Optimal global rigidity estimates in unitary invariant ensembles

Tom Claeys

joint work with Benjamin Fahs, Gaultier Lambert and Christian Webb

Integrability and Randomness in Mathematical Physics and Geometry

CIRM Luminy, April 8, 2019
Global rigidity of eigenvalues

Fundamental question in random matrix theory is to understand eigenvalue statistics of large random matrices

✓ Global statistics of eigenvalues: limiting eigenvalue distribution, macroscopic linear statistics ...

✓ Local statistics of eigenvalues: universal local correlations, extreme eigenvalue distribution

✓ In this talk: maximal fluctuation of eigenvalues around their classical positions
Global rigidity in the GUE

Classical GUE eigenvalue locations

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ be the eigenvalues of a GUE matrix $M$ of size $N \times N$, normalized such that the eigenvalue distribution converges to a semi-circle law on $[-1, 1]$.

(Equivalently, $M$ is Hermitian and the independent entries $M_{i,j}$ are iid (real on the diagonal, complex otherwise) Gaussians with variance $\frac{1}{4n}$.)

Classical locations $\kappa_1, \ldots, \kappa_N \in [-1, 1]$ are given by $\frac{2}{\pi} \int_{-1}^{\kappa_j} \sqrt{1 - x^2} \, dx = \frac{j}{N}$. 

\[ \frac{2}{\pi} \int_{-1}^{\kappa_j} \sqrt{1 - x^2} \, dx = \frac{j}{N}. \]
Global rigidity in the GUE

Global rigidity

What can we say for large $N$ about the distribution of the normalized maximal fluctuation of eigenvalues

$$M_N := \max_{j=1,\ldots,N} \left\{ \frac{2}{\pi} \sqrt{1 - \kappa_j^2} |\lambda_j - \kappa_j| \right\}?$$
Global rigidity in the GUE

Upper bound for generalized Wigner matrices (Erdos-Yau-Yin '12)

\[ \mathbb{P}\left( M_N \geq \frac{(\log N)^{\alpha \log \log N}}{N} \right) \leq C \exp \left( -c(\log N)^{\alpha' \log \log N} \right) \]

Lower bound for GUE (Gustavsson '05)

\[ 2\sqrt{2} \sqrt{1 - \kappa_j^2} \frac{N}{\sqrt{\log N}} (\lambda_j - \kappa_j) \to \mathcal{N}(0, 1) \]

for \( \delta \leq j \leq (1 - \delta)N \), which implies (non-optimal) lower bounds for \( M_N \).
Global rigidity in the GUE

Theorem (C-Fahs-Lambert-Webb '18)

For any $\epsilon > 0$, we have

$$
\lim_{N \to \infty} \mathbb{P} \left( \frac{(1 - \epsilon) \log N}{\pi N} < M_N < \frac{(1 + \epsilon) \log N}{\pi N} \right) = 1.
$$

Unitary invariant ensembles

A similar result holds for unitary invariant ensembles with eigenvalue distribution

$$
\frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 \prod_{1 \leq j \leq N} e^{-NV(\lambda_j)} \, d\lambda_j
$$

for real analytic $V$ with sufficient growth at $\pm \infty$. 
Global rigidity in unitary invariant ensembles

Equilibrium measure and classical locations

Semi-circle law is then replaced by the equilibrium measure $\mu_V$ minimizing

$$\int_{\mathbb{R} \times \mathbb{R}} \log |x - y|^{-1} d\mu(x)d\mu(y) + \int_{\mathbb{R}} V(x)d\mu(x).$$

We assume that $\mu_V$ is one-cut regular, and that the support is $[-1, 1]$ for convenience.

The classical locations $\kappa_1, \ldots, \kappa_N \in [-1, 1]$ are now defined by

$$\int_{-1}^{\kappa_j} d\mu_V(x) = \frac{j}{N}.$$
For any $\epsilon > 0$, we have

$$
\lim_{N \to \infty} \mathbb{P} \left( \frac{(1 - \epsilon) \log N}{\pi N} < \max \left\{ \frac{d\mu_V}{dx}(\kappa_j) |\lambda_j - \kappa_j| \right\} < \frac{(1 + \epsilon) \log N}{\pi N} \right) = 1.
$$
Global rigidity in unitary invariant ensembles

**Eigenvalue counting function**

We prove this via the extrema of the normalized eigenvalue counting function

$$h_N(x) = \sqrt{2\pi} \left( \sum_{1 \leq j \leq N} 1_{\lambda_j \leq x} - N \int_{-1}^{x} d\mu_V \right), \quad x \in \mathbb{R}.$$ 

Namely, we prove that for any $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P} \left[ (1 - \delta) \sqrt{2} \log N \leq \max_{x \in \mathbb{R}} \left\{ \pm h_N(x) \right\} \leq (1 + \delta) \sqrt{2} \log N \right] = 1.$$ 

Heuristically, we expect $h_N(\lambda_j) = \int_{\lambda_j}^{\kappa_j} d\mu_V(x) \approx \frac{d\mu_V}{dx}(\kappa_j)(\kappa_j - \lambda_j)$, which explains the connection between global rigidity and the maximum of the normalized eigenvalue counting function.
Extreme values of the eigenvalue counting function

Extreme of log-correlated fields

\( h_N \) behaves for large \( N \) like a stochastic process with log-correlations (Johansson '98)

How to estimate extrema of log-correlated processes? This question has been studied in different contexts.

✓ Riemann \( \zeta \) function and CUE (Fyodorov-Hiary-Keating '12, Arguin-Belius-Bourgade '16, Chhaibi-Madaule-Najnudel '16)
✓ Circular Beta Ensemble and Sine Beta process (Chhaibi-Madaule-Najnudel '16, Paquette-Zeitouni '16, Holcomb-Paquette '18)
✓ Characteristic polynomial in unitary invariant ensembles (Fyodorov-Simm '14, Lambert-Paquette '18)
Extreme values of the eigenvalue counting function

Multiplicative chaos

Powerful tools to study such extrema come from the theory of multiplicative chaos

✓ General theory (Kahane ’85, Rhodes-Vargas ’14, Berestycki ’15)
✓ Applied to Circular Unitary Ensemble (Fyodorov-Keating ’14, Webb ’15, Berestycki-Webb-Wong ’18, Lambert-Ostrovsky-Simm ’18)

Exponential moments

Crucial input for this method: good control of exponential moments

\[ \mathbb{E} e^{\gamma h_N(x)} \text{ and } \mathbb{E} e^{\gamma_1 h_N(x_1) + \gamma_2 h_N(x_2)} \text{ for large } N \]
Upper bound estimates

Upper bound for \( \max_{x \in I} \{ \pm h_N(x) \} \) can be obtained using an elementary one-moment method.

1. \[
\max_{x \in I} \{ \pm h_N(x) \} \leq \max_{j : \kappa_j \in I} \{ \pm h_N(\kappa_j) \} + 1.
\]

2. By a union bound and Markov’s inequality,

\[
\mathbb{P} \left( \max_{j : \kappa_j \in I} \{ h_N(\kappa_j) \} > Y \right) \leq \sum_{j : \kappa_j \in I} \mathbb{P} (h_N(\kappa_j) > Y) \leq \sum_{j : \kappa_j \in I} \frac{\mathbb{E} e^{\gamma h_N(\kappa_j)}}{e^{\gamma Y}}.
\]

3. Substitute large \( N \) asymptotics for \( \mathbb{E} e^{\gamma h_N(x)} \) and choose \( Y \) as big as possible such that rhs decays for some \( \gamma \).
Extreme values of the eigenvalue counting function

Upper bound estimates

\[ \mathbb{E} e^{\gamma h_N(x)} \] is a Hankel determinant with discontinuous weight \( e^{-NV(\lambda)} e^{\gamma 1_{\lambda \leq x}} \), and large \( N \) asymptotics for such Hankel determinants are known for \( x \in (-1 + \delta, 1 - \delta) \) (Its-Krasovsky '08 for GUE, Charlier '18 for one-cut regular unitary invariant ensembles):

\[ \mathbb{E} e^{\gamma h_N(x)} \leq C_{\gamma} N^{\gamma^2/2}, \quad x \in (-1 + \delta, 1 - \delta). \]

To extend this to all eigenvalues, we need a similar result for \( x \) close to \( \pm 1 \). We prove

\[ \mathbb{E} e^{\gamma h_N(x)} \leq C'_{\gamma} N^{\gamma^2/2} (1 - x^2)^{3\gamma^2/4}, \quad |x| \leq 1 - mN^{-2/3}. \]
Extreme values of the eigenvalue counting function

**Lower bound estimates**

Optimal lower bound estimates are much harder to obtain, and require to investigate the log-correlated structure of $h_N$.

**Log-correlated structure**

$h_N$ behaves for large $N$ (Johansson '98) like a Gaussian process $X(x)$ with logarithmic covariance kernel

$$
\Sigma(x, y) := \log \left| \frac{1 - xy + \sqrt{1 - x^2} \sqrt{1 - y^2}}{x - y} \right|.
$$
For studying the maximum of $h_N$, we prove that the random measure

$$d\mu^\gamma_N = \frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}} \, dx, \quad \gamma \in \mathbb{R}$$

converges weakly in distribution to a multiplicative chaos measure which can be formally written as (cf. Kahane '85, Rhodes-Vargas '10, Berestycki '17, Berestycki-Webb-Wong '17)

$$d\mu^\gamma(x) = \frac{e^{\gamma X(x)}}{\mathbb{E}e^{\gamma X(x)}} \, dx.$$
Multiplicative chaos

It will turn out that the extreme values of the limiting measure $\mu^\gamma$ will lead us to estimates for extreme values of $h_N$.

Heuristics

Heuristically, the random measure $d\mu_N^\gamma(x) = \frac{e^{\gamma h_N(x)}}{E e^{\gamma h_N(x)}} dx$ is expected to be dominated for $\gamma > 0$ by $x$-values where $h_N(x)$ is exceptionally large, namely $h_N(x) \geq \gamma \log N$ and it is natural to expect that the multiplicative chaos measure $\mu^\gamma$ will give us information about large values of $h_N(x)$.

For $|\gamma| > \sqrt{2}$, $\mu^\gamma = 0$, which suggests heuristically that values where $h_N(x) \geq (\sqrt{2} + \delta) \log N$ are unlikely to occur.
Consider the set of $\gamma$-thick points

\[ \mathcal{T}_N^{\pm \gamma} = \{ x \in [-1, 1] : \pm h_N(x) \geq \pm \gamma \log N \} . \]

This set contains points where $h_N(x)$ is of the order of its variance rather than its standard deviation. It follows from the multiplicative chaos convergence that for any $\gamma \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\}$, in probability,

\[ \lim_{N \to \infty} \frac{\log |\mathcal{T}_N^\gamma|}{\log N} = -\frac{\gamma^2}{2} . \]
Another consequence of the multiplicative chaos convergence is that

\[
\lim_{N \to \infty} \frac{1}{\log N} \log \left( \int_{-1}^{1} e^{\gamma h_N(x)} \, dx \right) = \begin{cases} 
\gamma^2 / 2 & \text{if } \gamma \leq \sqrt{2} \\
\sqrt{2\gamma} - 1 & \text{if } \gamma \geq \sqrt{2} 
\end{cases},
\]

in probability.

In the physics literature, this is called a freezing transition of the random energy landscape \( h_N \) (cf. Fyodorov-Bouchaud ’08, Fyodorov-Le Doussal-Russo ’12, Fyodorov-Keating ’14 for CUE).
Exponential moment estimates

Convergence to multiplicative chaos

The key technical input to prove convergence of $\mu_N^\gamma$ to $\mu$ consists of detailed asymptotic estimates as $N \to \infty$ for exponential moments of the form

$$\mathbb{E} e^{\gamma_1 h_N(x) + \gamma_2 h_N(y) + \sum_{j=1}^{N} W(\lambda_j)}.$$

These can also be written as Hankel determinants

$$D_N(x, y; \gamma_1, \gamma_2; W) = \det \left( \int_{\mathbb{R}} \lambda^{i+j} f(\lambda; x, y; \gamma_1, \gamma_2; W) d\lambda \right)_{i,j=0}^{N-1},$$

with

$$f(\lambda; x, y; \gamma_1, \gamma_2; W) = e^{\sqrt{2} \pi \gamma_1 \mathbb{1}_{\{\lambda \leq x\}} + \sqrt{2} \pi \gamma_2 \mathbb{1}_{\{\lambda \leq y\}} + W(\lambda) - NV(\lambda)}.$$

Asymptotics are known (Charlier ’18) for $x \neq y \in (-1, 1)$ fixed and for $W$ independent of $N$. 

Exponential moment estimates

Two merging singularities

\[ \log D_N(x_1, x_2; \gamma_1, \gamma_2; 0) = \log D_N(x_1; \gamma_1 + \gamma_2; 0) + \sqrt{2}\pi \gamma_2 N \int_{x_1}^{x_2} d\mu V \]
\[ - \gamma_1 \gamma_2 \max\{0, \log(|x_1 - x_2| N)\} + O(1), \]

as \( N \to \infty \), where the error term is uniform for \(-1 + \delta < x_1 < x_2 < 1 - \delta\), \(0 < x_2 - x_1 < \delta\) for \(\delta\) sufficiently small.

Method of proof

We prove this using a method similar to one sed for Toeplitz determinants with merging Fisher-Hartwig singularities (C-Krasovsky '15) and Hankel determinants with merging root singularities (C-Fahs '16), based on a Riemann-Hilbert approach.
Exponential moment estimates

\( N \)-dependent \( W \)

Assume that \( W = W_N \) is a sequence of functions which are analytic and uniformly bounded on a suitable domain which does not shrink too fast with \( N \).

\[
\log D_N(x_1, x_2; \gamma_1, \gamma_2; W_N) = \log D_N(x_1, x_2; \gamma_1, \gamma_2; 0)
+ N \int W_N d\mu_V + \frac{1}{2} \sigma(W_N)^2 + \sum_{j=1}^{2} \frac{\gamma_j}{\sqrt{2}} \sqrt{1 - x_j^2} UW_N(x_j) + o(1),
\]

as \( N \to \infty \), uniformly for \((x_1, x_2)\) in any fixed compact subset of \((-1, 1)^2\), where

\[
\sigma(f)^2 = \iint_{S^2} f'(x)f'(y) \frac{\Sigma(x, y)}{2\pi^2} \, dx \, dy, \quad (Uw)(x) = \frac{1}{\pi} \text{P.V.} \int_{-1}^{1} \frac{w(t)}{x-t} \frac{dt}{\sqrt{1-t^2}}.
\]
Exponential moment estimates

Finally, we need also asymptotics for Hankel determinants with one singularity tending to the edge \( \pm 1 \). This is needed for the upper bound estimate for the maximum of \( h_N \).

Singularity close to the edge

\[
\log \frac{D_N(x; \gamma; 0)}{D_N(x; 0; 0)} = \sqrt{2 \pi \gamma N} \int_{-1}^{x} d\mu_V(\xi) + \frac{\gamma^2}{2} \log N + \frac{3\gamma^2}{4} \log(1 - x^2) + \mathcal{O}(1),
\]

as \( N \to \infty \), with the error term uniform for all \( |x| \leq 1 - MN^{-2/3} \), with \( M \) sufficiently large.
Overview

Summary of the method

1. Hankel determinant asymptotics
   \[ \text{Convergence of } \frac{e^{\gamma h_N(x)}}{\mathbb{E} e^{\gamma h_N(x)}} \, dx \text{ to a multiplicative chaos measure } \mu^\gamma \]
   \[ \Rightarrow \text{Estimates for } \gamma\text{-thick points} \]
   \[ \Rightarrow \text{Estimates for the lower bound of } \max h_N \]

2. Hankel determinant asymptotics
   \[ \Rightarrow \text{Estimates for the upper bound of } \max h_N \text{ via one-moment method} \]

3. Estimates for extrema of \( h_N \)
   \[ \Rightarrow \text{Estimates for global rigidity of eigenvalues} \]
Simulations

Histogram of GUE eigenvalues for $N = 300$

Normalized eigenvalue counting function $h_N$ for $N = 300$. 
Simulations

\[
\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}} \quad \text{with } \gamma = 0.5
\]

\[
\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}} \quad \text{with } \gamma = 1.0
\]

\[
\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}} \quad \text{with } \gamma = 1.4
\]
Histogram of GUE eigenvalues for $N = 6000$

Normalized eigenvalue counting function $h_N$ for $N = 6000$. 

Simulations
Simulations

\[ \frac{e^{\gamma N(x)}}{\mathbb{E}e^{\gamma N(x)}} \text{ with } \gamma = 0.3 \]

\[ \frac{e^{\gamma N(x)}}{\mathbb{E}e^{\gamma N(x)}} \text{ with } \gamma = 0.5 \]

\[ \frac{e^{\gamma N(x)}}{\mathbb{E}e^{\gamma N(x)}} \text{ with } \gamma = 1.4 \]
Thank you for your attention!