Monodromy dependence of Painlevé tau functions

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CIRM, 08/04/2019

collaborations with
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Example 1: Sine kernel

Introduce

$$\tau(t) = \det\left(\mathbf{1} - K\big|_{(0,t)}\right), \qquad K(x,y) = \frac{\sin\frac{x-y}{2}}{\pi(x-y)}.$$

- $\tau(t)$ is a Painlevé V tau function: $\zeta(t) = t\frac{d}{dt}\ln\tau(t)$ satisfies

$$\left(t\zeta''\right)^2 + \left(t\zeta' - \zeta\right)\left(t\zeta' - \zeta + 4\zeta'^2\right) = 0. \qquad (\zeta\text{-PV})$$

- Asymptotics:

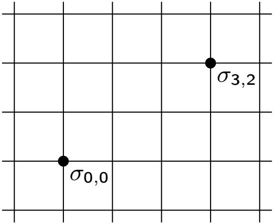
$$\begin{aligned}\tau(t\to 0) &= 1 - \frac{t}{2\pi} + \frac{t^4}{576\pi^2} + O\left(t^6\right), \\ \tau(t\to\infty) &= \tau_{\text{sine}} \cdot t^{-\frac{1}{4}} e^{-\frac{t^2}{32}}\left[1 + \frac{1}{2t^2} + O\left(t^{-4}\right)\right]\end{aligned}$$

Conjecture [Dyson, '76]:

$$\tau_{\rm sine} = 2^{\frac{7}{12}} e^{3\zeta'(-1)} = \sqrt{2}\, G\left(\tfrac{1}{2}\right) G\left(\tfrac{3}{2}\right).$$

- Barnes G-function is essentially defined by the recurrence relation $G(z+1) = \Gamma(z)\, G(z)$; it has integral and product representations, etc
- proved in [Ehrhardt, '04; Krasovsky, '04; Deift, Its, Krasovsky, Zhou, '06]

Example 2: 2D Ising model



- Nearest-neighbor interaction

$$\mathcal{H}[\sigma] = -J \sum_{i,j} \sigma_{i,j} \left(\sigma_{i+1,j} + \sigma_{i,j+1}\right),$$

of spin variables $\sigma_{i,j} = \pm 1$.

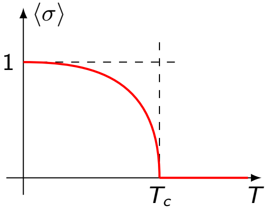
- Spin-spin correlation function:

$$\left\langle \sigma_{0,0}\sigma_{r_x,r_y} \right\rangle = \frac{\sum_{[\sigma]} \sigma_{0,0}\sigma_{r_x,r_y} e^{-\beta H[\sigma]}}{\sum_{[\sigma]} e^{-\beta H[\sigma]}}$$

- Phase transition at $s \equiv \sinh 2\beta J = 1$
- Spontaneous magnetization [Yang, '52]:

$$\langle\sigma\rangle = \begin{cases} \left(1 - s^{-4}\right)^{\frac{1}{8}}, & s > 1, \\ 0, & s < 1, \end{cases}$$

- Correlation length $\Lambda \sim 2^{-\frac{1}{2}} |s-1|^{-1}$



Scaling limit of the 2-point functions is described by

$$T \to T_c, \qquad \Lambda \to \infty, \qquad R = \sqrt{r_x^2 + r_y^2} \to \infty, \qquad \frac{R}{\Lambda} \to t,$$

$$\left\langle \sigma_{0,0}\sigma_{r_x,r_y} \right\rangle \to \Lambda^{-\frac{1}{4}} 2^{\frac{3}{8}} \tau_{\pm}(t), \qquad T \gtrless T_c,$$

Scaled correlations can be written in terms of Fredholm determinants & related to Painlevé functions [McCoy, Tracy, '73; Wu, McCoy, Tracy, Barouch, '76] (PV, PIII(D_6), PIII(D_8)). In particular, both $\zeta_{\pm} = t\frac{d}{dt}\ln \tau_{\pm}(t)$ satisfy

$$\left(t\zeta''\right)^2 = 4\left(\zeta - t\zeta'\right)^2 + 4\left(\zeta'\right)^2\left(\zeta - t\zeta'\right) + \left(\zeta'\right)^2 \qquad (\zeta\text{-PV})$$

- Long distances (form factor expansions):

$$\tau_+(t \to \infty) \sim \frac{e^{-t}}{\sqrt{2\pi t}}, \qquad \tau_-(t \to \infty) \sim 1.$$

- Short distances (conformal limit):

$$\tau_{\pm}(t \to 0) \sim \tau_{\text{Ising}} \cdot (2t)^{-\frac{1}{4}}.$$

Theorem [Tracy, '91]:

$$\tau_{\text{Ising}} = 2^{\frac{1}{12}} e^{3\zeta'(-1)} = G\left(\tfrac{1}{2}\right) G\left(\tfrac{3}{2}\right).$$

- alternative proof in [Bothner, '17]
- constant factors in the asymptotics of tau functions (connection constants) were computed for many other (special!) Fredholm determinant solutions of Painlevé equations:
  - correlator of twist fields in sine-Gordon field theory at the free-fermion point [Basor, Tracy, '91]
  - Airy kernel [Tracy, Widom, '92; Deift, Its, Krasovsky, '06]
  - Bessel kernel [Tracy, Widom, '93]
  - confluent hypergeometric kernel [Deift, Krasovsky, Vasilevska, '10]
  - hypergeometric kernel [Lisovyy, '09]
  - ...

Summary:

- Ising scaled correlator = specific PV tau function
- it has Fredholm determinant representation
- its asymptotics at one singular point ($t \to \infty$) is "easy"
- the asymptotics at the other singular point ($t \to 0$) is difficult (connection constant)

Questions:

- Can the general solutions of Painlevé equations be written as Fredholm determinants?
- How to compute the relevant connection constants?

In this talk, I will mainly focus on the Painlevé VI case.

Isomonodromic tau function [Jimbo, Miwa, Ueno, '79]

Consider a system of linear ODEs with rational coefficients

$$\frac{d\Phi}{dz} = A(z)\,\Phi, \qquad A, \Phi \in \mathrm{Mat}_{N\times N}$$

- Laurent expansions of $A(z)$ at singularities

$$A(z) = \begin{cases} \dfrac{A_\nu}{(z-a_\nu)^{r_\nu+1}} + O\left((z-a_\nu)^{-r_\nu}\right) & \text{as } z \to a_\nu, \\ -z^{r_\infty-1}A_\infty + O\left(z^{r_\infty-2}\right) & \text{as } z \to \infty, \end{cases}$$

  where $r_1, \ldots, r_n, r_\infty \in \mathbb{Z}_{\geq 0}$.

- assume A_ν are diagonalizable,

$$A_\nu = G_\nu \Theta_{\nu,-r_\nu} G_\nu^{-1}, \qquad \Theta_{\nu,-r_\nu} = \mathrm{diag}\left\{\theta_{\nu,1}, \ldots, \theta_{\nu,N}\right\}.$$

  and non-resonant ($\theta_{\nu,k}$ are distinct whenever $r_\nu = 0$).

At each singular point, there is a unique formal solution

$$\Phi^{(\nu)}_{\text{form}}(z) = G_\nu \hat{\Phi}^{(\nu)}(z)\, e^{\Theta_\nu(z)}, \qquad \hat{\Phi}^{(\nu)}(z) = \mathbf{1} + \sum_{k=1}^{\infty} g_{\nu,k}(z-a_\nu)^k,$$

where $\Theta_\nu(z)$ are diagonal and have the form

$$\Theta_\nu(z) = \sum_{k=-r_\nu}^{-1} \frac{\Theta_{\nu,k}}{k}(z-a_\nu)^k + \Theta_{\nu,0} \ln(z-a_\nu).$$

Isomonodromic times:

- positions of singularities a_ν
- diagonal elements $\Theta_{\nu,k\neq 0}$

Monodromy data:

- Stokes matrices relating canonical solutions in different sectors at a_ν
- formal monodromy exponents $\Theta_{\nu,0}$
- connection matrices relating canonical solutions at different singularities

Theorem [Jimbo, Miwa, Ueno, '79]:
Let us collectively denote the times by $\mathcal{T}$. The 1-form

$$\omega_{\mathrm{JMU}} = - \sum_{\nu=1,\ldots,n,\infty} \mathrm{res}_{z=a_\nu} \operatorname{Tr} \left(\hat{\Phi}^{(\nu)}(z)^{-1} \partial_z \hat{\Phi}^{(\nu)}(z) \, d_{\mathcal{T}} \Theta_\nu(z) \right)$$

is closed when restricted to isomonodromic family of $A(z)$. It thus defines the isomonodromic tau function by

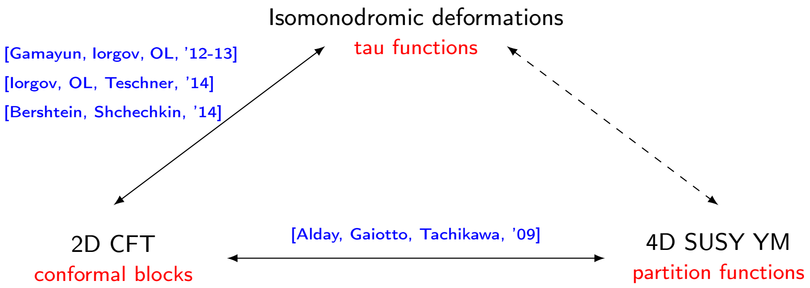
$$d_{\mathcal{T}} \ln \tau_{\mathrm{JMU}} = \omega_{\mathrm{JMU}}$$

Example. For $A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$ (4 simple poles $0, t, 1, \infty$)

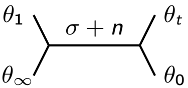
$$\partial_t \ln \tau_{\mathrm{JMU}} = \frac{\operatorname{Tr} A_0 A_t}{t} + \frac{\operatorname{Tr} A_t A_1}{t-1}.$$

For 2×2 matrices, this is Painlevé VI tau function.

Aim: Extend Jimbo-Miwa-Ueno differential to the space of monodromy data (the space of parameters and initial conditions for Painlevé).



$$\tau_{\rm VI}(t) = \mathcal{N}_0 \sum_{n\in\mathbb{Z}} e^{in\eta}$$



$$= \mathcal{N}_0 \, (1-t)^{2\theta_t\theta_1} \sum_{n\in\mathbb{Z}} e^{in\eta} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\sigma+n,\vec{\theta}) \, t^{(\sigma+n)^2-\theta_0^2-\theta_t^2+|\lambda|+|\mu|}$$

$$= \mathcal{N}_0 \, t^{\sigma^2-\theta_0^2-\theta_t^2} \det(\mathbf{1}+K) \qquad \leftarrow \ \textbf{Task 1}$$

- explicit integrable (2×2 or 4×4) matrix kernel K involving ${}_2F_1$ functions; acts on vector-valued functions on a circle (and not on an interval!)

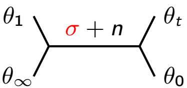
Asymptotic behaviors of τ_{VI}:

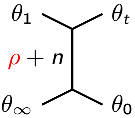
$$\tau_{\mathrm{VI}}(t) \simeq \begin{cases} \tilde{\mathcal{N}}_0\, t^{\sigma^2-\theta_0^2-\theta_t^2} & \text{as } t \to 0, \\ \tilde{\mathcal{N}}_1\, (1-t)^{\rho^2-\theta_1^2-\theta_t^2} & \text{as } t \to 1. \end{cases}$$

- σ, ρ are 2 Painlevé VI integration constants, related to monodromy of the associated 4-point Fuchsian system

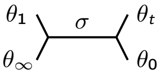
Task 2 $\to$ compute the connection coefficient $\tilde{\mathcal{N}}_1/\tilde{\mathcal{N}}_0$

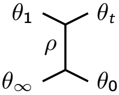
Remark. Tau function can be expanded in different channels (there are different Fredholm determinant representations, adapted for asymptotic analysis near different critical points):

$$\tau_{\mathrm{VI}}(t) = \mathcal{N}_0 \sum_{n\in\mathbb{Z}} e^{in\eta}$$ 

$$= \mathcal{N}_1 \sum_{n\in\mathbb{Z}} e^{in\mu}$$ 

This allows to relate the connection coefficient to the $c=1$ fusion matrix,



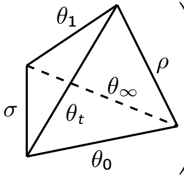
$$= \int_\Gamma d\rho\;\; F\begin{bmatrix} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{bmatrix};\begin{matrix} \rho \\ \sigma \end{matrix}\Big]$$ 

$d\rho$

It turns out $\ln \frac{\mathcal{N}_1}{\mathcal{N}_0}$ coincides (up to an elementary correction) with the generating function of the canonical transformation between two pairs of Darboux coordinates on $\mathrm{Hom}\left(\pi_1\left(C_{0,4}\right), \mathrm{SL}\left(2,\mathbb{C}\right)\right)/\mathrm{SL}\left(2,\mathbb{C}\right)$: σ, η and ρ, μ.

This in its turn coincides (again up to an elementary correction) with the complexified volume of the hyperbolic tetrahedron with dihedral angles σ, ρ, $\theta_{0,t,1,\infty}$.

$$\ln \frac{\mathcal{N}_1}{\mathcal{N}_0} \sim \mathrm{Vol}\left(\right.$$



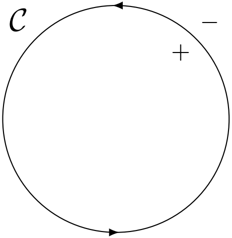
$$\left. \right) \sim \ln \prod_{k=1}^{8} \frac{G\left(1+z_k\right)}{G\left(1-z_k\right)}$$

- z_k's are explicit elementary (though complicated) functions of monodromy parameters
- conjecture in [Iorgov, OL, Tykhyy, '13]
- proved in [Its, OL, Prokhorov, '16]

Riemann-Hilbert setup

- let $\mathcal{C} \subset \mathbb{C}$ be a circle centered at the origin
- pick a loop $J(z) \in \mathrm{Hom}(\mathcal{C}, \mathrm{GL}_N(\mathbb{C}))$
- $J(z)$ continues into an annulus $\mathcal{A} \supset \mathcal{C}$

$$J(z) = \sum_{k \in \mathbb{Z}} J_k z^k,$$



Two Riemann-Hilbert problems:

$$\text{direct}: \quad J(z) = \Psi_-(z)^{-1} \Psi_+(z)$$
$$\text{dual}: \quad J(z) = \bar{\Psi}_+(z)\, \bar{\Psi}_-(z)^{-1}$$

Main definition: The tau function of RHPs defined by $(\mathcal{C}, J)$ is defined as Fredholm determinant

$$\tau[J] = \det_{H_+}\left(\Pi_+ J^{-1}\Pi_+ J\,\Pi_+\right),$$

where $H = L^2\left(\mathcal{C}, \mathbb{C}^N\right)$ and Π_+ is the orthogonal projection on H_+ along H_-.

Properties:

- dual RHP is solvable iff the operator $P := \Pi_+ J^{-1}\Pi_+$ is invertible on H_+, in which case $P^{-1} = \bar{\Psi}_+\Pi_+\bar{\Psi}_-^{-1}\Pi_+$
- likewise, for direct RHP and $Q := \Pi_+ J\,\Pi_+$, with $Q^{-1} = \Psi_+^{-1}\Pi_+\Psi_-\Pi_+$
- if either direct or dual RHP is not solvable, then $\tau[J] = 0$

Example: scalar case $(N = 1)$

- direct and dual factorization coincide
- $J(z) = (1 - t_1 z)^{\nu_1}(1 - t_2/z)^{\nu_2}$ with $|z| = 1$ and $|t_1|, |t_2| < 1$, then

$$\tau[J] = (1 - t_1 t_2)^{\nu_1\nu_2}$$

Remark. $\tau[J]$ appears [Widom, '76] in the asymptotics of determinant of block Toeplitz matrix with symbol J,

$$T_K[J] = \begin{pmatrix} J_0 & J_{-1} & \dots & J_{-K+1} \\ J_1 & J_0 & \dots & J_{-K+2} \\ \vdots & \vdots & \ddots & \vdots \\ J_{K-1} & J_{K-2} & \dots & J_0 \end{pmatrix}.$$

In this context, $\tau[J]$ is called Widom's constant.

- strong Szegő for $N = 1$: $\tau[J] = \exp \sum_{k=1}^{\infty} k (\ln J)_k (\ln J)_{-k}$
- no nice general formula for $N \geq 2$

If the direct RHP is solvable, then $\tau[J]$ may also be written as

$$\tau[J] = \det_H (\mathbf{1} + K), \qquad K = \begin{pmatrix} 0 & \mathsf{a}_{+-} \\ \mathsf{a}_{-+} & 0 \end{pmatrix},$$

where $\mathsf{a}_{\pm\mp} = \Psi_\pm \Pi_\pm \Psi_\pm^{-1} - \Pi_\pm : H_\mp \to H_\pm$ are integral operators

$$(\mathsf{a}_{\pm\mp} f)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{a}_{\pm\mp}(z, z') \, f(z') \, dz',$$

with block integrable kernels

$$\mathsf{a}_{\pm\mp}(z, z') = \pm \frac{\mathbf{1} - \Psi_\pm(z) \, \Psi_\pm(z')^{-1}}{z - z'}.$$

In applications to Painlevé:

- $\Psi_\pm$ (direct factorization) are given and define the jump $J = \Psi_-{}^{-1} \Psi_+$
- $\Psi_\pm$ are expressed via classical special functions (Gauss, Kummer & Bessel for PVI, PV, PIII's)
- dual factorization ($\bar{\Psi}_\pm$ in $J = \bar{\Psi}_+ \bar{\Psi}_-^{-1}$) is the problem to be solved

Variational formula

Theorem: Let $(z,t)\mapsto J(z,t)$ be a smooth family of $\mathrm{GL}(N,\mathbb{C})$-loops which depend on an extra parameter t and admit direct & dual factorization. Then

$$\partial_t \ln \tau[J] = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathrm{Tr}\left\{J^{-1}\partial_t J\left[\partial_z\bar{\Psi}_-\,\bar{\Psi}_-^{-1}+\Psi_+^{-1}\,\partial_z\Psi_+\right]\right\}dz.$$

Proof.

$$\partial_t \ln \det_{H_+} PQ = \mathrm{Tr}_{H_+}\left(\partial_t P\,P^{-1}+Q^{-1}\partial_t Q\right)=$$

$$=\mathrm{Tr}_H\left(\Pi_+J^{-1}\partial_t J\,\left(\bar{\Psi}_-\Pi_-\bar{\Psi}_-^{-1}-\Pi_-\right)+\left(\Psi_+^{-1}\Pi_+\Psi_+-\Pi_+\right)J^{-1}\partial_t J\right)$$

Given $\tilde{d}(z,z')=\dfrac{\Psi_+(z)^{-1}\,\Psi_+(z')-\mathbf{1}}{z-z'}$, we have $\tilde{d}(z,z)=\Psi_+^{-1}\,\partial_z\Psi_+.$ □

- due to [Widom, '74]; rediscovered by [Its, Jin, Korepin, '06]
- related results in the study of dependence of isomonodromic tau functions on monodromy [Bertola, '09]

Corollary: in isomonodromic RHPs,

Widom's constant $\tau[J]$ $\simeq$ Jimbo-Miwa-Ueno tau function

Example: 4-point Fuchsian system

4 regular singularities at $0, t, 1, \infty$:

$$\partial_z \Phi = \Phi A(z), \qquad A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

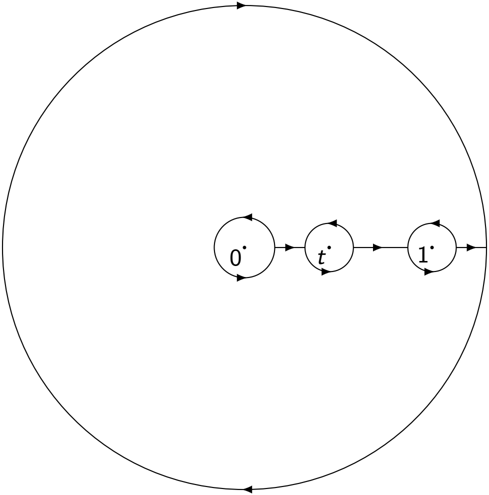
- arbitrary rank: $A_{0,t,1} \in \mathrm{Mat}_{N\times N}(\mathbb{C})$
- generic case: $A_{0,t,1}$ and $A_\infty := -A_0 - A_t - A_1$ are diagonalizable
- fix the diagonalizations $A_\nu = G_\nu^{-1} \Theta_\nu G_\nu$ with diagonal Θ_ν
- eigenvalues of A_ν are assumed distinct mod $\mathbb{Z}$

There exist unique fundamental solutions $\Phi^{(\nu)}(z)$, holomorphic on the universal covering of $\mathbb{C} \backslash \{0, t, 1\}$ and such that

$$\Phi^{(\nu)}(z) = \begin{cases} (\nu - z)^{\Theta_\nu} G^{(\nu)}(z), & \text{for } \nu = 0, t, 1, \\ (-z)^{-\Theta_\infty} G^{(\infty)}(z), & \text{for } \nu = \infty, \end{cases}$$

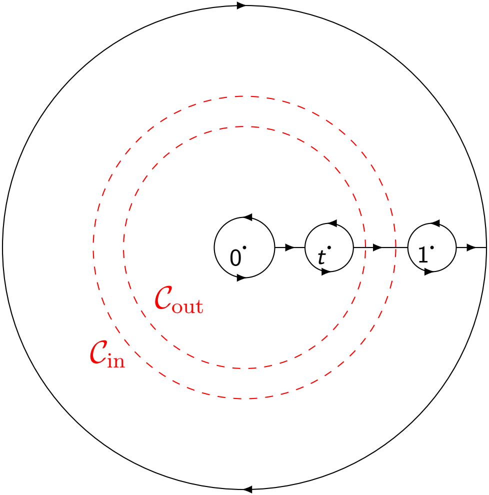
where $G^{(\nu)}(z)$ is holomorphic and invertible in a finite open disk around $z = \nu$ and satisfies $G^{(\nu)}(\nu) = G_\nu$.

Dual RHP$_1$ for $\tilde{\Psi}$



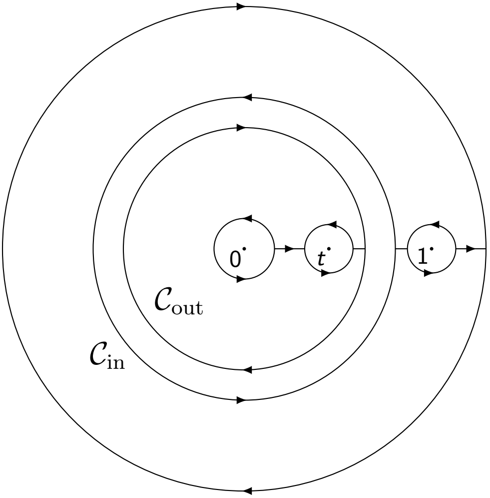
$$\tilde{\Psi}(z) = \begin{cases} G^{(\nu)}(z), & z \in D_\nu, \\ \Phi(z), & z \notin \mathbb{R}_{\geq 0} \cup \bar{D}_0 \cup \bar{D}_t \cup \bar{D}_1 \cup \bar{D}_\infty. \end{cases}$$

Dual RHP$_1$ for $\tilde{\Psi}$

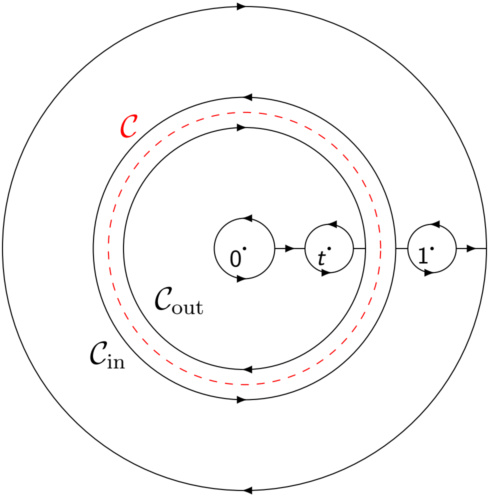


$$\hat{\Psi}(z) = \begin{cases} (-z)^{-\mathfrak{S}}\,\tilde{\Psi}(z), & z \in \mathcal{A}, \\ \tilde{\Psi}(z), & z \notin \bar{\mathcal{A}}. \end{cases}$$

Dual RHP_2 for $\hat{\Psi}$

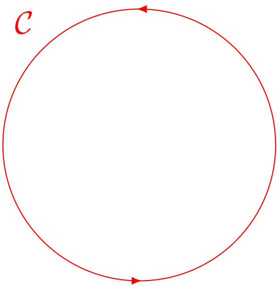


Dual RHP$_2$ for $\hat{\Psi}$



$$\bar{\Psi}(z) = \begin{cases} \Psi_+(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi_-(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

Dual RHP_3 for $\bar{\Psi}$



$$\bar{\Psi}(z) = \begin{cases} \Psi_+(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi_-(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

- contour $\mathcal{C}$ (single circle !), smooth jump $J : \mathcal{C} \to \mathrm{GL}(N, \mathbb{C})$ given by

$$J(z) = \Psi_-(z)^{-1}\Psi_+(z) = \bar{\Psi}_+(z)\,\bar{\Psi}_-(z)^{-1}$$

- we are in the previously described setup!

Substitute into Widom's differentiation formula

$$\partial_s \ln \tau[J] = \frac{1}{2\pi i} \oint_C \operatorname{Tr} J^{-1} \partial_s J \left(\partial_z \bar{\Psi}_- \bar{\Psi}_-^{-1} + \Psi_+^{-1} \partial_z \Psi_+ \right) dz.$$

the expression for the jump $J = \Phi_i^{-1}\Phi_e$ and the dual/direct factorizations,

$$\bar{\Psi}_- = \Phi_e^{-1}\Phi, \quad \bar{\Psi}_+ = \Phi_i^{-1}\Phi, \quad \Psi_- = (-z)^{-\mathfrak{S}}\,\Phi_i, \quad \Psi_+ = (-z)^{-\mathfrak{S}}\,\Phi_e,$$

and use that $\partial_z \Phi = \Phi A(z)$. This gives

$$\partial_s \ln \tau[J] = \frac{1}{2\pi i} \oint_C \operatorname{Tr} \Big\{ A(z)\,\Phi^{-1}\Phi_i \partial_s \left(\Phi_i^{-1}\Phi\right) - A(z)\,\Phi^{-1}\Phi_e \partial_s \left(\Phi_e^{-1}\Phi\right)$$
$$- \frac{\mathfrak{S}}{z} (-z)^{-\mathfrak{S}}\,\Phi_i \partial_s \left(\Phi_i^{-1} (-z)^{\mathfrak{S}}\right) + \frac{\mathfrak{S}}{z} (-z)^{-\mathfrak{S}}\,\Phi_e \partial_s \left(\Phi_e^{-1} (-z)^{\mathfrak{S}}\right) \Big\}\, dz$$

Red terms contribute via the residues at $z = 0, t$, and blue ones via the residues at $z = 1, \infty$.

The log-derivative then reduces to

$$\partial_s \ln \tau\,[J] = \sum_{\nu=0,t,1,\infty} \mathrm{Tr}\, \Theta_\nu \partial_s G_\nu G_\nu^{-1}$$

$$- \sum_{\nu=0,t,\infty} \mathrm{Tr}\, \Theta_{\nu,i} \partial_s G_{\nu,i} G_{\nu,i}^{-1} - \sum_{\nu=0,1,\infty} \mathrm{Tr}\, \Theta_{\nu,e} \partial_s G_{\nu,e} G_{\nu,e}^{-1}$$

where Θ_ν are exponents of the 4-point solution,

$$\Theta_{0,i} = \Theta_0, \quad \Theta_{t,i} = \Theta_t, \quad \Theta_{\infty,i} = \mathfrak{S},$$
$$\Theta_{0,e} = \mathfrak{S}, \quad \Theta_{1,e} = \Theta_1, \quad \Theta_{\infty,e} = \Theta_\infty,$$

and $G_{\nu,i}$, $G_{\nu,e}$ are 3-point counterparts of G_ν.

For $s = t$ (isomonodromic time):

- 1st line is nothing but the Jimbo-Miwa-Ueno definition of τ
- 2nd line corresponds to tau functions of auxiliary 3-point systems

We then obtain

$$\tau_{\mathrm{JMU}}(t) = t^{\frac{1}{2}\operatorname{Tr}\left(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2\right)} \tau[J].$$

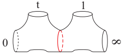
- Recall that

$$\tau[J] = \det(\mathbf{1} + K), \qquad K = \begin{pmatrix} 0 & \mathsf{a}_{+-} \\ \mathsf{a}_{-+} & 0 \end{pmatrix},$$

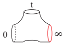
$$\mathsf{a}_{\pm\mp}(z, z') = \pm \frac{\mathbf{1} - \Psi_{\pm}(z)\,\Psi_{\pm}(z')^{-1}}{z - z'}.$$

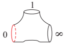
- $\tau_{\mathrm{JMU}}(t)$ for 4-point system written via auxiliary 3-point solutions
- hypergeometric representations for $N = 2 \Longrightarrow$ PVI tau function !

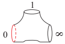
Schematically,

$\tau_{\mathrm{JMU}}\Bigg($ 

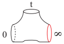
$\Bigg) =$

$\tau_{\mathrm{JMU}}\Bigg($ 

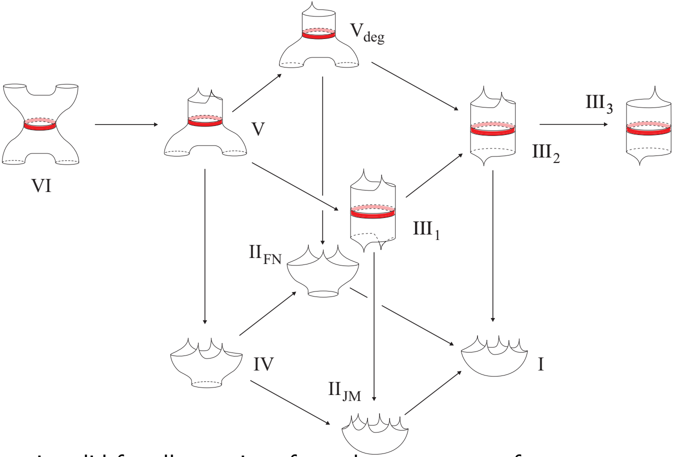
$\Bigg)\ \tau_{\mathrm{JMU}}\Bigg($ 

$\Bigg)\ \det \left(\begin{matrix} \mathbf{1} & a_{+-}\Big(\end{matrix}\right.$ 

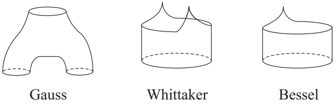
$\Big)$

$\left. \begin{matrix} a_{-+}\Big(\end{matrix}\right.$ 

$\Big) \quad \mathbf{1} \Bigg)$



The same is valid for all equations from the upper part of Chekhov-Mazzocco-Rubtsov geometric confluence diagram, since PVI, PV, $PIII_{1,2,3}$ surfaces (Riemann-Hilbert contours) may be cut into solvable pieces:



Connection coefficient

Considering a different pants decomposition which combines t and 1 instead of t and 0, we obtain a different Fredholm determinant representation, which is better adapted for the asymptotic analysis of the regime $t \to 1$.

$$\bar{\tau}_{\mathrm{JMU}}(t) = (1-t)^{\frac{1}{2}\operatorname{Tr}\left(\bar{\mathfrak{S}}^2 - \Theta_1^2 - \Theta_t^2\right)} \tau\left[\bar{J}\right].$$

It is of course proportional to the previous tau function $\tau_{\mathrm{JMU}}(t)$, and their ratio is the connection coefficient that we want to compute.

Corollary: For any monodromy parameter s,

$$\begin{aligned}
\partial_s \ln \frac{\tilde{\mathcal{N}}_1}{\tilde{\mathcal{N}}_0} = & \sum_{\nu=0,t,\infty} \operatorname{Tr} \bar{\Theta}_{\nu,i} \partial_s \bar{G}_{\nu,i} \bar{G}_{\nu,i}^{-1} + \sum_{\nu=0,1,\infty} \operatorname{Tr} \bar{\Theta}_{\nu,e} \partial_s \bar{G}_{\nu,e} \bar{G}_{\nu,e}^{-1} \\
& - \sum_{\nu=0,t,\infty} \operatorname{Tr} \Theta_{\nu,i} \partial_s G_{\nu,i} G_{\nu,i}^{-1} - \sum_{\nu=0,1,\infty} \operatorname{Tr} \Theta_{\nu,e} \partial_s G_{\nu,e} G_{\nu,e}^{-1} \\
& + \frac{1}{2} \ln t \, \partial_s \operatorname{Tr}\left(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2\right) - \frac{1}{2} \ln(1-t) \, \partial_s \operatorname{Tr}\left(\bar{\mathfrak{S}}^2 - \Theta_1^2 - \Theta_t^2\right)
\end{aligned}$$

Theorem [Its, OL, Prokhorov, '16]

For generic monodromy data,

$$\frac{\tilde{\mathcal{N}}_1}{\tilde{\mathcal{N}}_0} = \prod_{\epsilon,\epsilon'=\pm} \frac{G\left(1+\epsilon\overline{\sigma}+\epsilon'\theta_t-\epsilon\epsilon'\theta_1\right)G\left(1+\epsilon\overline{\sigma}+\epsilon'\theta_0-\epsilon\epsilon'\theta_\infty\right)}{G\left(1+\epsilon\sigma+\epsilon'\theta_t+\epsilon\epsilon'\theta_0\right)G\left(1+\epsilon\sigma+\epsilon'\theta_1+\epsilon\epsilon'\theta_\infty\right)}\times$$

$$\times\prod_{\epsilon=\pm}\frac{G(1+2\epsilon\sigma)}{G(1+2\epsilon\overline{\sigma})}\prod_{k=1}^{4}\frac{\hat{G}(\varsigma+\nu_k)}{\hat{G}(\varsigma+\lambda_k)}$$

Here $G\left(z\right)$ denotes the Barnes G-function, $\hat{G}\left(z\right)=\frac{G(1+z)}{G(1-z)}$, the parameters $\nu_{1\ldots 4}$ and $\lambda_{1\ldots 4}$ are defined by

$$\begin{aligned}
\nu_1 &= \sigma+\theta_0+\theta_t, & \lambda_1 &= \theta_0+\theta_t+\theta_1+\theta_\infty,\\
\nu_2 &= \sigma+\theta_1+\theta_\infty, & \lambda_2 &= \sigma+\overline{\sigma}+\theta_0+\theta_1,\\
\nu_3 &= \overline{\sigma}+\theta_0+\theta_\infty, & \lambda_3 &= \sigma+\overline{\sigma}+\theta_t+\theta_\infty,\\
\nu_4 &= \overline{\sigma}+\theta_t+\theta_1, & \lambda_4 &= 0,
\end{aligned}$$

and the quantity ς is determined by

$$e^{2\pi i\varsigma}=\frac{2\cos 2\pi\left(\sigma-\overline{\sigma}\right)-2\cos 2\pi\left(\theta_0+\theta_1\right)-2\cos 2\pi\left(\theta_\infty+\theta_t\right)+\operatorname{Tr} M_0M_1}{\sum_{k=1}^{4}\left(e^{2\pi i(\nu_\Sigma-\nu_k)}-e^{2\pi i(\nu_\Sigma-\lambda_k)}\right)},$$

with $2\nu_\Sigma=\sum_{k=1}^{4}\nu_k=\sum_{k=1}^{4}\lambda_k$.

Some open problems

- 3-point auxiliary solutions are known for 4-point Fuchsian systems of higher rank N whose 2 singularities have special spectral type $(N-1,1)$. It is then in principle possible to find explicitly the log-differential of the connection coefficient. Is it possible to integrate it? What would be the higher-rank analog of the tetrahedron?
- In the generic case, the 3-point solutions for $N > 2$ are not available. Can we at least find an interpretation of the connection constant in terms of Poisson geometry of the $\mathrm{SL}(N,\mathbb{C})$ character variety of $C_{0,4}$? Affirmative answer [Bertola, Korotkin, '19]
- Connections constants for PI are computed in [OL, Roussillon, '16]. Their evaluation for PII-PV with generic parameters is wide open; for PV conjectural expressions are available [OL, Nagoya, Roussillon, '18].

For $N=2$:

$$\mathsf{a}_{+-}(z,z') = \frac{(1-z')^{2\theta_1}\begin{pmatrix} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{--}(z)\end{pmatrix}\begin{pmatrix} K_{--}(z') & -K_{+-}(z') \\ -K_{-+}(z') & K_{++}(z')\end{pmatrix} - \mathbf{1}}{z-z'},$$

$$\mathsf{a}_{-+}(z,z') = \frac{\mathbf{1} - \left(1-\frac{t}{z'}\right)^{2\theta_t}\begin{pmatrix} \bar{K}_{++}(z) & \bar{K}_{+-}(z) \\ \bar{K}_{-+}(z) & \bar{K}_{--}(z)\end{pmatrix}\begin{pmatrix} \bar{K}_{--}(z') & -\bar{K}_{+-}(z') \\ -\bar{K}_{-+}(z') & \bar{K}_{++}(z')\end{pmatrix}}{z-z'}$$

with

$$K_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix} \theta_1+\theta_\infty\pm\sigma, \theta_1-\theta_\infty\pm\sigma \\ \pm 2\sigma\end{matrix} ; z\right],$$

$$K_{\pm\mp}(z) = \pm\frac{\theta_\infty^2-(\theta_1\pm\sigma)^2}{2\sigma(1\pm 2\sigma)}\, z\, {}_2F_1\left[\begin{matrix} 1+\theta_1+\theta_\infty\pm\sigma, 1+\theta_1-\theta_\infty\pm\sigma \\ 2\pm 2\sigma\end{matrix} ; z\right],$$

$$\bar{K}_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix} \theta_t+\theta_0\mp\sigma, \theta_t-\theta_0\mp\sigma \\ \mp 2\sigma\end{matrix} ; \frac{t}{z}\right],$$

$$\bar{K}_{\pm\mp}(z) = \mp\, t^{\mp 2\sigma} e^{\mp i\eta}\frac{\theta_0^2-(\theta_t\mp\sigma)^2}{2\sigma(1\mp 2\sigma)}\,\frac{t}{z}\, {}_2F_1\left[\begin{matrix} 1+\theta_t+\theta_0\mp\sigma, 1+\theta_t-\theta_0\mp\sigma \\ 2\mp 2\sigma\end{matrix} ; \frac{t}{z}\right].$$