$$

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$$S_k^{\mathbb{R}}(n) = Q_{k+1}^{\mathbb{R}}(n) - (4n-2)Q_k^{\mathbb{R}}(n) - 8k(2k-1)Q_{k-1}^{\mathbb{R}}(n)$$
$$S_k^{\mathbb{H}}(n) = 2Q_{k+1}^{\mathbb{H}}(n) - (16n+4)Q_k^{\mathbb{H}}(n) - 16k(2k-1)Q_{k-1}^{\mathbb{H}}(n)$$

Moments for $\beta = 1$ and $\beta = 4$

Theorem (Cunden, FM, O'Connell and Simm, 2019)

The quantities $S_k^{\mathbb{R}}(n)$ and $S_k^{\mathbb{H}}(n)$ have Meixner polynomial factors:

$$\begin{aligned} S_k^{\mathbb{R}}(n) &= -3n(n-1)\,(2k-1)!!\, M_{n-2}(k;3,-1) \\ &= -3n(n-1)\,(2k-1)!!\, M_k(n-2;3,-1) \\ S_k^{\mathbb{H}}(n) &= -6n(2n+1)\,(2k-1)!!\, M_{2n-1}(k;3,-1) \\ &= -6n(2n+1)\,(2k-1)!!\, M_k(2n-1;3,-1). \end{aligned}$$

In particular, $S_k^{\mathbb{R}}(n)/(2k-1)!!$ and $S_k^{\mathbb{H}}(n)/(2k-1)!!$ are polynomials invariant up to a change of sign under the reflection $k \to -3-k$, with complex zeros on the vertical line $\mathrm{Re}(k) = -3/2$.

Moments for $\beta = 1$ and $\beta = 4$

Theorem (Cunden, FM, O' Connell, Simm, 2019)

We have the following duality between GOE and GSE

$$Q_k^{\mathbb{R}}(2n+1) = 2^{k+1} Q_k^{\mathbb{H}}(n) + 4^k \Gamma\left(k + 1/2\right) f_k(n)$$

where

$$f_k(n) = \frac{i^n n!}{\Gamma\left(n + 1/2\right)} P_n^{(1/4)}(-i(k+1/4); \pi/2)$$

and $P_n^{(\lambda)}(x, \phi)$ is a Meixner-Pollaczek polynomial.

Conclusions

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Cunden, F.D., Mezzadri, F., O'Connell, N., Simm N., *Commun. Math. Phys.* (2019)