# Log-gases on a quadratic lattice via discrete loop equations

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## Log-gases on a quadratic lattice

Fix $S \subset \mathbb{R}$, $N > 0$ and

$$w : S \to \mathbb{R}_{\geqslant 0}, \quad w(x) \geqslant 0.$$

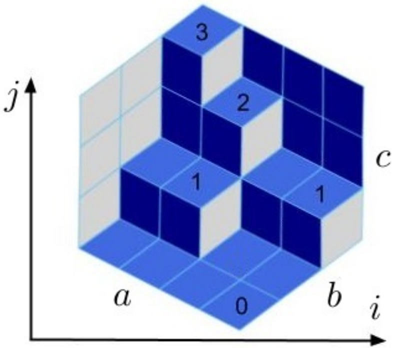
We consider a class of probability measures $\mathbb{P}_N(S, w)$ on all $N$-point subsets of $S$ of the form

$$\mathbb{P}_N(\ell_1, \ell_2, \ldots, \ell_n) \propto \prod_{1 \leqslant i < j \leqslant N} (\ell_i - \ell_j)^2 \cdot \prod_{i=1}^{N} w(\ell_i),$$

where $\ell_i \in S$ for $i = 1, \ldots, N$,

where $S = \{q^{-x} + uq^x : 0 \leqslant x \leqslant M\}$ with $q \in (0,1)$, $x, M \in \mathbb{Z}_{\geqslant 0}$ and $u \in [0,1)$.

# Tiling model




Lozenge tilings of a hexagon can be viewed as stepped surfaces.

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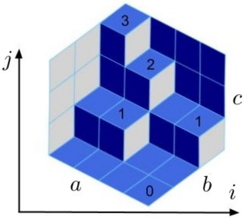
Consider the probability
measure on the set of tilings defined by

$$\mathcal{P}(\mathcal{T}) = \frac{\omega(\mathcal{T})}{Z(a,b,c)}, \text{ where } \omega(\mathcal{T}) = \prod_{\Diamond \in \mathcal{T}} \omega(\Diamond).$$



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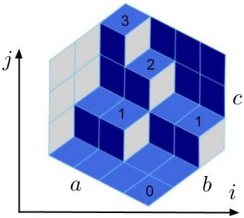


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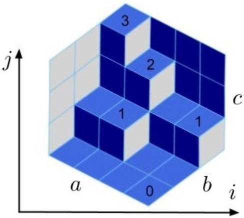


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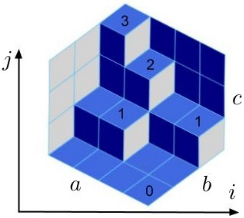


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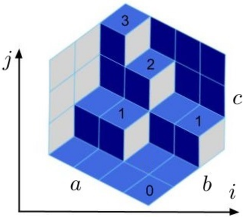


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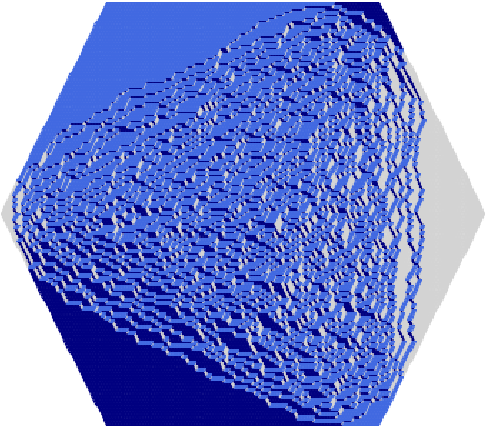
$$\omega(\mathcal{T}) = const(a,b,c) \cdot q^{-\textbf{volume}}.$$

- Let $j$ be the coordinate of $\diamond$. Set $\omega(\diamond) = \kappa q^{j} - \kappa^{-1} q^{-j}$, $1 > q > 0$.

# Limit shape



# Waterfall

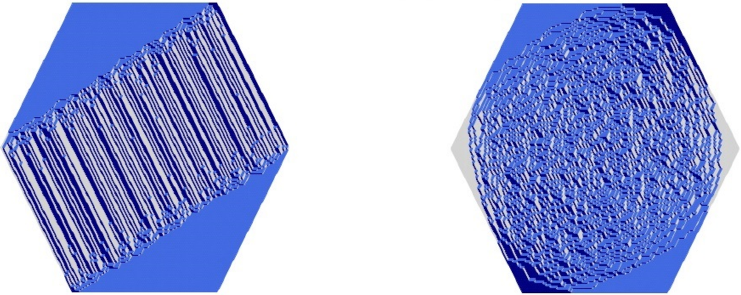
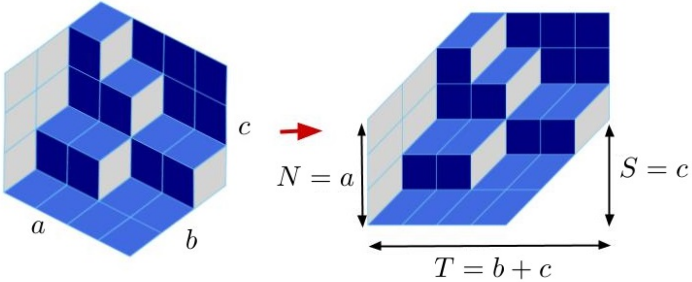


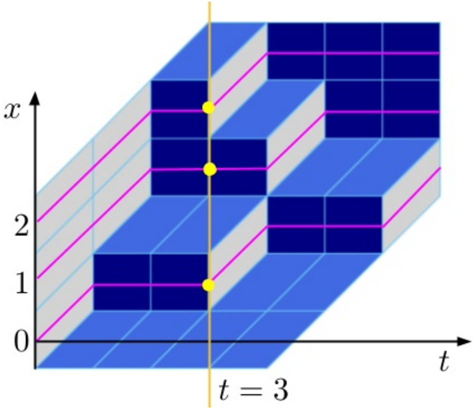
Figure: A simulation for $a = 80$, $b = 80$, $c = 80$. On the left picture the parameters are $\kappa^2 = -1$, $q = 0.8$, and on the right picture the parameters are $\kappa^2 = -1$, $q = 0.98$.

# Affine transformation



# Tiling model

It establishes a bijection between tilings and non-intersecting paths:




Let $C(t) = (x_1, x_2, \ldots, x_N)$ be the positions of nodes.

# q-Racah ensemble

Theorem (Borodin, Gorin, Rains '2009)

$$\mathrm{Prob}\{C(t) = (x_1, \ldots, x_N)\} = C \cdot \prod_{0 \leqslant i < j \leqslant M} (\sigma(x_i) - \sigma(x_j))^2 \prod_{i=1}^{N} w_t(x_i),$$

*where* $\sigma(x_i) = q^{-x_i} + u(\kappa, N, S, T) q^{x_i}$ *and* $w_t(x)$ *is the weight function of the q-Racah polynomial ensemble up to a factor not depending on* $x$.

# Limit shape for domino and lozenze tilings

- **Law of Large Numbers** for the height function [Cohn–Larsen–Propp '98], [Cohn–Kenyon–Propp '01], [Kenyon–Okounkov '07]

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- **Central Limit Theorem**: convergence of the global fluctuations of the height function [Kenyon '01], [Borodin–Ferrari '08], [Petrov '13], [Duse–Metcalfe '14], [Bufetov–Gorin '17], [Bufetov-K. '17]

# Regularity assumptions

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  - $M_N \geqslant N - 1$
  - $\max\left(N^2 \left|q_N - \mathtt{q}^{1/N}\right|, |M_N - N \cdot \mathtt{M}|, N|u_N - \mathtt{u}|\right) \leqslant A_1$, for some $A_1 > 0$.

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- Denote

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N}\delta\left(\ell_i\right),$$

where $(\ell_1, \ldots, \ell_N)$ is $\mathbb{P}_N -$ distributed with parameters $q_N, u_N, M_N$.

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- For some positive constants $A_2, A_3 > 0$

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- $V(s)$ is differentiable and for some $A_4 > 0$

$$\left|V'(s)\right| \leqslant A_4 \cdot \left[1 + |\log|s - 1 - \mathtt{u}|| + |\log|s - \mathtt{q}^{-\mathtt{M}} - \mathtt{u}\mathtt{q}^{\mathtt{M}}||\right],$$

for $s \in \left[1 + \mathtt{u}, \mathtt{q}^{-\mathtt{M}} + \mathtt{u}\mathtt{q}^{\mathtt{M}}\right]$.

# Law of Large Numbers

## Theorem (Dimitrov-K. '18)

*There is a deterministic, compactly supported and absolutely continuous probability measure* $\mu(x)dx$ *such that* $\mu_N$ *concentrate (in probability) near* $\mu$*. More precisely, for any Lipschitz function* $f(x)$ *defined on a real neighborhood of the interval* $[1+\mathtt{u}, \mathtt{q}^{-\mathtt{M}} + \mathtt{u}\mathtt{q}^{\mathtt{M}}]$ *and each* $\varepsilon > 0$ *the random variables*

$$N^{1/2-\varepsilon} \left| \int_{\mathbb{R}} f(x)\mu_N(dx) - \int_{\mathbb{R}} f(x)\mu(x)dx \right|$$

*converge to* 0 *in probability and in the sense of moments.*

## Theorem (Dimitrov-K. '18)

*Take $m \geqslant 1$ polynomials $f_1, \dots, f_m \in \mathbb{R}[x]$ and define*

$$\mathcal{L}_{f_i} = N \int_{\mathbb{R}} f_j(x)\mu_N(dx) - N\mathbb{E}_{\mathbb{P}_N}\left[\int_{\mathbb{R}} f_j(x)\mu_N(dx)\right] \text{ for } i = 1, \dots, m.$$

*Assume that the limit measure has one band, then under technical assumptions the random variables $\mathcal{L}_{f_i}$ converge jointly in the sense of moments to an $m$-dimensional centered Gaussian vector $X = (X_1, \dots, X_m)$ with covariance*

$$Cov(X_i, X_j) = \frac{1}{(2\pi i)^2} \oint_{\Gamma} \oint_{\Gamma} f_i(s) f_j(t) \mathcal{C}(s,t) ds dt, \text{ where}$$

$$\mathcal{C}(s,t) = \frac{1}{2(s-t)^2}\left(1 - \frac{(s-\alpha_1)(t-\beta_1) + (t-\alpha_1)(s-\beta_1)}{2\sqrt{(s-\alpha_1)(s-\beta_1)}\sqrt{(t-\alpha_1)(t-\beta_1)}}\right),$$

*where $\Gamma$ is a positively oriented contour that encloses the interval $[1+\mathrm{u}, \mathrm{q}^{-\mathrm{M}} + \mathrm{u}\mathrm{q}^{\mathrm{M}}]$.*

# General $\beta$

Let $q \in (0,1)$, $u \in [0,1)$, $M \geqslant N-1 \in \mathbb{Z}$, $\theta > 0$.

- Recall

$$\Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty} \text{ and satisfies } \frac{\Gamma_q(x+1)}{\Gamma_q(x)} = \frac{1-q^x}{1-q},$$

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- Let $\ell_i = q^{-\lambda_i} + uq^{\lambda_i}$, where

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- Denote $\mathfrak{X}_N^\theta := \{(\ell_1,\ldots,\ell_N)\}$.

# General $\beta$

We consider probability measures on $\mathfrak{X}_N^\theta$

$$\mathbb{P}_N{}^\theta(\ell_1,\ldots,\ell_N) = \frac{1}{Z_N}\prod_{1\leqslant i<j\leqslant N} q^{2\theta\lambda_j}\frac{\Gamma_q(\lambda_j-\lambda_i+1)\Gamma_q(\lambda_j-\lambda_i+\theta)}{\Gamma_q(\lambda_j-\lambda_i)\Gamma_q(\lambda_j-\lambda_i+1-\theta)}\times$$

$$\times\prod_{1\leqslant i<j\leqslant N}\frac{\Gamma_q(\lambda_j+\lambda_i+v+1)\Gamma_q(\lambda_j+\lambda_i+v+\theta)}{\Gamma_q(\lambda_j+\lambda_i+v)\Gamma_q(\lambda_j+\lambda_i+v+1-\theta)}\cdot\prod_{i=1}^N w(\ell_i),$$

$v$ is such that $q^u = v$.

# General $\beta$ analogue

Observe as $q \to 1^-$ and $q^x \to y \in [0, 1)$

$$\frac{\Gamma_q(x+\alpha)}{\Gamma_q(x)} = (1-q)^{-\alpha}\frac{(q^x;q)}{(q^{x+\alpha};q)_\infty} \sim (1-q)^{-\alpha}(1-y)^{\alpha}.$$

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Setting $q^{-\lambda_i} = y_i$, $\ell_i = y_i + u/y_i$ for $i = 1, \dots, N$ we get

$$\prod_{1\leqslant i<j\leqslant N} y_j^{2\theta}(1-y_i y_j^{-1})^{2\theta} \cdot (1-u y_i^{-1} y_j^{-1})^{2\theta} = \prod_{1\leqslant i<j\leqslant N} (\ell_j - \ell_i)^{2\theta}.$$

# LLN and CLT for discrete log-gases

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- **Law of Large numbers** [Johansson '00, '02], [Feral '08]
- **Central limit theorem**
  - special cases [Borodin-Ferrari '08], [Breuer–Duits '13], [Dolega–Feray '15]
  - general potential with general $\beta$ [Borodin–Gorin–Guionnet '15]

# How is CLT usually proved?

Johansson's CLT proof in RMT is based on **loop equations**

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$$[\mathbb{E}G_N(z)]^2+\frac{2}{\beta}V'(z)[\mathbb{E}G_N(z)]+(\text{analytic})=\frac{1}{N}(\dots),$$

obtained by a clever integration by parts.

# Assumptions on the weight

- Assume there is an open set $\mathcal{M} \subset \mathbb{C} \backslash \{0, \pm\sqrt{\mathrm{u}}\}$, such that for large $N$

$$\left( \left[ q_N^1, q_N^{-M_N - 1} \right] \cup \left[ u_N q_N^{M_N + 1}, u_N q_N^{-1} \right] \right) \subset \mathcal{M}.$$

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- Suppose there exist analytic functions $\phi_N^+, \phi_N^-$ on $\mathcal{M}$ such that for $z \in \mathcal{M}$ and $\sigma_N(z) = z + u_N z^{-1}$ the following hold

$$\frac{w(\sigma_N(z); N)}{w(\sigma_N(q_N z); N)} = \frac{\phi_N^+(z)}{\phi_N^-(z)}, \qquad \phi_N^+(q_N^{-M_N-1}) = \phi_N^-(1) = 0$$

# Nekrasov's equation

## Theorem

*Define*

$$R_N(z) = \Phi_N^-(z) \cdot \mathbb{E}_{\mathbb{P}_N}\left[\prod_{i=1}^{N} \frac{\sigma_N(q_N z) - \ell_i}{\sigma_N(z) - \ell_i}\right] + \\ +\Phi_N^+(z) \cdot \mathbb{E}_{\mathbb{P}_N}\left[\prod_{i=1}^{N} \frac{\sigma_N(z) - \ell_i}{\sigma_N(q_N z) - \ell_i}\right],$$

$$\text{where} \quad \Phi^-(z) = q(qz^2 - u)(z^2 - u)\phi^-(z),$$

$$\Phi^+(z) = (qz^2 - u)(q^2 z^2 - u)\phi^+(z) \text{ and } \sigma(z) = z + uz^{-1}.$$

*Then $R(z)$ is analytic in the same complex neighborhood $\mathcal{M}$. If $\Phi^{\pm}(z)$ are polynomials of degree at most $d$, then $R(z)$ is also a polynomial of degree at most $d$.*

# Computations

$$\prod_{i=1}^{N}\frac{\sigma(qz)-\sigma(q^{-\ell_i})}{\sigma(z)-\sigma(q^{-\ell_i})} = \exp\left(\sum_{i=1}^{N}\left(\log\left(\frac{(qz-q^{-\ell_i})}{(z-q^{-\ell_i})}\cdot\frac{(z-uq^{\ell_i-1})}{(z-uq^{\ell_i})}\right)\right)\right)$$

$$= \exp\left(\sum_{i=1}^{N}\left(\log\left(1+\frac{(q-1)z}{z-q^{-\ell_i}}\right)+\log\left(1+\frac{(q^{-1}-1)uz^{-1}}{uz^{-1}-q^{\ell_i}}\right)\right)\right).$$

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$$= \exp\left(\sum_{i=1}^{N}\left(\log\left(1 + \frac{(q-1)z}{z - q^{-\ell_i}}\right) + \log\left(1 + \frac{(q^{-1} - 1)uz^{-1}}{uz^{-1} - q^{\ell_i}}\right)\right)\right).$$

Now we set $q = \mathrm{q}^{\frac{1}{N}}$.

$$\prod_{i=1}^{N} \frac{\sigma(qz) - \sigma(q^{-\ell_i})}{\sigma(z) - \sigma(q^{-\ell_i})} = \mathrm{q} \exp\Big[(z - uz^{-1}) G_\mu(z) + $$

$$+ \frac{1}{N}\left(\Delta G_N(z) + \frac{z \log \mathrm{q}}{2} \frac{\partial}{\partial z}\left((z - uz^{-1}) G_N(z)\right)\right) + \cdots\Big],$$

where $\Delta G_N(z) = N(G_N(z) - G_\mu(z))$.

# Functional equation on the Stieltjes transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(x)dx}{z - x}$$

$$\lim_{N\to\infty} \mathbb{E}_{\mathbb{P}_N}\left[\prod_{i=1}^{N} \frac{\sigma_N(q_N z) - \ell_i}{\sigma_N(z) - \ell_i}\right] = \exp\left(\mathfrak{G}(z)\right)$$

$$\text{with } \mathfrak{G}(z) = \log(\mathrm{q}) \cdot (z - \mathrm{u}z^{-1}) \cdot G_\mu(z + \mathrm{u}z^{-1}).$$

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with $\mathfrak{G}(z) = \log(\mathrm{q}) \cdot (z - \mathrm{u}z^{-1}) \cdot G_\mu(z + \mathrm{u}z^{-1})$.

$$R_\mu(z) = \Phi^-(z)\mathrm{q}^{(z-u/z)G_\mu(z+u/z)} + \Phi^+(z)\mathrm{q}^{-(z-u/z)G_\mu(z+u/z)}.$$

## Proposition

*The density of the limit measure* $\mu$ *for* $x \in (0, \mathtt{M})$ *is given by*

$$\mu(x) = \frac{1}{\pi} \arccos\left( \frac{R(q^{-x})}{\sqrt{\Phi^{+}(q^{-x})\Phi^{-}(q^{-x})}} \right)$$

*If the expression inside the arccosine is greater than* $1$*, then we set* $\mu(x) = 0$ *and if it is less than* $-1$*, then we set* $\mu(x) = 1$*.*

# Discrete RHP

## Definition

Let $\omega(z)$ be a $2 \times 2$ matrix-valued function and $\mathfrak{X} \subset \mathbb{R}$ finite. An analytic function

$$m \colon \mathbb{C} \backslash \mathfrak{X} \to \mathrm{Mat}(2, \mathbb{C})$$

solves a DRHP $(\mathfrak{X}, \omega)$ if the entries of $m$ are meromorphic with at most simple poles at the points of $\mathfrak{X}$ and its residues at these points are given by the jump residue condition

$$\operatorname*{Res}_{z=x} m(z) = \lim_{z \to x} m(z)\omega(z), \;\; x \in \mathfrak{X}.$$

## Proposition

*The DRHP $(S,\omega)$ has a unique solution $\mathfrak{m}(z)$ such that*

$$\omega(\psi) = \begin{bmatrix} 0 & w(\psi) \\ 0 & 0 \end{bmatrix}, \quad \mathfrak{m}(\psi)\cdot\begin{bmatrix} \psi^{-N} & 0 \\ 0 & \psi^{N} \end{bmatrix} = I + O\left(\frac{1}{\psi}\right), z \to \infty.$$

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$$\omega(\psi) = \begin{bmatrix} 0 & w(\psi) \\ 0 & 0 \end{bmatrix}, \quad \mathfrak{m}(\psi) \cdot \begin{bmatrix} \psi^{-N} & 0 \\ 0 & \psi^{N} \end{bmatrix} = I + O\left(\frac{1}{\psi}\right), z \to \infty.$$

*Moreover,*

$$A(z) = \mathfrak{m}(\sigma(qz)) \begin{bmatrix} \Phi^{-}(z) & 0 \\ 0 & \Phi^{+}(z) \end{bmatrix} \mathfrak{m}^{-1}(\sigma(z)) \text{ is analytic and}$$

$$\mathrm{Tr}[A] = \Phi^{-}(z) \cdot \mathbb{E}_{\mathbb{P}_N}\left[\prod_{i=1}^{N} \frac{\sigma(qz) - \ell_i}{\sigma(z) - \ell_i}\right] + \Phi^{+}(z) \cdot \mathbb{E}_{\mathbb{P}_N}\left[\prod_{i=1}^{N} \frac{\sigma(z) - \ell_i}{\sigma(qz) - \ell_i}\right].$$

# Special cases of $R_\mu(z)$

Recall that monic orthogonal polynomials satisfy difference equation:

$$A_N(z)P_N(\sigma(z)) = B_N(z)P_N(\sigma(q^{-1}z)) + C_N(z)P_N(qz)).$$

$$\text{Then } R_\mu(z) = \lim_{N\to\infty} A_N(z).$$

# Nekrasov's equation

## Theorem (Dimitrov-K. '18)

*Define*

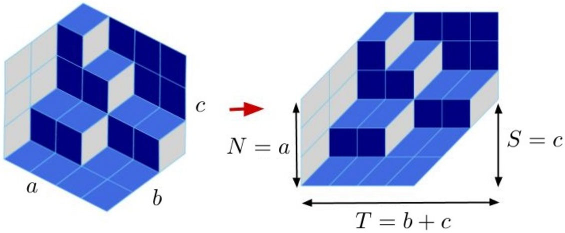
$$R(z) = \Phi^-(z) \cdot \mathbb{E}_{\mathbb{P}_N^\theta}\left[\prod_{i=1}^{N} \frac{\sigma(q^\theta z) - \ell_i}{\sigma(z) - \ell_i}\right] + \\ + \Phi^+(z) \cdot \mathbb{E}_{\mathbb{P}_N^\theta}\left[\prod_{i=1}^{N} \frac{\sigma(q^{1-\theta} z) - \ell_i}{\sigma(qz) - \ell_i}\right],$$

$$\textit{where} \quad \Phi^-(z) = q^\theta (q^{2-\theta} z^2 - u)(z^2 - u)\phi^-(z),$$

$$\Phi^+(z) = (q^\theta z^2 - u)(q^2 z^2 - u)\phi^+(z) \textit{ and } \sigma(z) = z + uz^{-1}.$$

*Then $R(z)$ is analytic in the same complex neighborhood $\mathcal{M}^\theta$. If $\Phi^\pm(z)$ are polynomials of degree at most $d$, then $R(z)$ is also a polynomial of degree at most $d$.*

# Limit shape for the tilings




Let real numbers $\mathtt{N}, \mathtt{T}, \mathtt{S}, \mathtt{q}$ and $\mathtt{k}$ be such that

$$\mathtt{N}, \mathtt{T}, \mathtt{S}, \mathtt{q} > 0, \quad \mathtt{k} \geqslant 0, \quad \mathtt{q} < 1, \quad \mathtt{N} < \mathtt{T}, \quad \mathtt{S} < \mathtt{T}, \quad \mathtt{k}^2 \mathtt{q}^{-\mathtt{T}} < 1.$$

# Limit shape

## Theorem (Dimitrov-K '18)

*Let*

$$N = \mathtt{N}\varepsilon^{-1} + O(1), \quad T = \mathtt{T}\varepsilon^{-1} + O(1), \quad S = \mathtt{S}\varepsilon^{-1} + O(1),$$

$$q = \mathtt{q}^{\varepsilon} + O(\varepsilon^2), \quad \kappa = \mathtt{k} + O(\varepsilon).$$

*Then for any point* $(t, x)$ *in the hexagon* $\mathcal{P}$ *with parameters* $\mathtt{N}, \mathtt{T}, \mathtt{S}$ *and* $\eta > 0$ *there exists an explicit function* $\hat{h}(t, x)$ *such that*

$$\lim_{\varepsilon \to 0^+} \mathbb{P}_{\varepsilon}\left(\left|\varepsilon \cdot h\left(\lfloor t\varepsilon^{-1} \rfloor, \lfloor x\varepsilon^{-1} \rfloor + 1/2\right) - \hat{h}(t, x)\right| > \eta\right) = 0.$$

# Limit shape

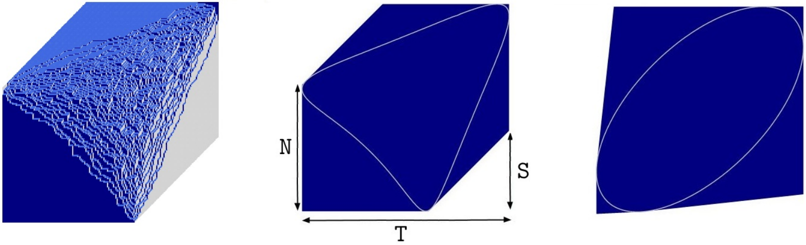



Figure: The left part shows a simulation of a tiling. The middle part shows the hexagon $\mathcal{P}$ and the liquid region $\mathcal{D}$ is the region inside the gray curve. The right part denotes the image of $\mathcal{P}$ and $\mathcal{D}$ under the map $(t, x) \to (\mathtt{q}^{-t}, \mathtt{q}^{-x} + \mathtt{k}^2\mathtt{q}^{-\mathtt{S}-t+x})$

# Fluctuations

- Given a random point configuration $\{(t, x_k^t)\}$ on slice $t$ define $(U, V)$

$$U(t,k) = q^{-t} \text{ and } V(t,k) = q^{-x_k^t} + \kappa^2 q^{x_k^t - S - t}$$

for $0 \leqslant t \leqslant T$ and $1 \leqslant k \leqslant N$.

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- Define a random height function $\mathcal{H}$ for the new particle system as pushforward of $h$.
- This transform bijectively maps liquid region $\mathfrak{D}$ to a new region $\mathfrak{D}'$.

## Theorem (Dimitrov-K. '18)

*Fix* $w \in (1, \mathrm{q}^{-\mathrm{T}})$ *and let* $t(\varepsilon)$ *be a sequence of integers such that* $q^{-t(\varepsilon)} = w + O(\varepsilon)$. *There exists a diffeomorfism*

$$\Omega : \mathfrak{D}' \to \mathbb{H}$$

*such that for any polinomials* $f_i \in \mathbb{R}$, $i = 1, \ldots, m$

$$\int_{\mathbb{R}} \sqrt{\pi} \left( \mathcal{H}(q^{-t}, v) - \mathbb{E}_{\mathbb{P}_\varepsilon} \left[ \mathcal{H}(q^{-t}, v) \right] \right) f_i(v) dv$$

*converge jointly to a Gaussian vector with mean zero and covariance*

$$\mathbb{E}[X_i X_j] = \int_{a(u)}^{b(u)} \int_{a(u)}^{b(u)} f_i(x) f_j(y) \left( -\frac{1}{2\pi} \log \left| \frac{\Omega(u,x) - \Omega(u,y)}{\Omega(u,x) - \overline{\Omega}(u,y)} \right| \right) dxdy.$$

# Diffeomorphism

$\Omega(w,v)$ is a unique solution $z(w,v) \in \mathbb{H}$ for $(w,v) \in \mathcal{D}'$ of

$$a_2(w,v)z^2 + a_1(w,v)z + a_0(w,v) = 0, \text{ where}$$

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- $a_0 = (w-1)(\mathtt{q}^{-\mathtt{T}} - \mathtt{q}^{\mathtt{N}})(\mathtt{q}^{-\mathtt{S}} - 1)(1 - \mathtt{k}^2\mathtt{q}^{-\mathtt{T}+\mathtt{N}});$
- $a_1 = v\mathtt{q}^{\mathtt{N}}(\mathtt{q}^{-\mathtt{T}} - 1) + \big(u(\mathtt{q}^{-\mathtt{S}} - \mathtt{q}^{\mathtt{N}}) - \mathtt{q}^{-\mathtt{S}+\mathtt{N}} - \mathtt{q}^{-\mathtt{T}} + 2\mathtt{q}^{\mathtt{N}}\big) + $
  $+ w\mathtt{k}^2\mathtt{q}^{\mathtt{N}}(\mathtt{q}^{-\mathtt{T}} + \mathtt{q}^{-\mathtt{S}+\mathtt{N}} - 2\mathtt{q}^{-\mathtt{S}-\mathtt{T}}) + \mathtt{k}^2\mathtt{q}^{-\mathtt{T}+\mathtt{N}}(\mathtt{q}^{-\mathtt{S}} - \mathtt{q}^{\mathtt{N}})).$
- $a_2 = \mathtt{q}^{\mathtt{N}}(v - 1 - \mathtt{k}^2\mathtt{q}^{-\mathtt{S}}w);$

# Assumptions

- Assume

$$\Phi_N^-(z) = q_N(z^2 - u_N)\phi_N^-(z) = \Phi^-(z) + \varphi_N^-(z) + O\left(N^{-2}\right) \text{ and}$$
$$\Phi_N^+(z) = (q_N^2 z^2 - u_N)\phi_N^+(z) = \Phi^+(z) + \varphi_N^+(z) + O\left(N^{-2}\right),$$

where $\varphi_N^{\pm}(z) = O(N^{-1})$ and the constants in the big $O$ notation are uniform over $z$ in compact subsets of $\mathcal{M}$. All the aforementioned functions are holomorphic in $\mathcal{M}$.

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  where $\varphi_N^{\pm}(z) = O(N^{-1})$ and the constants in the big $O$ notation are uniform over $z$ in compact subsets of $\mathcal{M}$. All the aforementioned functions are holomorphic in $\mathcal{M}$.

- Assume there exists unique maximal interval

$$[\alpha_1, \beta_1] \subset [1 + \mathtt{u}, \mathtt{q}^{-\mathtt{M}} + \mathtt{u}\mathtt{q}^{\mathtt{M}}]$$

  such that $0 < \mu(x) < f_{\mathtt{q}}(\sigma_{\mathtt{q}}^{-1}(x))^{-1}$ on $[\alpha_1, \beta_1]$, where $\sigma_{\mathtt{q}}(x) = \mathtt{q}^{-x} + \mathtt{u}\mathtt{q}^{x}$ and $f_{\mathtt{q}}(x) = \frac{d}{dx}\sigma_{\mathtt{q}}\left(\mathtt{q}^{-x}\right)$.