Optimal global rigidity estimates in unitary invariant ensembles

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Global rigidity of eigenvalues

Random matrix eigenvalues

Fundamental question in random matrix theory is to understand eigenvalue statistics of large random matrices

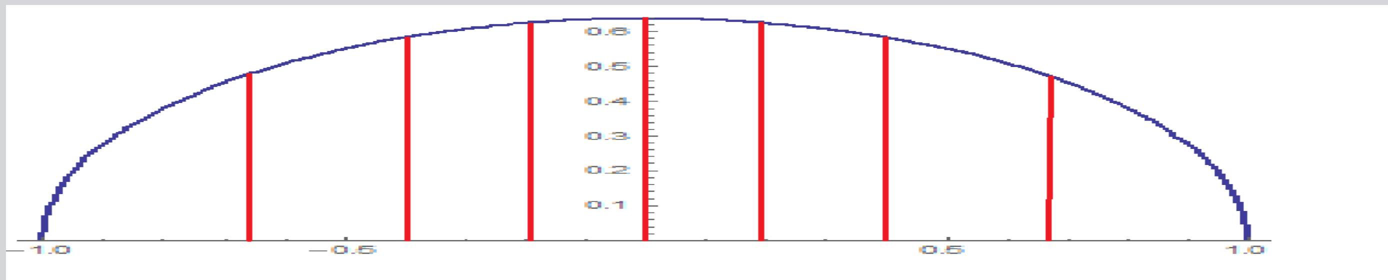
- ✓ Global statistics of eigenvalues: limiting eigenvalue distribution, macroscopic linear statistics ...
- ✓ Local statistics of eigenvalues: universal local correlations, extreme eigenvalue distribution
- ✓ In this talk: maximal fluctuation of eigenvalues around their classical positions

Global rigidity in the GUE

Classical GUE eigenvalue locations

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ be the eigenvalues of a GUE matrix M of size $N \times N$, normalized such that the eigenvalue distribution converges to a semi-circle law on $[-1, 1]$.
(Equivalently, M is Hermitian and the independent entries $M_{i,j}$ are iid (real on the diagonal, complex otherwise) Gaussians with variance $\frac{1}{4n}$.)

Classical locations $\kappa_1, \ldots, \kappa_N \in [-1, 1]$ are given by $\frac{2}{\pi} \int_{-1}^{\kappa_j} \sqrt{1 - x^2} dx = \frac{j}{N}$.



Global rigidity in the GUE

Global rigidity

What can we say for large N about the distribution of the normalized maximal fluctuation of eigenvalues

$$M_N := \max_{j=1,\ldots,N} \left\{ \frac{2}{\pi} \sqrt{1-\kappa_j^2}\, |\lambda_j - \kappa_j| \right\}?$$

Global rigidity in the GUE

Upper bound for generalized Wigner matrices (Erdos-Yau-Yin '12)

$$\mathbb{P}\left(M_N \geq \frac{(\log N)^{\alpha \log\log N}}{N}\right) \leq C \exp\left(-c(\log N)^{\alpha' \log\log N}\right)$$

Lower bound for GUE (Gustavsson '05)

$$2\sqrt{2}\sqrt{1-\kappa_j^2}\,\frac{N}{\sqrt{\log N}}(\lambda_j - \kappa_j) \to \mathcal{N}(0,1)$$

for $\delta \leq j \leq (1-\delta)N$, which implies (non-optimal) lower bounds for M_N.

Global rigidity in the GUE

Theorem (C-Fahs-Lambert-Webb '18)

For any $\epsilon > 0$, we have

$$\lim_{N\to\infty} \mathbb{P}\left((1-\epsilon)\frac{\log N}{\pi N} < M_N < (1+\epsilon)\frac{\log N}{\pi N} \right) = 1.$$

Unitary invariant ensembles

A similar result holds for unitary invariant ensembles with eigenvalue distribution

$$\frac{1}{Z_N} \prod_{1\leq i<j\leq N} \left|\lambda_i - \lambda_j\right|^2 \prod_{1\leq j\leq N} e^{-NV(\lambda_j)}\, d\lambda_j$$

for real analytic V with sufficient growth at $\pm\infty$.

Global rigidity in unitary invariant ensembles

Equilibrium measure and classical locations

Semi-circle law is then replaced by the equilibrium measure μ_V minimizing

$$\int_{\mathbb{R}\times\mathbb{R}} \log|x-y|^{-1} d\mu(x)d\mu(y) + \int_{\mathbb{R}} V(x)d\mu(x).$$

We assume that μ_V is one-cut regular, and that the support is $[-1,1]$ for convenience.

The classical locations $\kappa_1, \dots, \kappa_N \in [-1,1]$ are now defined by $\int_{-1}^{\kappa_j} d\mu_V(x) = \frac{j}{N}$.

Global rigidity in unitary invariant ensembles

Theorem (C-Fahs-Lambert-Webb '18)

For any $\epsilon > 0$, we have

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{(1-\epsilon)\log N}{\pi N} < \max\left\{ \frac{d\mu_V}{dx}(\kappa_j)|\lambda_j - \kappa_j| \right\} < \frac{(1+\epsilon)\log N}{\pi N} \right) = 1.$$

Eigenvalue counting function

We prove this via the extrema of the normalized eigenvalue counting function

$$h_N(x) = \sqrt{2}\pi\left(\sum_{1\le j\le N} \mathbf{1}_{\lambda_j \le x} - N\int_{-1}^{x} d\mu_V\right), \qquad x \in \mathbb{R}.$$

Namely, we prove that for any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left[(1-\delta)\sqrt{2}\log N \le \max_{x\in\mathbb{R}}\{\pm h_N(x)\} \le (1+\delta)\sqrt{2}\log N\right] = 1.$$

Heuristically, we expect $h_N(\lambda_j) = \int_{\lambda_j}^{\kappa_j} d\mu_V(x) \approx \frac{d\mu_V}{dx}(\kappa_j)(\kappa_j - \lambda_j)$, which explains the connection between global rigidity and the maximum of the normalized eigenvalue counting function.

Extreme values of the eigenvalue counting function

Extreme of log-correlated fields

h_N behaves for large N like a stochastic process with log-correlations (Johansson '98)

How to estimate extrema of log-correlated processes? This question has been studied in different contexts.

- ✓ Riemann ζ function and CUE (Fyodorov-Hiary-Keating '12, Arguin-Belius-Bourgade '16, Chhaibi-Madaule-Najnudel '16)
- ✓ Circular Beta Ensemble and Sine Beta process (Chhaibi-Madaule-Najnudel '16, Paquette-Zeitouni '16, Holcomb-Paquette '18)
- ✓ Characteristic polynomial in unitary invariant ensembles (Fyodorov-Simm '14, Lambert-Paquette '18)

Extreme values of the eigenvalue counting function

Multiplicative chaos

Powerful tools to study such extrema come from the theory of multiplicative chaos

- ✓ General theory (Kahane '85, Rhodes-Vargas '14, Berestycki '15)
- ✓ Applied to Circular Unitary Ensemble (Fyodorov-Keating '14, Webb '15, Berestycki-Webb-Wong '18, Lambert-Ostrovsky-Simm '18)

Exponential moments

Crucial input for this method: good control of exponential moments $\mathbb{E}e^{\gamma h_N(x)}$ and $\mathbb{E}e^{\gamma_1 h_N(x_1)+\gamma_2 h_N(x_2)}$ for large N

Extreme values of the eigenvalue counting function

Upper bound estimates

Upper bound for $\max_{x\in I}\{\pm h_N(x)\}$ can be obtained using an elementary one-moment method.

1. $$\max_{x\in I}\{\pm h_N(x)\} \le \max_{j:\kappa_j\in I}\{\pm h_N(\kappa_j)\} + 1.$$

2. By a union bound and Markov's inequality,

$$\mathbb{P}\left(\max_{j:\kappa_j\in I}\{h_N(\kappa_j)\} > Y\right) \le \sum_{j:\kappa_j\in I}\mathbb{P}\left(h_N(\kappa_j) > Y\right) \le \sum_{j:\kappa_j\in I}\frac{\mathbb{E}e^{\gamma h_N(\kappa_j)}}{e^{\gamma Y}}.$$

3. Substitute large N asymptotics for $\mathbb{E}e^{\gamma h_N(x)}$ and choose Y as big as possible such that rhs decays for some γ.

Extreme values of the eigenvalue counting function

Upper bound estimates

$\mathbb{E}e^{\gamma h_N(x)}$ is a Hankel determinant with discontinuous weight $e^{-NV(\lambda)}e^{\gamma \mathbf{1}_{\lambda \le x}}$, and large N asymptotics for such Hankel determinants are known for $x \in (-1+\delta, 1-\delta)$ (Its-Krasovsky '08 for GUE, Charlier '18 for one-cut regular unitary invariant ensembles):

$$\mathbb{E}e^{\gamma h_N(x)} \le C_\gamma N^{\frac{\gamma^2}{2}}, \quad x \in (-1+\delta, 1-\delta).$$

To extend this to all eigenvalues, we need a similar result for x close to ± 1. We prove

$$\mathbb{E}e^{\gamma h_N(x)} \le C'_\gamma N^{\frac{\gamma^2}{2}}(1-x^2)^{\frac{3\gamma^2}{4}}, \quad |x| \le 1 - mN^{-2/3}.$$

Extreme values of the eigenvalue counting function

Lower bound estimates

Optimal lower bound estimates are much harder to obtain, and require to investigate the log-correlated structure of h_N.

Log-correlated structure

h_N behaves for large N (Johansson '98) like a Gaussian process $X(x)$ with logarithmic covariance kernel

$$\Sigma(x,y) := \log\left|\frac{1 - xy + \sqrt{1-x^2}\sqrt{1-y^2}}{x-y}\right|.$$

Multiplicative chaos

Maximum of the eigenvalue counting function

For studying the maximum of h_N, we prove that the random measure

$$d\mu_N^\gamma = \frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}} dx, \qquad \gamma \in \mathbb{R}$$

converges weakly in distribution to a multiplicative chaos measure which can be formally written as (cf. Kahane '85, Rhodes-Vargas '10, Berestycki '17, Berestycki-Webb-Wong '17)

$$d\mu^\gamma(x) = \frac{e^{\gamma X(x)}}{\mathbb{E}e^{\gamma X(x)}} dx.$$

Multiplicative chaos

Extreme values

It will turn out that the extreme values of the limiting measure μ^γ will lead us to estimates for extreme values of h_N.

Heuristics

Heuristically, the random measure $d\mu_N^\gamma(x) = \frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}dx$ is expected to be dominated for $\gamma > 0$ by x-values where $h_N(x)$ is exceptionally large, namely $h_N(x) \geq \gamma \log N$ and it is natural to expect that the multiplicative chaos measure μ^γ will give us information about large values of $h_N(x)$.

For $|\gamma| > \sqrt{2}$, $\mu^\gamma = 0$, which suggests heuristically that values where $h_N(x) \geq (\sqrt{2} + \delta) \log N$ are unlikely to occur.

Multiplicative chaos

Multiplicative chaos and γ-thick points

Consider the set of γ-thick points

$$\mathscr{T}_N^{\pm\gamma} = \{x \in [-1,1] : \pm h_N(x) \geq \pm\gamma \log N\}\,.$$

This set contains points where $h_N(x)$ is of the order of its variance rather than its standard deviation. It follows from the multiplicative chaos convergence that for any $\gamma \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\}$, in probability,

$$\lim_{N\to\infty} \frac{\log|\mathscr{T}_N^{\gamma}|}{\log N} = -\frac{\gamma^2}{2}.$$

Multiplicative chaos

Freezing transition

Another consequence of the multiplicative chaos convergence is that

$$\lim_{N\to\infty} \frac{1}{\log N} \log\left(\int_{-1}^{1} e^{\gamma h_N(x)}\,dx\right) = \begin{cases} \gamma^2/2 & \text{if } \gamma \le \sqrt{2} \\ \sqrt{2}\gamma - 1 & \text{if } \gamma \ge \sqrt{2} \end{cases},$$

in probability.

In the physics literature, this is called a freezing transition of the random energy landscape h_N (cf. Fyodorov-Bouchaud '08, Fyodorov-Le Doussal-Russo '12, Fyodorov-Keating '14 for CUE).

Exponential moment estimates

Convergence to multiplicative chaos

The key technical input to prove convergence of μ_N^γ to μ consists of detailed asymptotic estimates as $N \to \infty$ for exponential moments of the form

$$\mathbb{E} e^{\gamma_1 h_N(x) + \gamma_2 h_N(y) + \sum_{j=1}^N W(\lambda_j)}.$$

These can also be written as Hankel determinants

$$D_N(x, y; \gamma_1, \gamma_2; W) = \det \left(\int_{\mathbb{R}} \lambda^{i+j} f(\lambda; x, y; \gamma_1, \gamma_2; W) d\lambda \right)_{i,j=0}^{N-1},$$

with $f(\lambda; x, y; \gamma_1, \gamma_2; W) = e^{\sqrt{2}\pi\gamma_1 \mathbf{1}_{\{\lambda \le x\}} + \sqrt{2}\pi\gamma_2 \mathbf{1}_{\{\lambda \le y\}} + W(\lambda) - NV(\lambda)}$.

Asymptotics are known (Charlier '18) for $x \neq y \in (-1, 1)$ fixed and for W independent of N.

Exponential moment estimates

Two merging singularities

$$\log D_N(x_1,x_2;\gamma_1,\gamma_2;0) = \log D_N(x_1;\gamma_1+\gamma_2;0) + \sqrt{2}\pi\gamma_2 N \int_{x_1}^{x_2} d\mu_V$$
$$- \gamma_1\gamma_2 \max\{0, \log(|x_1 - x_2|N)\} + \mathcal{O}(1),$$

as $N \to \infty$, where the error term is uniform for $-1+\delta < x_1 < x_2 < 1-\delta$, $0 < x_2 - x_1 < \delta$ for δ sufficiently small.

Method of proof

We prove this using a method similar to one sed for Toeplitz determinants with merging Fisher-Hartwig singularities (C-Krasovsky '15) and Hankel determinants with merging root singularities (C-Fahs '16), based on a Riemann-Hilbert approach.

Exponential moment estimates

N-dependent W

Assume that $W = W_N$ is a sequence of functions which are analytic and uniformly bounded on a suitable domain which does not shrink too fast with N.

$$\log D_N(x_1, x_2; \gamma_1, \gamma_2; W_N) = \log D_N(x_1, x_2; \gamma_1, \gamma_2; 0)$$
$$+ N \int W_N d\mu_V + \frac{1}{2}\sigma(W_N)^2 + \sum_{j=1}^{2} \frac{\gamma_j}{\sqrt{2}} \sqrt{1 - x_j^2}\,\mathcal{U}W_N(x_j) + o(1),$$

as $N \to \infty$, uniformly for (x_1, x_2) in any fixed compact subset of $(-1,1)^2$, where

$$\sigma(f)^2 = \iint_{\S^2} f'(x) f'(y) \frac{\Sigma(x,y)}{2\pi^2} dx dy, \; (\mathcal{U}w)(x) = \frac{1}{\pi} \mathrm{P.V.} \int_{-1}^{1} \frac{w(t)}{x-t} \frac{dt}{\sqrt{1-t^2}}.$$

Exponential moment estimates

Finally, we need also asymptotics for Hankel determinants with one singularity tending to the edge ± 1. This is needed for the upper bound estimate for the maximum of h_N.

Singularity close to the edge

$$\log \frac{D_N(x;\gamma;0)}{D_N(x;0;0)} = \sqrt{2}\pi\gamma N \int_{-1}^{x} d\mu_V(\xi) + \frac{\gamma^2}{2}\log N + \frac{3\gamma^2}{4}\log(1-x^2) + \mathcal{O}(1),$$

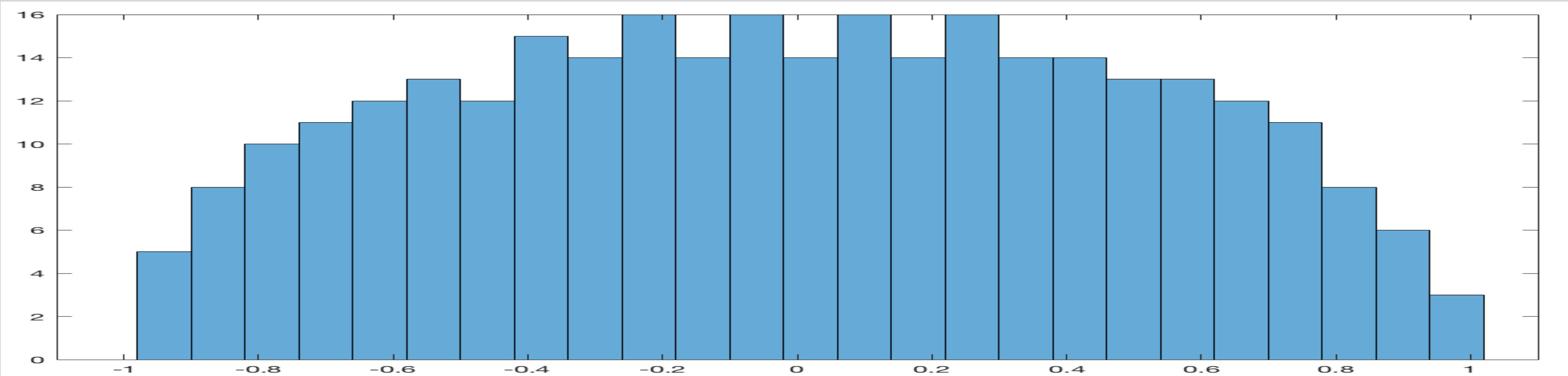
as $N \to \infty$, with the error term uniform for all $|x| \leq 1 - MN^{-2/3}$, with M sufficiently large.

Summary of the method

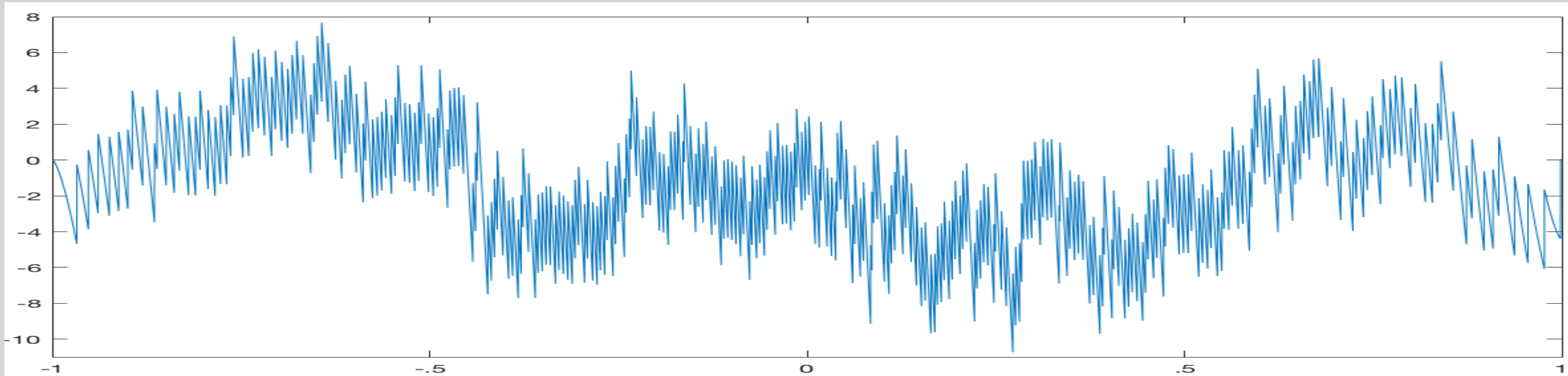
1. Hankel determinant asymptotics
   $\Longrightarrow$ Convergence of $\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}dx$ to a multiplicative chaos measure μ^γ
   $\Longrightarrow$ Estimates for γ-thick points
   $\Longrightarrow$ Estimates for the lower bound of $\max h_N$
2. Hankel determinant asymptotics
   $\Longrightarrow$ Estimates for the upper bound of $\max h_N$ via one-moment method
3. Estimates for extrema of h_N
   $\Longrightarrow$ Estimates for global rigidity of eigenvalues

Simulations

Histogram of GUE eigenvalues for $N = 300$

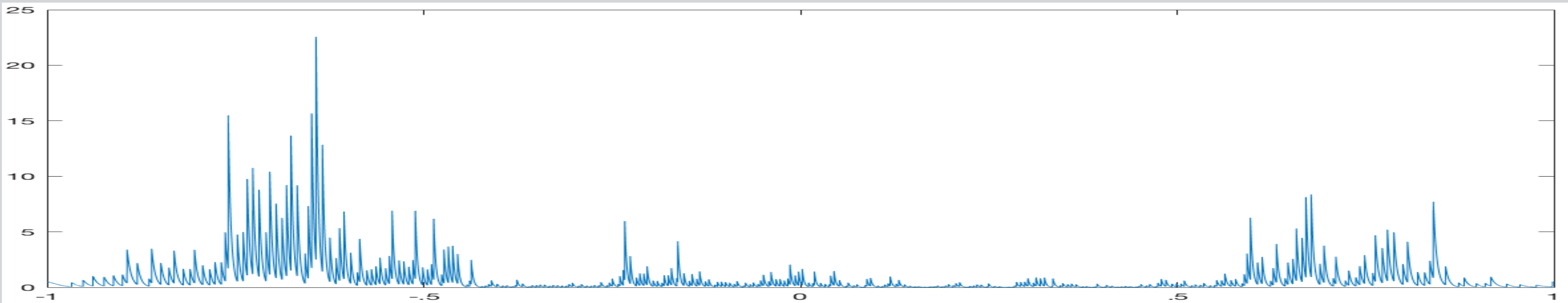


Normalized eigenvalue counting function h_N for $N = 300$.

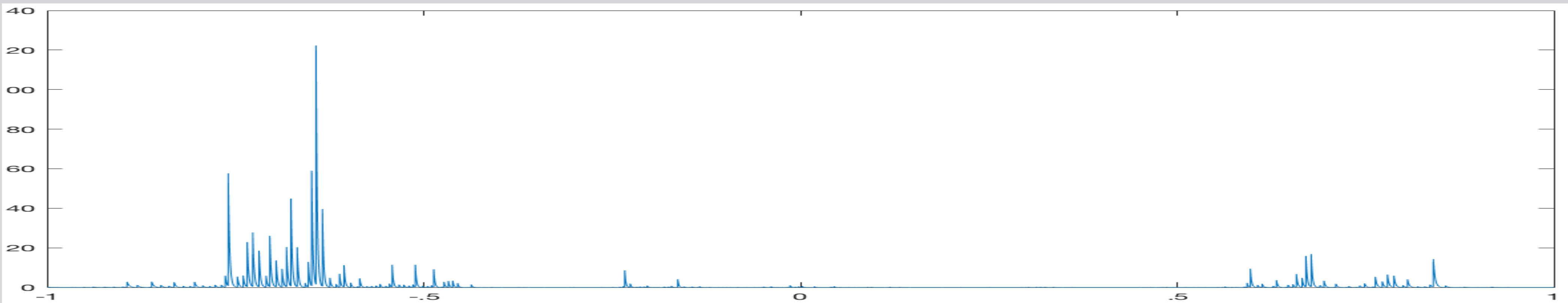


Simulations

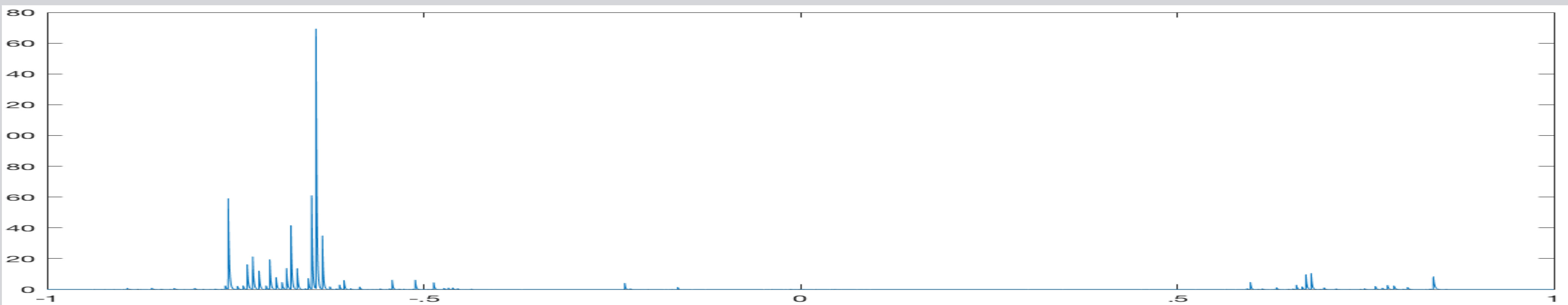
$\dfrac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ with $\gamma = 0.5$



$\dfrac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ with $\gamma = 1.0$

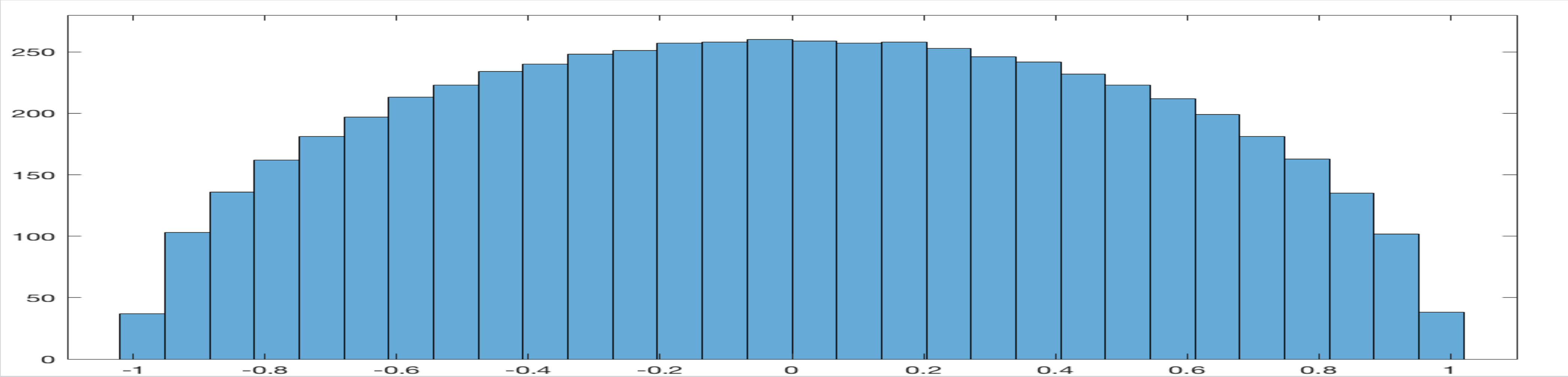


$\dfrac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ with $\gamma = 1.4$

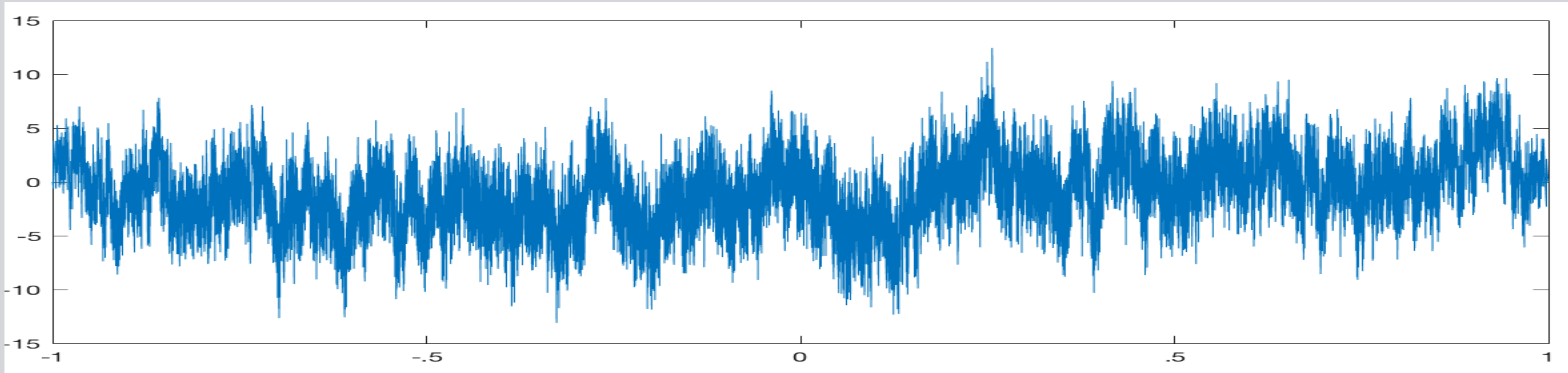


Simulations

Histogram of GUE eigenvalues for $N = 6000$

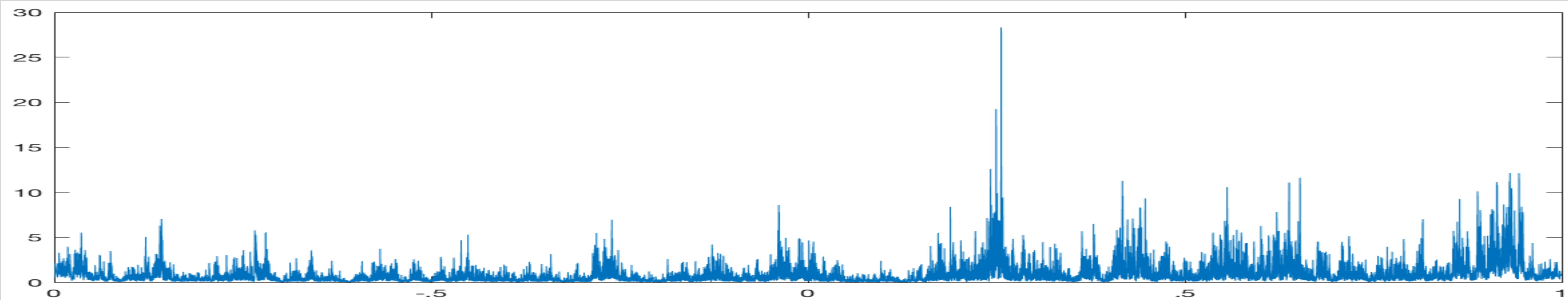


Normalized eigenvalue counting function h_N for $N = 6000$.

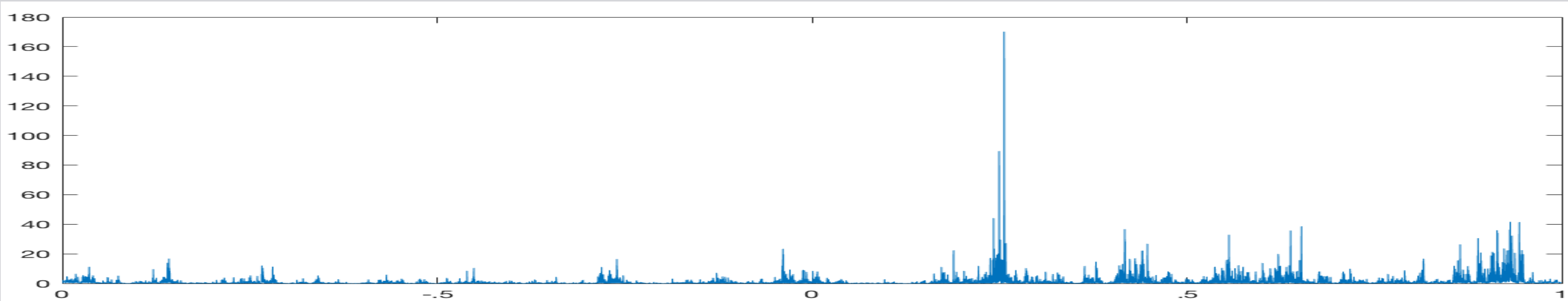


Simulations

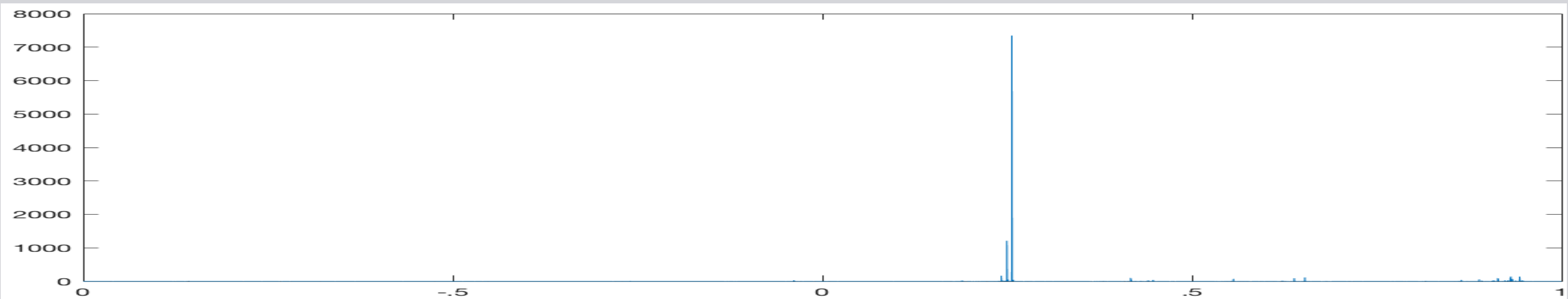
$\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ **with** $\gamma = 0.3$



$\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ **with** $\gamma = 0.5$



$\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ **with** $\gamma = 1.4$



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Thank you for your attention!