Tilings of a hexagon and non-hermitian orthogonality on a contour

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joint work with

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Integrability and Randomness in

Mathematical Physics and Geometry

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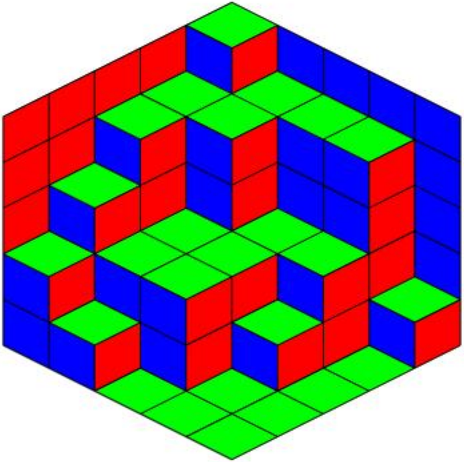
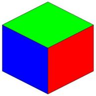
Outline

1. Hexagon tilings

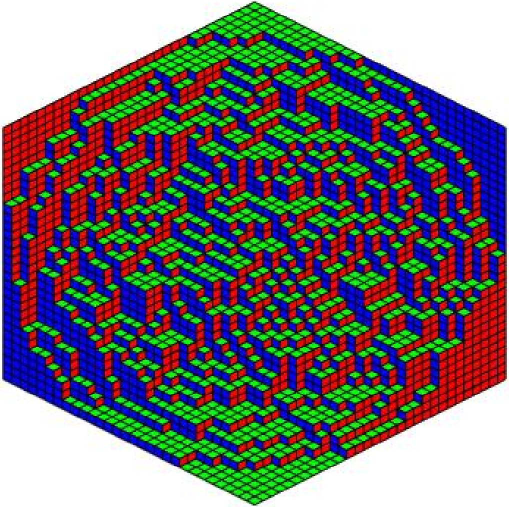
Lozenge tiling of a hexagon





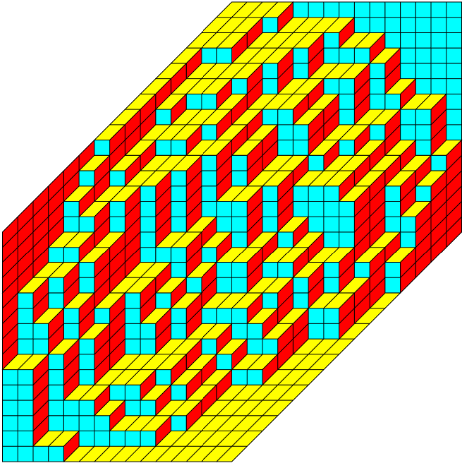
three types of lozenges

Large random tiling



Arctic circle phenomenon

Affine change of coordinates



Hexagon with **corner points** **at** $(0,0)$**,** $(N,0)$**,** $(2N,N)$**,** $(2N,2N)$**,** $(N,2N)$**, and** $(0,N)$**.**

Non uniform model

Probability of tiling: $$\mathbb{P}(\mathcal{T}) = \frac{W(\mathcal{T})}{\sum_{\mathcal{T}'} W(\mathcal{T}')}$$

Weight on a tiling: $$W(\mathcal{T}) = \prod_{\square \in \mathcal{T}} w(\square)$$

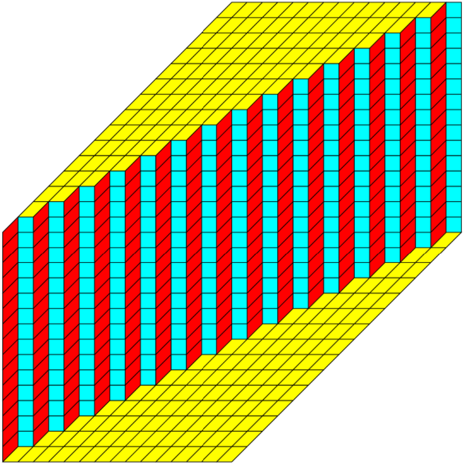
Weight of $\square$ depends on its position:

$$w(\square) = \begin{cases} \alpha, & \text{if } \square \text{ is in odd numbered column} \\ 1, & \text{if } \square \text{ is in even numbered column} \end{cases}$$

$\alpha = 1$ **is the usual uniform model**

$\alpha < 1$ **means punishment if $\square$ is in an odd column**

Case $\alpha = 0$



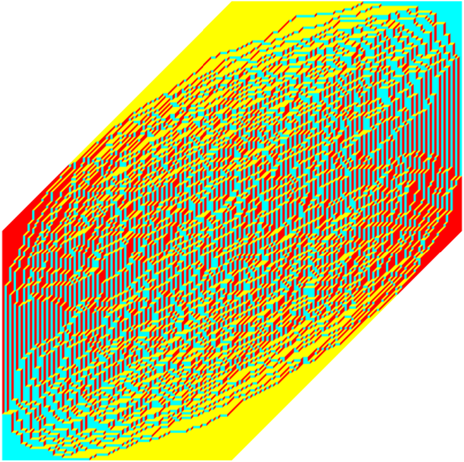
Only ground state in case $\alpha = 0$

Small $\alpha > 0$

Liquid region consists of two ellipses if $\alpha > 0$ is small

Special frozen region with two tiles in the middle

Larger $\alpha > 0$

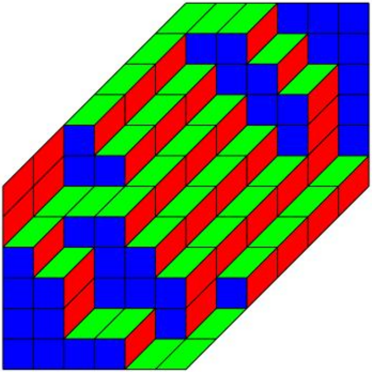


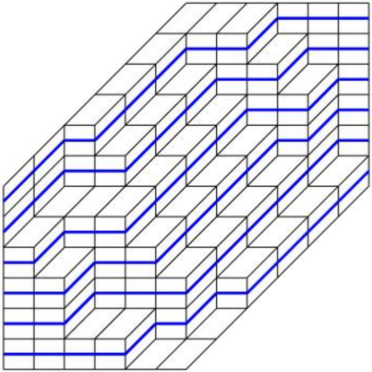
Liquid region is bounded by more complicated curve

Special frozen region is broken. It no longer goes all the way from left to right.

2. Non-intersecting paths

Non-intersecting paths



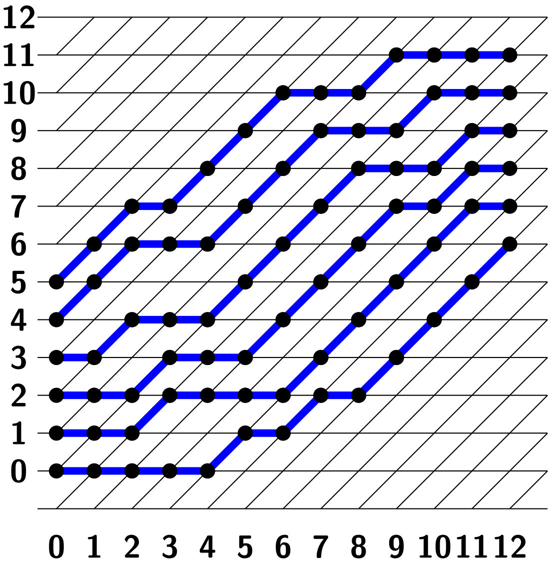


Tiling is equivalent to N non-intersecting paths

starting at $(0,0)$, . . . , $(0, N-1)$ and

ending at $(2N, N)$, . . . , $(2N, 2N-1)$

Paths fit on a directed graph



Weighted graph

Weight of a path system $w(P_1,\ldots,P_N)=\prod_{j=1}^{N}\prod_{e\in P_j} w(e)$

Weight of an edge

$$w(e)=\begin{cases}\alpha, & \text{if } e \text{ is a horizontal edge in an odd numbered column}\\ 1, & \text{otherwise}\end{cases}$$

Interacting particle system is determinantal

[Lindstrom Gessel Viennot]

LGV formula

Probability for particle configuration $(x_j^{(m)})_{j,m}$ **is**

$$\frac{1}{Z_N} \prod_{m=1}^{2N} \det \left[T_m \left(x_i^{(m-1)}, x_j^{(m)} \right) \right]_{i,j=0,\ldots,N-1}$$

with transition matrices

$$T_m(x,y) = \begin{cases} \alpha & \text{if } y = x \text{ and } m \text{ is odd} \\ 1 & \text{if } y = x \text{ and } m \text{ is even} \\ 1 & \text{if } y = x + 1 \\ 0 & \text{otherwise} \end{cases}$$

Sum formula for correlation kernel **[Eynard Mehta]**

Transition matrices are Toeplitz

Observation:

T_m **is an infinite Toeplitz matrix with symbol**

$$a_m(z) = \begin{cases} z + \alpha, & \textbf{if } m \textbf{ is odd} \\ z + 1, & \textbf{if } m \textbf{ is even} \end{cases}$$

Theorem [Duits-K]; (scalar version)

Correlation kernel is

$$K(x_1, y_1; x_2, y_2) = -\frac{\chi_{x_1>x_2}}{2\pi i} \oint_\gamma \prod_{m=x_2+1}^{x_1} a_m(z) \cdot z^{y_2-y_1-1} dz$$

$$+\frac{1}{(2\pi i)^2} \oint_\gamma \frac{dz}{z} \oint_\gamma \frac{dw}{w^{2N}} \prod_{m=x_2+1}^{N} a_m(w) \cdot R_N(w,z) \cdot \prod_{m=1}^{x_1} a_m(z) \cdot \frac{w^{y_2}}{z^{y_1}}$$

R_N **is the reproducing kernel for orthogonal polynomials on** γ **with weight**

$$W(z) = \frac{1}{z^{2N}} \prod_{m=1}^{2N} a_m(z) = \frac{(z+1)^N (z+\alpha)^N}{z^{2N}}$$

Orthogonal polynomials

$$\frac{1}{2\pi i}\oint_\gamma p_n(z) z^k \frac{(z+1)^N (z+\alpha)^N}{z^{2N}} dz = \kappa_n \delta_{k,n}, \qquad k = 0, \ldots, n-1$$

with reproducing kernel

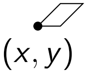
$$R_N(w,z) = \sum_{n=0}^{N-1} \frac{p_n(w) p_n(z)}{\kappa_n}$$
$$= \kappa_N^{-1} \frac{p_N(z) p_{N-1}(w) - p_{N-1}(z) p_N(w)}{z-w}$$

Non-hermitian orthogonality! Existence of OP is not automatic but can be proved for degrees $n \leq 2N$

OP is Jacobi polynomials $P_n^{(-2N,2N)}$ **in case** $\alpha = 1$

3. Tile probabilities

Probabilities for lozenges (one-point functions)

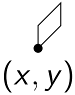
$\mathbb{P}\Big($ 

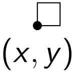
$\Big)$

$$= 1 - K(x, y; x, y)$$

$$= 1 - \frac{1}{(2\pi i)^2} \oint_{\gamma} \oint_{\gamma} R_N(w, z) \frac{(w+1)^N (w+\alpha)^N}{w^{2N}} \times \frac{(z+1)^{\lfloor \frac{x}{2} \rfloor} (z+\alpha)^{\lfloor \frac{x+1}{2} \rfloor}}{(w+1)^{\lfloor \frac{x}{2} \rfloor} (w+\alpha)^{\lfloor \frac{x+1}{2} \rfloor}} \frac{w^y}{z^y} \frac{dwdz}{z}.$$

with similar double contour integral formulas for

$\mathbb{P}\Big($ 

$\Big)$ **and** $\mathbb{P}\Big($ 

$\Big)$

Large N limit

Suppose x and y vary with N such that

$$\lim_{N\to\infty} \frac{x}{N} = 1+\xi, \qquad \lim_{N\to\infty} \frac{y}{N} = 1+\eta$$

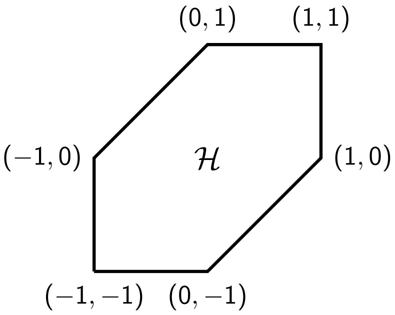
(ξ,η) are coordinates for the hexagon $\mathcal{H}$

Double contour integral has relevant saddle point

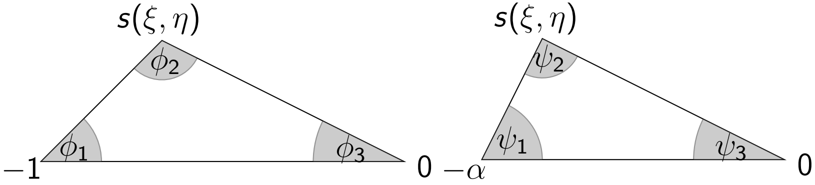
$$s(\xi,\eta)$$

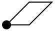
Liquid region is characterized by

$$\operatorname{Im} s(\xi,\eta) > 0$$

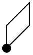


Main result on limiting tile probabilities



$\lim_{N\to\infty} \mathbb{P}\Big($  $(x,y)\Big)$

$$= \frac{\phi_3}{\pi} = \frac{\psi_3}{\pi} = 1 - \frac{1}{\pi}\arg s(\xi,\eta),$$

$\lim_{N\to\infty} \mathbb{P}\Big($  $(x,y)\Big)$

$$= \begin{cases} \dfrac{\phi_1}{\pi}, & x \textbf{ odd,} \\ \dfrac{\psi_1}{\pi}, & x \textbf{ even,} \end{cases}$$

$\lim_{N\to\infty} \mathbb{P}\Big($  $(x,y)\Big)$

$$= \begin{cases} \dfrac{\phi_2}{\pi}, & x \textbf{ odd,} \\ \dfrac{\psi_2}{\pi}, & x \textbf{ even,} \end{cases}$$

4. Saddle points

Saddle point equation

Asymptotic analysis of the orthogonal polynomials.

If $p_N(z) \approx e^{g(z)N}$ **then (very roughly)**

$$R_N(w,z) \approx e^{(g(w)+g(z))N}$$

and the integrand of the double integral is

$$\approx e^{g(z)N}(z+1)^{\frac{1+\xi}{2}N}(z+\alpha)^{\frac{1+\xi}{2}N}z^{-(1+\eta)N} \\ \times e^{g(w)N}(w+1)^{\frac{1-\xi}{2}N}(w+\alpha)^{\frac{1-\xi}{2}N}w^{-(1-\eta)N}$$

Saddle point equations

$$g'(z) + \frac{1+\xi}{2(z+1)} + \frac{1+\xi}{2(z+\alpha)} - \frac{1+\eta}{z} = 0$$

$$g'(w) + \frac{1-\xi}{2(w+1)} + \frac{1-\xi}{2(w+\alpha)} - \frac{1-\eta}{2} = 0$$

g-function

g **function typically takes the form**

$$g(z) = \int \log(z - s) d\mu_0(s)$$

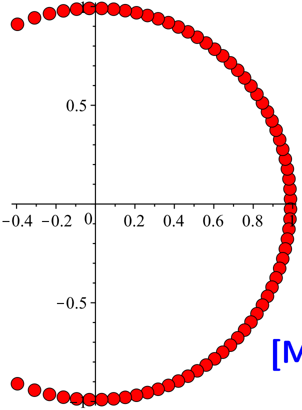
where μ_0 is the weak limit of the normalized zero counting measures of the orthogonal polynomials

$$\frac{1}{N} \sum_{p_N(z)=0} \delta_z \xrightarrow{*} \mu_0$$

Where are the zeros of the orthogonal polynomials?

Zeros of orthogonal polynomials: $\alpha = 1$

Zeros of $P_N^{(-2N,2N)}$ cluster as $N \to \infty$ to an arc on the unit circle.

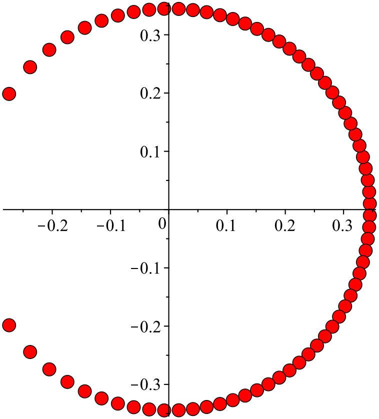


[Martínez-Finkelshtein Orive]

[Driver Duren]

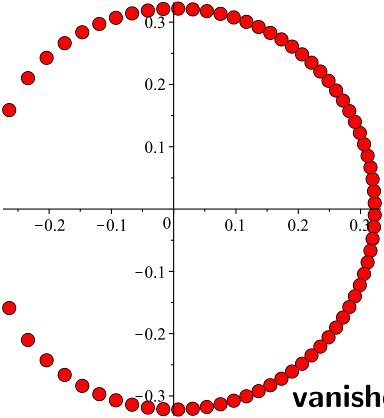
Zeros of orthogonal polynomials: $1/9 < \alpha < 1$

Zeros of P_N cluster as $N \to \infty$ to an arc on the circle of radius $\sqrt{\alpha}$.



Zeros of orthogonal polynomials: $\alpha = 1/9$

The circular arc closes at $\alpha = 1/9$



The density of zeros vanishes quadratically at $-1/3$

Local behavior in terms of Lax pair solutions for Hastings-McLeod solution of Painlevé II

5. Equilibrium measure

Equilibrium conditions

Take $V(z) = 2\log z - \log(z+1) - \log(z+\alpha)$

μ_0 **should be probability measure on contour** γ_0 **going around** 0 **such that** $g(z) = \int \log(z-s)d\mu_0(s)$ **satisfies**

$$\operatorname{Re}\left[g_+(z) + g_-(z) - V(z) + \ell\right] \begin{cases} = 0, & \textbf{for } z \in \operatorname{supp}(\mu_0), \\ \leq 0, & \textbf{for } z \in \gamma_0 \setminus \operatorname{supp}(\mu_0), \end{cases}$$

$\operatorname{Im}\left[g_+(z) + g_-(z) - V(z)\right]$ **is constant on each connected component of** $\operatorname{supp}(\mu_0)$,

μ_0 **is equilibrium measure of** γ_0 **in external field** $\operatorname{Re} V$

γ_0 **is a contour with the** S**-property** **[Stahl]**

Rational function Q_α

Since $g'_+ + g'_- = V'$ **on the support**

$$\left[\int \frac{d\mu_0(s)}{z-s} - \frac{V'(z)}{2}\right]^2 = Q_\alpha(z)$$ **is a rational function**

Rational function Q_α

Since $g'_+ + g'_- = V'$ **on the support**

$$\left[\int \frac{d\mu_0(s)}{z-s} - \frac{V'(z)}{2}\right]^2 = Q_\alpha(z)$$ **is a rational function**

If $\alpha \geq 1/9$ **then**

$$Q_\alpha(z) = \frac{(z+\sqrt{\alpha})^2(z-z_+(\alpha))(z-z_-(\alpha))}{z^2(z+1)^2(z+\alpha)^2}$$

with $z_\pm(\alpha) = \sqrt{\alpha}\, e^{\pm i\theta_\alpha}$ **for some** $\frac{2\pi}{3} \leq \theta_\alpha \leq \pi$

If $\alpha < 1/9$ **then**

$$Q_\alpha(z) = \frac{(z-z_+(\alpha))^2(z-z_-(\alpha))^2}{z^2(z+1)^2(z+\alpha)^2}$$

with real $-1 < z_-(\alpha) < -\sqrt{\alpha} < z_+(\alpha) < -\alpha$

Liquid/frozen regions

Saddle point equation

$$Q_\alpha(z) = \left(-\frac{\xi}{2(z+1)} - \frac{\xi}{2(z+\alpha)} + \frac{\eta}{z}\right)^2$$

becomes a degree four polynomial equation in z.

It has four solutions (four saddles).

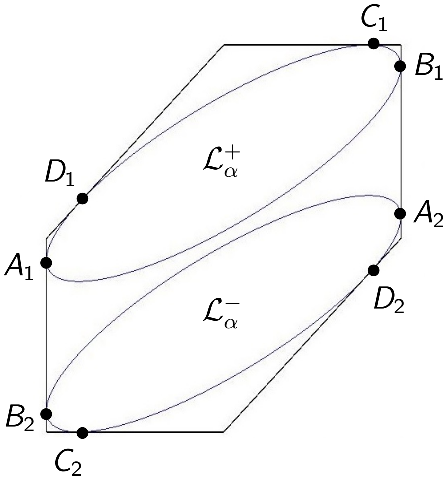
Lemma

If (ξ, η) belongs to the hexagon, then at least two saddles in $(-1, -\alpha)$.
Hence at most one saddle in $\mathbb{C}^+$.

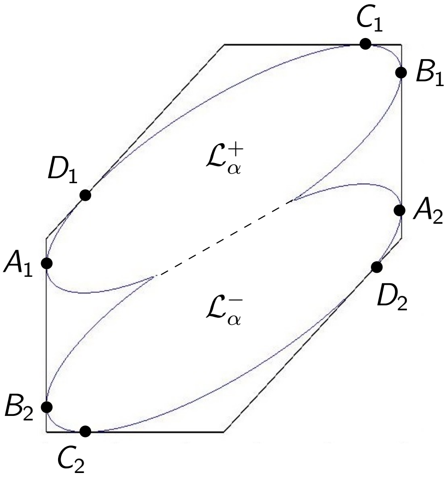
Liquid region $\mathcal{L}_\alpha$: there is a saddle $z = s(\xi, \eta)$ in $\mathbb{C}^+$.

Otherwise frozen region: all saddles are real.

Liquid region for $\alpha < \frac{1}{9}$



Liquid region for $\alpha > \frac{1}{9}$



Transition at $\alpha = \frac{1}{9}$: tangent ellipses and tacnode...

6. Riemann-Hilbert problem

RH problem for orthogonal polynomials

RH problem **[Fokas Its Kitaev]**

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & \frac{(z+1)^N (z+\alpha)^N}{z^{2N}} \\ 0 & 1 \end{pmatrix} \text{ on } \gamma_0$$

$$Y(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix} \text{ as } z \to \infty$$

Reproducing kernel in terms of solution of RH problem

$$R_N(w,z) = \frac{1}{z-w} \begin{pmatrix} 0 & 1 \end{pmatrix} Y^{-1}(w) Y(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Transformation

First transformation in RH analysis

$$T(z) = \begin{pmatrix} e^{N\frac{\ell}{2}} & 0 \\ 0 & e^{-N\frac{\ell}{2}} \end{pmatrix} Y(z) \begin{pmatrix} e^{-N(g(z)+\frac{\ell}{2})} & 0 \\ 0 & e^{N(g(z)+\frac{\ell}{2})} \end{pmatrix}$$

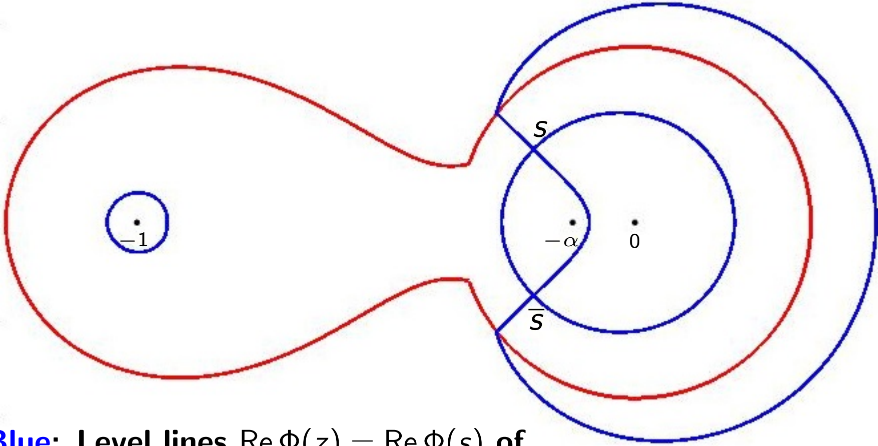
Steepest descent analysis as in
[Deift Kriecherbauer McLaughlin Venakides Zhou]

Main outcome

$T(z)$ **and** $T^{-1}(z)$ **remain bounded as** $N \to \infty$**, uniformly for** z **away from the branch points.**

7. Deformation of contours

Possible contours for $(\xi, \eta) \in \mathcal{L}_\alpha^-$, $\xi < 0$

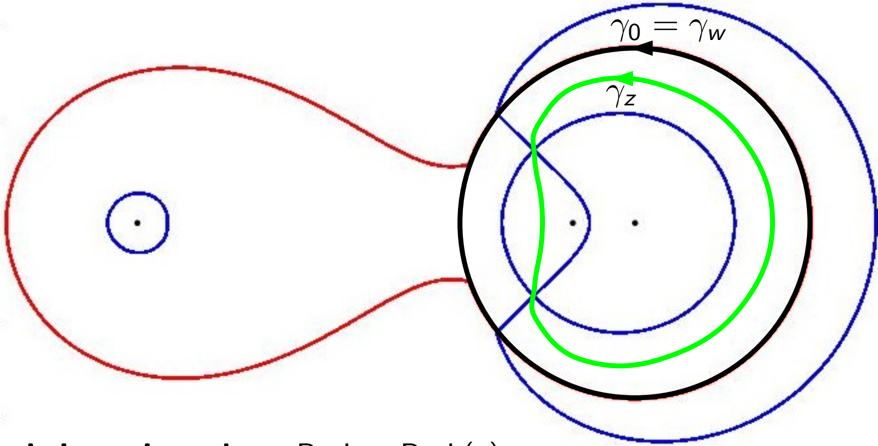


Blue: Level lines $\mathrm{Re}\,\Phi(z) = \mathrm{Re}\,\Phi(s)$ **of**

$$\Phi(z) = g(z) + \frac{1+\xi}{2(z+1)} + \frac{1+\xi}{2(z+\alpha)} - \frac{1+\eta}{z}$$

Figure is for $\alpha = \frac{1}{8}$.

More contours



γ_z **is in region where** $\mathrm{Re}\,\Phi < \mathrm{Re}\,\Phi(s)$

Algebraic identity

$$R_N(w,z)\frac{(w+1)^N(w+\alpha)^N}{w^{2N}}$$

$$= \begin{pmatrix}1 & 0\end{pmatrix} T_-^{-1}(w)T(z)\begin{pmatrix}1\\0\end{pmatrix} e^{N(g(z)-g_-(w))}$$

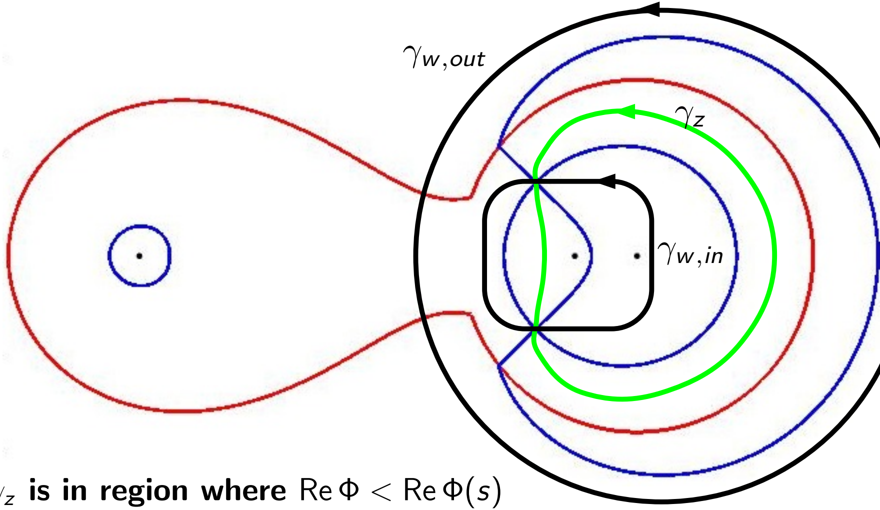
$$- \begin{pmatrix}1 & 0\end{pmatrix} T_+^{-1}(w)T(z)\begin{pmatrix}1\\0\end{pmatrix} e^{N(g(z)-g_-(w))}$$

Deform first term to outside and second term to inside

Integrand for third type lozenge is (essentially)

$$\begin{pmatrix}1 & 0\end{pmatrix} T^{-1}(w)T(z)\begin{pmatrix}1\\0\end{pmatrix} e^{N(\Phi(z)-\Phi(w))}\frac{1}{z(z-w)}$$

Deformation of γ_w



γ_z **is in region where** $\mathrm{Re}\,\Phi < \mathrm{Re}\,\Phi(s)$

$\gamma_{w,in}$ **and** $\gamma_{w,out}$ **are in region where** $\mathrm{Re}\,\Phi > \mathrm{Re}\,\Phi(s)$

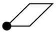
Deforming γ_w **to** $\gamma_{w,in}$ **we may go across a pole at** $w = z$

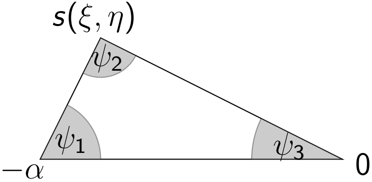
Pole contribution

Remaining double integrals are small

$$\frac{1}{(2\pi i)^2}\int_{\gamma_z} dz \int_{\gamma_{w,in}\cup\gamma_{w,out}} dw \begin{pmatrix}1 & 0\end{pmatrix} T^{-1}(w)T(z)\begin{pmatrix}1\\0\end{pmatrix}$$
$$\times\, e^{N(\Phi(z)-\Phi(w))}\frac{1}{z(z-w)} \to 0 \quad \textbf{as } N\to\infty.$$

Contributions from pole crossings combine to

$$1 - \lim_{N\to\infty}\mathbb{P}\Bigg($$

$$(x,y)\Bigg) = \frac{1}{2\pi i}\int_{\bar{s}}^{s}\frac{dz}{z} = \frac{1}{\pi}\arg s(\xi,\eta) = 1 - \frac{\psi_3}{\pi}$$



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in preparation (coming soon...)

M. Duits and A.B.J. Kuijlaars
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J. Eur. Math. Soc. (to appear), arXiv:1712.05636

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