Random Orthogonal Polynomials: From matrices to point processes

Diane Holcomb, KTH

Integrability and Randomness in Math Physics
CIRM, April 2019

Outline

1. OPUCs and matrices
2. Random Orthogonal Polynomials and β-ensembles
3. Counting functions and a nice CLT (Killip)
4. The Sine_β limit process via it's counting function
5. Results for Sine_β
6. OPUCs and Dirac Operators (if there's time)

OPUCs (part I)

For any measure on the unit circle ($\partial\mathbb{D}$), we can associate a family of orthogonal polynomials, $\Phi_0(z), \Phi_1(z), \Phi_2(z), \ldots$.

OPUCs (part I)

For any measure on the unit circle ($\partial\mathbb{D}$), we can associate a family of orthogonal polynomials, $\Phi_0(z), \Phi_1(z), \Phi_2(z), \ldots$.

There exists a bijection between measures on the unit circle and sequences of Verblunsky coefficients.

$$\mu \leftrightarrow \{\alpha_k\}_{k=0}^{\infty}$$

where the α_k's give recurrence coefficients that may be used to build the OPUCs that are orthogonal with respect to the measure μ.

OPUCs (part I)

For any measure on the unit circle ($\partial\mathbb{D}$), we can associate a family of orthogonal polynomials, $\Phi_0(z), \Phi_1(z), \Phi_2(z), \ldots$.

There exists a bijection between measures on the unit circle and sequences of Verblunsky coefficients.

$$\mu \leftrightarrow \{\alpha_k\}_{k=0}^{\infty}$$

where the α_k's give recurrence coefficients that may be used to build the OPUCs that are orthogonal with respect to the measure μ.

Particularly in the case where μ has finite support we may study the orthogonal polynomials to obtain information about the measure. If the measure is random this can be more useful that studying the measure directly.

OPUCs (part II): The Szegö Recursion

Suppose that $\Phi_0(z), \Phi_1(z), \ldots$ are a family of OPUCs associated to a measure μ on $\partial\mathbb{D}$.

Define: $\Phi_k^*(z) = z^k \overline{\Phi}_k(\frac{1}{z})$.
Then:

$$\Phi_{k+1}(z) = z\Phi_k(z) - \bar{\alpha}_k \Phi_k^*(z)$$
$$\Phi_{k+1}^*(z) = \Phi_k^*(z) - \alpha_k z \Phi_k(z)$$

$$\begin{bmatrix} \Phi_{k+1}(z) \\ \Phi_{k+1}^*(z) \end{bmatrix} = \begin{bmatrix} z & -\bar{\alpha}_k \\ -\alpha_k z & 1 \end{bmatrix} \begin{bmatrix} \Phi_k(z) \\ \Phi_k^*(z) \end{bmatrix} = T_k \begin{bmatrix} \Phi_k(z) \\ \Phi_k^*(z) \end{bmatrix}$$

Using this notation we can write

$$\begin{bmatrix} \Phi_{k+1}(z) \\ \Phi_{k+1}^*(z) \end{bmatrix} = T_k \cdots T_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

OPUCs and Matrices

Suppose that U_n is an $n \times n$ unitary matrix. We can define a spectral measure μ_n by

$$\int_{\partial \mathbb{D}} f(z) d\mu_n(z) = \langle f(U_n)\mathbf{e}_1, \mathbf{e}_1 \rangle$$

OPUCs and Matrices

Suppose that U_n is an $n \times n$ unitary matrix. We can define a spectral measure μ_n by

$$\int_{\partial\mathbb{D}} f(z) d\mu_n(z) = \langle f(U_n)\mathbf{e}_1, \mathbf{e}_1 \rangle$$

In this case we have that If the measure $\mu_n = \sum_{k=1}^{n} q_k \delta_{z_k}$ and there exists a bijection

$$(\{z_k\}_{k=1}^{n}, \{q_k\}_{k=1}^{n-1}) \leftrightarrow \{\alpha_k\}_{k=0}^{n-1}$$

with $\alpha_k \in \mathbb{D}$ for $k \leq n-1$ and $|\alpha_{n-1}| = 1$.

OPUCs and Matrices

Suppose that U_n is an $n \times n$ unitary matrix. We can define a spectral measure μ_n by

$$\int_{\partial \mathbb{D}} f(z) d\mu_n(z) = \langle f(U_n)\mathbf{e}_1, \mathbf{e}_1 \rangle$$

In this case we have that If the measure $\mu_n = \sum_{k=1}^{n} q_k \delta_{z_k}$ and there exists a bijection

$$(\{z_k\}_{k=1}^{n}, \{q_k\}_{k=1}^{n-1}) \leftrightarrow \{\alpha_k\}_{k=0}^{n-1}$$

with $\alpha_k \in \mathbb{D}$ for $k \leq n-1$ and $|\alpha_{n-1}| = 1$.
The associated Verblunsky coefficients $\{\alpha_k\}_{k=0}^{n-1}$ allow us to generate

$$\Phi_0(z), \Phi_1(z), ..., \Phi_n(z) = \det(U_n - zI)$$

Notice that $\Phi_n(z)$ is not actually orthogonal to the previous polynomials with respect to μ_n.

Random Matrix Ensembles

Recall that if we choose O_n and U_n according to Haar measure on the orthogonal and unitary groups respectively, then the eigenvalues of O_n or U_n have joint distribution given by

$$f(\theta_1, ..., \theta_n) = \frac{1}{Z_{n,\beta}} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^{\beta}. \tag{0.1}$$

for $\beta = 1, 2$.

Random Matrix Ensembles

Recall that if we choose O_n and U_n according to Haar measure on the orthogonal and unitary groups respectively, then the eigenvalues of O_n or U_n have joint distribution given by

$$f(\theta_1, ..., \theta_n) = \frac{1}{Z_{n,\beta}} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^{\beta}. \tag{0.1}$$

for $\beta = 1, 2$.

If we study μ_n defined as the spectral measure at $\mathbf{e}_1$ then

$$\mu_n = \sum_{k=1}^{n} q_k \delta_{e^{i\theta_k}}, \qquad \text{where} \qquad \sum q_k = 1.$$

and the weights $\{q_k\}$ are independent from the $\{\theta_k\}$ with

$$(q_1, ..., q_n) \sim \text{Dirichlet}(\tfrac{\beta}{2}, ..., \tfrac{\beta}{2})$$

Verblunsky's for the β-circular ensemble

The joint density on the previous slide defines an n-point measure on the unit circle (or $[-\pi, \pi]$) for any $\beta > 0$. A set of angles with joint density

$$f(\theta_1, ..., \theta_n) = \frac{1}{Z_{n,\beta}} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^{\beta}. \tag{0.2}$$

is called the β-circular ensemble

Theorem (Killip-Nenciu)

Let $\mu_n = \sum_{k=1}^n q_k \delta_{e^{i\theta_k}}$ with $\{\theta_k\}$ having β-circular distribution and $(q_1, ..., q_n) \sim Dirichlet(\frac{\beta}{2}, ..., \frac{\beta}{2})$. then the associated Verblunsky coefficients will be independent with rotationally invariant distribution and

$$|\alpha_k|^2 \sim \begin{cases} Beta\left(1, \frac{\beta}{2}(n-k-1)\right) & k < n-1 \\ 1 & k = n-1 \end{cases}$$

Finding a counting function

For a measure μ supported on n points we can use the Szegö recursion to define the function $\Phi_n(z)$ (not an OPUC) which is 0 on the support of μ.

$$\begin{aligned} e^{ix} \in \operatorname{supp} \mu &\iff \Phi_n(e^{ix}) = 0 \\ &\iff e^{ix}\Phi_{n-1}(e^{ix}) = \overline{\alpha}_{n-1}\Phi^*_{n-1}(e^{ix}) \end{aligned}$$

On $\partial\mathbb{D}$ the definition of Φ^*_k becomes $\Phi^*_k(e^{ix}) = e^{ixk}\overline{\Phi_k(e^{ix})}$:

$$e^{ix} \in \operatorname{supp} \mu \iff \arg \overline{\alpha}_{n-1} = 2\arg(\Phi_{n-1}(e^{ix})) - x(n-2).$$

Finding a counting function

For a measure μ supported on n points we can use the Szegö recursion to define the function $\Phi_n(z)$ (not an OPUC) which is 0 on the support of μ.

$$
\begin{aligned}
e^{ix} \in \text{supp } \mu \iff & \Phi_n(e^{ix}) = 0 \\
\iff & e^{ix}\Phi_{n-1}(e^{ix}) = \overline{\alpha}_{n-1}\Phi_{n-1}^*(e^{ix})
\end{aligned}
$$

On $\partial\mathbb{D}$ the definition of Φ_k^* becomes $\Phi_k^*(e^{ix}) = e^{ixk}\overline{\Phi_k(e^{ix})}$:

$$
e^{ix} \in \text{supp } \mu \iff \arg \overline{\alpha}_{n-1} = 2\arg(\Phi_{n-1}(e^{ix})) - x(n-2).
$$

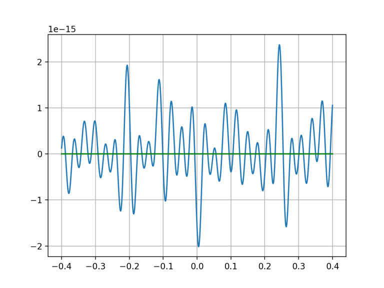
More generally define

$$
\omega_k(x) = 2\arg(\Phi_k(e^{ix})) - x(k-1),
$$

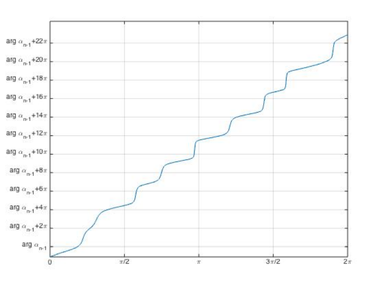
then...

$$
N([0,x]) = \left\lfloor \frac{\omega_{n-1}(x) - \arg \overline{\alpha}_{n-1}}{2\pi} \right\rfloor
$$

The counting function from $\omega_{n-1}(x)$ for Circular β



$P_{350}(x)$ for $n = 1000$



$\omega_{n-1}(x)$ for $n = 12$, $\beta = 4$.

Counting functions are useful!

Theorem (Killip)

Let $N_n(a,b)$ be the number of points of an n-point β-circular ensemble that lie in the arc between a and b, then

$$\sqrt{\frac{\pi^2\beta}{2\log n}}\left[N_n(a,b)-\frac{n(b-a)}{2\pi}\right]\Rightarrow\mathcal{N}(0,1).$$

Counting functions are useful!

Theorem (Killip)

Let $N_n(a,b)$ be the number of points of an n-point β-circular ensemble that lie in the arc between a and b, then

$$\sqrt{\frac{\pi^2\beta}{2\log n}}\left[N_n(a,b) - \frac{n(b-a)}{2\pi}\right] \Rightarrow \mathcal{N}(0,1).$$

Rotational invariance means we can study $[a,b]=[0,x]$. We can compute

$$\omega_k(x) - \omega_{k-1}(x) = 2\arg(1+\tilde{\alpha}_k) + x$$

Where $\tilde{\alpha}_k \overset{d}{=} \alpha_k$ (only for a fixed x)

$$\omega_{n-1}(x) - nx = 2\sum_{k=0}^{n-1}\arg(1+\tilde{\alpha}_k)$$

Counting functions are useful!

Theorem (Killip)

Let $N_n(a,b)$ be the number of points of an n-point β-circular ensemble that lie in the arc between a and b, then

$$\sqrt{\frac{\pi^2\beta}{2\log n}}\left[N_n(a,b)-\frac{n(b-a)}{2\pi}\right]\Rightarrow \mathcal{N}(0,1).$$

Rotational invariance means we can study $[a,b]=[0,x]$. We can compute

$$\omega_k(x)-\omega_{k-1}(x)=2\arg(1+\tilde{\alpha}_k)+x$$

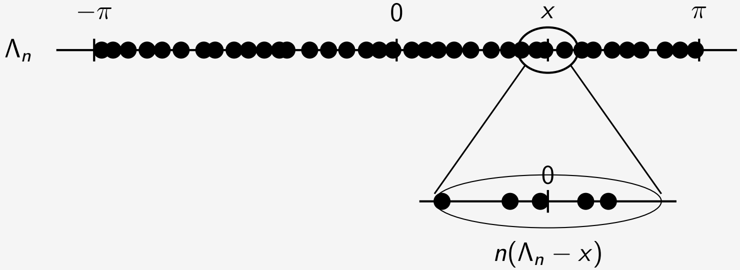
Where $\tilde{\alpha}_k \stackrel{d}{=} \alpha_k$ (only for a fixed x)

$$\omega_{n-1}(x)-nx=2\sum_{k=0}^{n-1}\arg(1+\tilde{\alpha}_k)$$

If we reverse the order of the Verblunsky coefficients we get that $\omega_{n-1}(x)-nx$ is a martinage in n. The martingale central limit theorem will give the theorem.

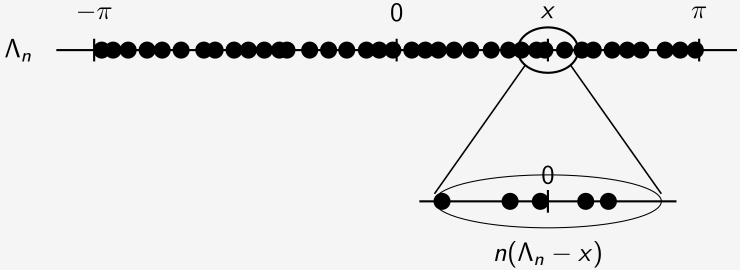
Local limits for Circular β

What if we want to see the local interaction between eigenvalues?



Local limits for Circular β

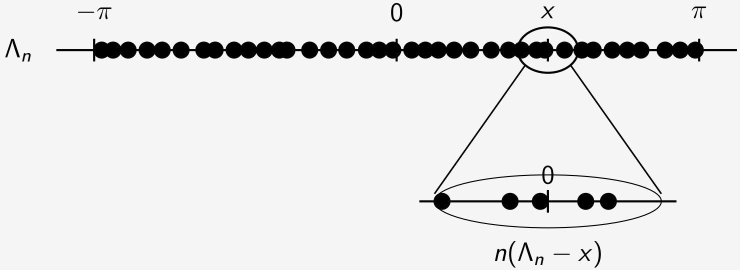
What if we want to see the local interaction between eigenvalues?



Rotational invariance means that we will see the same type of structure everywhere in the spectrum (on the circle).

Local limits for Circular β

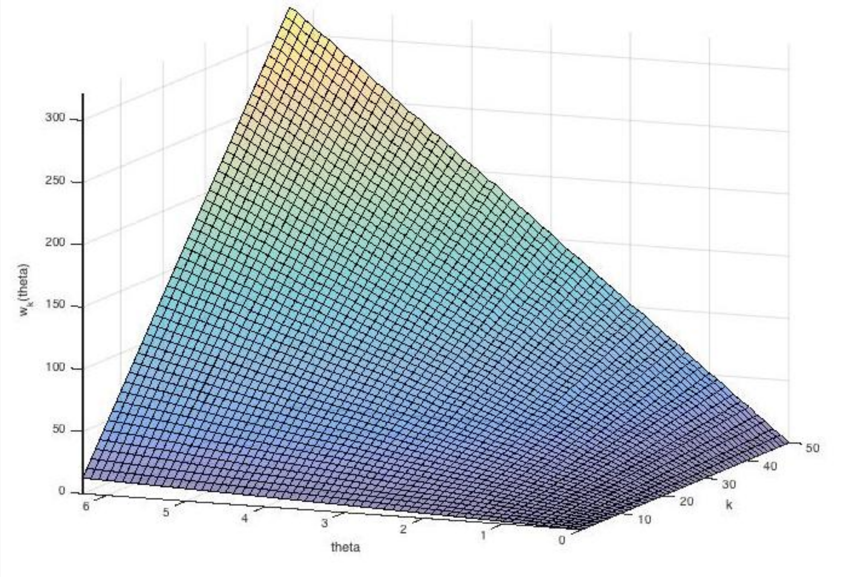
What if we want to see the local interaction between eigenvalues?



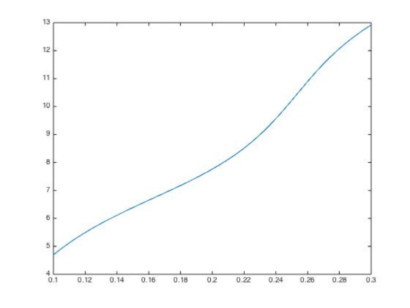
Rotational invariance means that we will see the same type of structure everywhere in the spectrum (on the circle).

We will focus near 0 which means we need to look at $\omega_{n-1}(x/n)$ in order to see the counting function.

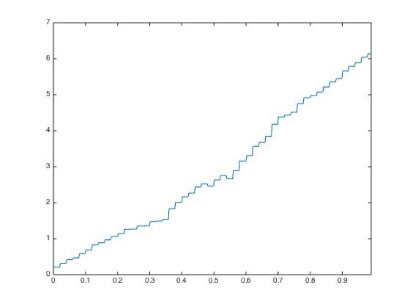
Seeing the local limit structure at a finite level



Seeing the local limit structure at a finite level



$\omega_{40}(x)$ on $[0.1, 0.3]$ for $\beta = 4$



$\omega_{\lfloor 50t \rfloor}(\frac{5}{50})$ on $[0, .99]$ for $\beta = 4$

The bulk limit

Theorem (Killip-Stoiciu, Valkó-Virág)

Let $\{\cdots < x_{-1} < 0 < x_0 < x_1 < \cdots\}$ have β-circular distribution (in the argument), then

$$\{\ldots, nx_{-1}, nx_0, nx_1, \ldots\} \Rightarrow \mathrm{Sine}_\beta \qquad \text{as } n \to \infty.$$

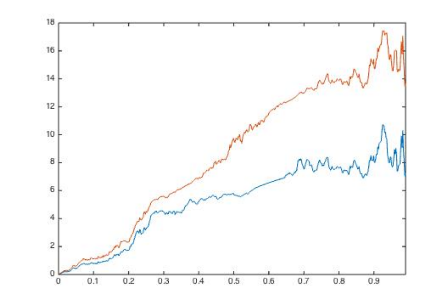
Sine_β *may be characterized by its counting function which has distribution* $N_\beta(\lambda) = \lim_{t\to\infty} \frac{\alpha_\lambda(t)}{2\pi}$ *where*

$$d\alpha_\lambda = \lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} dt + \mathrm{Re}[(e^{-i\alpha_\lambda} - 1) d(B^{(1)} + iB^{(2)})], \qquad \alpha_\lambda(0) = 0.$$

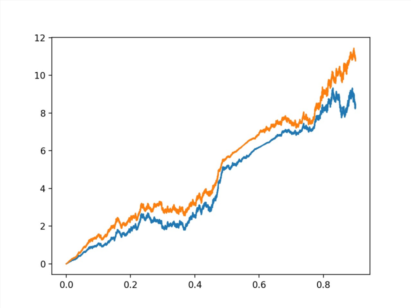
Morally: $\hat{\alpha}_\lambda(t) = \alpha_\lambda(-\frac{4}{\beta} log(1-t)) \approx \omega_{\lfloor nt \rfloor}(\lambda/n)$. Under this time change $\hat{\alpha}_\lambda(0) = 0, t \in [0,1)$

$$d\hat{\alpha}_\lambda(t) = \lambda dt + \frac{2}{\sqrt{\beta(1-t)}} \mathrm{Re}[(e^{-i\hat{\alpha}_\lambda} - 1) d(B^{(1)} + iB^{(2)})].$$

Moral proof by picture



$\omega_{\lfloor 500t \rfloor}(\frac{10}{500})$ and $\omega_{\lfloor 500t \rfloor}(\frac{14}{500})$ for $\beta = 4$



$\hat{\alpha}_{10}(t)$ and $\hat{\alpha}_{14}(t)$ for $\beta = 4$

What about Sine_2

Recall that for $\beta = 2$ there is a beautiful integrable structure. Sine_2 is a determinantal process with kernel function

$$K(x,y) = \frac{\sin(x-y)}{x-y}.$$

This description is very good for some types of questions:

What about Sine_2

Recall that for $\beta = 2$ there is a beautiful integrable structure. Sine_2 is a determinantal process with kernel function

$$K(x, y) = \frac{\sin(x - y)}{x - y}.$$

This description is very good for some types of questions:

What is the probability of seeing no points in a large interval?

$$P(N(\lambda) = 0) = (\kappa_\beta + o(1))\lambda^{-1/4} \exp\left(-\frac{\beta}{64}\lambda^2 + \left(\frac{\beta}{8} - \frac{1}{4}\right)\lambda\right)$$

Widom (1994), Deift, Its, and Zhou (1997), Krasovsky (2004), Ehrhardt (2006), Deift et al. (2007)

What about Sine_2

Recall that for $\beta = 2$ there is a beautiful integrable structure. Sine_2 is a determinantal process with kernel function

$$K(x, y) = \frac{\sin(x - y)}{x - y}.$$

This description is very good for some types of questions:

What is the probability of seeing no points in a large interval?

$$P(N(\lambda) = 0) = (\kappa_\beta + o(1))\lambda^{-1/4} \exp\left(-\frac{\beta}{64}\lambda^2 + \left(\frac{\beta}{8} - \frac{1}{4}\right)\lambda\right)$$

Widom (1994), Deift, Its, and Zhou (1997), Krasovsky (2004), Ehrhardt (2006), Deift et al. (2007)

Here you can not do as well with the counting function machinery (as of yet), but what about other questions?

Natural Questions for Sine_β (or Sine_2) counting function

1. What can we say about the distribution of the number of points in a large interval?
   - Large Gaps (Valkó, Virág)
   - Large deviations (H., Valkó)
   - Central limit theorem (Krichevsky, Valkó, Virág)

2. What is the probability of an overcrowded interval? (Holcomb, Valkó)

Natural Questions for Sine_β (or Sine_2) counting function

1. What can we say about the distribution of the number of points in a large interval?
   - Large Gaps (Valkó, Virág)
   - Large deviations (H., Valkó)
   - Central limit theorem (Krichevsky, Valkó, Virág)

2. What is the probability of an overcrowded interval? (Holcomb, Valkó)

3. Point process limits as $\beta \to 0$? (Allez, Dumaz)

Natural Questions for Sine_β (or Sine_2) counting function

1. What can we say about the distribution of the number of points in a large interval?
   - Large Gaps (Valkó, Virág)
   - Large deviations (H., Valkó)
   - Central limit theorem (Krichevsky, Valkó, Virág)

2. What is the probability of an overcrowded interval? (Holcomb, Valkó)

3. Point process limits as $\beta \to 0$? (Allez, Dumaz)

4. Maximum deviation of the counting function from its norm (H., Paquette)

Natural Questions for Sine$_\beta$ (or Sine$_2$) counting function

1. What can we say about the distribution of the number of points in a large interval?
   - Large Gaps (Valkó, Virág)
   - Large deviations (H., Valkó)
   - Central limit theorem (Krichevsky, Valkó, Virág)

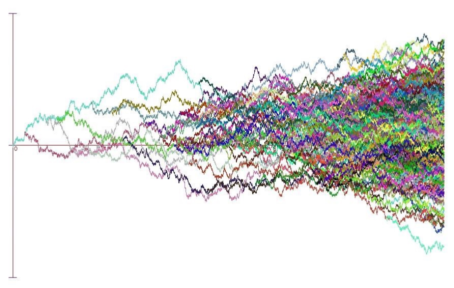
2. What is the probability of an overcrowded interval? (Holcomb, Valkó)

3. Point process limits as $\beta \to 0$? (Allez, Dumaz)

4. Maximum deviation of the counting function from its norm (H., Paquette)

5. Other questions on Sine$_\beta$ (Dereudre, Hardy, Leblé, Maïda, Chhaibi, Najnudel)

Log-correlated fields and branching processes



Branching Brownian Motion
(Borrowed from Matt Roberts)

Models with log-correlated structure: Branching Random walk, Branching Brownian motion, log-correlated Gaussian field, characteristic polynomials of random matrices.

A few people who have worked in the area: Derrida-Spohn, Hu-Shi, Aídékon-Shi, Arguin-Zindy

Full results for log-correlated Gaussian fields: Ding-Roy-Zeitouni

Log-correlated fields and Circular β

Conjecture (Fyodorov, Hiary, Keating)

For $\beta = 2$, and K_1, K_2 independent Gumble distributions

$$\sup_{z \in \partial \mathbb{D}} \log |\Phi_n(z)| - (\log n - \frac{3}{4} \log \log n) \to \frac{1}{2}(K_1 + K_2)$$

- 1st term: Arguin, Belius, Bourgade (2017)
- 2nd term: Paquette-Zeitouni (2017)
- tightness of the distribution ($\beta > 0$): Chhaibi, Mandaule, Najnudel (2018)

Recall that we said that $\hat{\alpha}_\lambda(t)$ was morally $2 \arg \Phi_{\lfloor nt \rfloor}(e^{i\lambda/n}) + t\lambda$. This gives that $2\mathrm{Im} \log \Phi_n(e^{i\lambda/n})$ is comparable to $2\pi N(\lambda) - \lambda$. For Sine_β the analogous question is

$$\sup_{|\lambda| \le x} (N_\beta(\lambda) - N_\beta(-\lambda) - \frac{\lambda}{\pi}) - C_\beta(\log x - \frac{3}{4} \log \log x) \Rightarrow ?$$

The Result

Theorem (H., Paquette)

$$\max_{0 \le \lambda \le x} \frac{N(\lambda) - N(-\lambda) - \frac{\lambda}{\pi}}{\log x} \to \frac{2}{\sqrt{\beta}\pi} \quad \textit{in probability as } x \to \infty.$$

The Result

Theorem (H., Paquette)

$$\max_{0\le\lambda\le x} \frac{N(\lambda)-N(-\lambda)-\frac{\lambda}{\pi}}{\log x} \to \frac{2}{\sqrt{\beta}\pi} \quad \textit{in probability as } x\to\infty.$$

Notice that

$$N(\lambda)-N(-\lambda)-\frac{\lambda}{\pi} = \frac{1}{2\pi}\mathrm{Re}\int_0^\infty (e^{-i\alpha_\lambda(t)}-e^{-i\alpha_{-\lambda}(t)})dZ = \frac{1}{2\pi}M_\lambda(\infty)$$

Conjecture

$$\max_{0\le\lambda\le x} M_\lambda(\infty) - \frac{2}{\sqrt{\beta}\pi}(\log x - \frac{3}{4}\log\log x) \Rightarrow \xi.$$

OPUCs and Dirac Operators

Recall that for OPUCs we had the Szegr̋ecursion

$$\begin{bmatrix} \Phi_{k+1}(z) \\ \Phi_{k+1}^*(z) \end{bmatrix} = \begin{bmatrix} z & -\bar{\alpha}_k \\ -\alpha_k z & 1 \end{bmatrix} \begin{bmatrix} \Phi_k(z) \\ \Phi_k^*(z) \end{bmatrix} = T_k \begin{bmatrix} \Phi_k(z) \\ \Phi_k^*(z) \end{bmatrix}$$

We can write

$$T_k = \begin{bmatrix} 1 & -\bar{\alpha}_k \\ -\alpha_k & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} = A_k Z.$$

OPUCs and Dirac Operators

Recall that for OPUCs we had the Szegő recursion

$$\begin{bmatrix} \Phi_{k+1}(z) \\ \Phi_{k+1}^*(z) \end{bmatrix} = \begin{bmatrix} z & -\bar{\alpha}_k \\ -\alpha_k z & 1 \end{bmatrix} \begin{bmatrix} \Phi_k(z) \\ \Phi_k^*(z) \end{bmatrix} = T_k \begin{bmatrix} \Phi_k(z) \\ \Phi_k^*(z) \end{bmatrix}$$

We can write

$$T_k = \begin{bmatrix} 1 & -\bar{\alpha}_k \\ -\alpha_k & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} = A_k Z.$$

Let $M_k = A_{k-1}A_{k-2}\cdots A_0$ then we can look at the evolution of

$$\begin{aligned} \begin{bmatrix} f_{k+1}(\lambda) \\ f_{k+1}^*(\lambda) \end{bmatrix} &= e^{-i\lambda(k+1)/2} M_k^{-1} \begin{bmatrix} \Phi_k(e^{i\lambda}) \\ \Phi_k^*(e^{i\lambda}) \end{bmatrix} \\ &= \begin{bmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{bmatrix}^{M_{k-1}} \begin{bmatrix} f_k(\lambda) \\ f_k^*(\lambda) \end{bmatrix} \end{aligned}$$

where $A^B = B^{-1}AB$.

OPUC's and Dirac Operators II

Theorem (Valkó, Virág)

Let μ_n be supported on n points and define $M(t) = M_{\lfloor m_n t \rfloor}$ for $t \in [0, n/m_n)$ and consider the differential operator τ acting on functions $g : [0, n/m_n) \to \mathbb{C}^2$ given by

$$\tau g = 2 \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}^{M_t} g'(t), \quad g(0) \parallel \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad g(\tfrac{n}{m_n}) \parallel M_{n-1}^{-1} \begin{bmatrix} \bar{\alpha}_{n-1} \\ 1 \end{bmatrix}.$$

The eigenvalues of τ are

$$\left\{ \lambda \in \mathbb{R} : e^{i \frac{\lambda}{m_n}} \text{ is in the support of } \mu_n \right\}$$

Idea of Proof

Then the solution of the eigenvalue equation $\tau g = \mu g$ satisfies

$$g'(t) = \begin{bmatrix} i\mu/2 & 0 \\ 0 & -i\mu/2 \end{bmatrix}^{M(t)} g(t)$$

which we can solve explicitly on the intervals $[\frac{k}{m_n}, \frac{k+1}{m_n})$ giving us

$$g(\tfrac{k+1}{m_n}) = \begin{bmatrix} e^{i\frac{\mu}{2m_n}} & 0 \\ 0 & e^{-i\frac{\mu}{2m_n}} \end{bmatrix}^{M(k/m_n)} g(\tfrac{k}{m_n}).$$

Idea of Proof

Then the solution of the eigenvalue equation $\tau g = \mu g$ satisfies

$$g'(t) = \begin{bmatrix} i\mu/2 & 0 \\ 0 & -i\mu/2 \end{bmatrix}^{M(t)} g(t)$$

which we can solve explicitly on the intervals $[\frac{k}{m_n}, \frac{k+1}{m_n})$ giving us

$$g(\tfrac{k+1}{m_n}) = \begin{bmatrix} e^{i\frac{\mu}{2m_n}} & 0 \\ 0 & e^{-i\frac{\mu}{2m_n}} \end{bmatrix}^{M(k/m_n)} g(\tfrac{k}{m_n}).$$

Recall

$$\begin{bmatrix} f_{k+1}(\lambda) \\ f^*_{k+1}(\lambda) \end{bmatrix} = \begin{bmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{bmatrix}^{M_{k-1}} \begin{bmatrix} f_k(\lambda) \\ f^*_k(\lambda) \end{bmatrix}$$

At this point we can see that $g_\mu(k/m_n) = f_k(\mu/m_n)$

Thank You!