Representations of classical Lie groups: two regimes of growth

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Plan

Three $(2+\varepsilon)$ settings: 1) Large unitary groups, 2) Unitarily invariant large random Hermitian matrices, 3) large symplectic and orthogonal groups.
Three limit regimes:

- 1) Random tilings; tensor products of representations; free probability 2) free probability 3) Random tilings with symmetry.
- 1) Infinite-dimensional unitary group 2) Unitarily invariant measures on infinite Hermitian matrices 3) Infinite-dimensional symplectic and orthogonal groups.
- Intermediate regime.

Free probability (in random matrices).

- Let A be a $N \times N$ Hermitian matrix with eigenvalues $\{a_i\}_{i=1}^N$. Let
$$m[A] := \frac{1}{N} \sum_{i=1}^{N} \delta\left(a_i\right)$$
be the *empirical* measure of A.
- For each $N = 1, 2, \ldots$ take two sets of real numbers $a(N) = \{a_i(N)\}_{i=1}^N$ and $b(N) = \{b_i(N)\}_{i=1}^N$.
- Let $\mathcal{A}(N)$ be the uniformly ("Haar distributed") random $N \times N$ Hermitian matrix with fixed eigenvalues $a(N)$ and let $\mathcal{B}(N)$ be the uniformly ("Haar distributed") random $N \times N$ Hermitian matrix with fixed eigenvalues $b(N)$ such that $\mathcal{A}(N)$ and $\mathcal{B}(N)$ are independent.

Free convolution

Suppose that as $N \to \infty$ the empirical measures of $\mathcal{A}(N)$ and $\mathcal{B}(N)$ weakly converge to probability measures $\mathbf{m}^1$ and $\mathbf{m}^2$, respectively.

Theorem (Voiculescu, 1991)

The random empirical measure of the sum $\mathcal{A}(N) + \mathcal{B}(N)$ *converges (weak convergence; in probability) to a deterministic measure* $\mathbf{m}^1 \boxplus \mathbf{m}^2$ *which is the free convolution of* $\mathbf{m}^1$ *and* $\mathbf{m}^2$*.*

$diag(a_1, \ldots, a_N)$ – diagonal matrix with eigenvalues $a_1, \ldots, a_N$.

$$HC(a_1, \ldots, a_N; b_1, \ldots, b_N) := \int_{U(N)} \exp\left(Tr\left(diag(a_1, \ldots, a_N) U_N diag(b_1, \ldots, b_N) U_N^* \right)\right) dU_N$$

Harish-Chandra-Itzykson-Zuber integral

$$HC(a_1, \ldots, a_N; b_1, \ldots, b_N) = const \frac{\det\left(\exp(a_i b_j)\right)_{i,j=1}^N}{\prod_{i<j}(a_i - a_j)\prod_{i<j}(b_i - b_j)}$$

One can prove the theorem of Voiculescu with the use of the following asymptotic result of Guionnet-Maida'04:

r is fixed, $N \to \infty$

$$\frac{1}{N} \log HC(x_1, \ldots, x_r, 0, \ldots, 0; \lambda_1, \ldots, \lambda_N) \to \Psi(x_1) + \cdots + \Psi(x_r)$$

$$\iff \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i}{N}\right) \to \mu, \qquad \text{weak convergence}$$

Functions convergence in a small neighborhood of $(1, 1, \ldots, 1) \in \mathbb{R}^r$.

$$\Psi'(x) = R_\mu^{free}(x).$$

Representations of $U(N)$

- Let $U(N)$ denote the group of all $N \times N$ unitary matrices.
- A *signature* of length N is a N-tuple of integers $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$.
  For example, $\lambda = (5, 3, 3, 1, -2, -2)$ is a signature of length 6.
- It is known that all irreducible representations of $U(N)$ are parameterized by signatures ($=$ highest weights). Let π^λ be an irreducible representation of $U(N)$ corresponding to λ.
- The character of π^λ is the Schur function

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det_{i,j=1,\ldots,N}\left(x_i^{\lambda_j+N-j}\right)}{\prod_{1\leq i<j\leq N}(x_i - x_j)}$$

Vershik-Kerov, 70's

- Given a finite-dimensional representation π of some group (e.g. $S(n), U(N), Sp(2N), SO(N)$) we can decompose it into irreducible components:

$$\pi = \bigoplus_{\lambda} c_\lambda \pi^\lambda,$$

where non-negative integers c_λ are multiplicities, and λ ranges over labels of irreducible representations.

- This decomposition can be identified with a probability measure ρ^π on labels

$$\rho^\pi(\lambda) := \frac{c_\lambda \dim(\pi^\lambda)}{\dim(\pi)}.$$

Tensor product

- Let λ and μ be signatures of length N. π^λ and π^μ — irreducible representations of $U(N)$.
- We consider the decomposition of the (Kronecker) tensor product $\pi^\lambda \otimes \pi^\mu$ into irreducible components

$$\pi^\lambda \otimes \pi^\mu = \bigoplus_\eta c_\eta^{\lambda,\mu} \pi^\eta,$$

  where η runs over signatures of length N.
- $$m[\lambda] := \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right).$$

- Assume that two sequences of signatures $\lambda = \lambda(N)$ and $\mu = \mu(N)$ satisfy

  $$m[\lambda] \xrightarrow[N\to\infty]{} m_1, \qquad m[\mu] \xrightarrow[N\to\infty]{} m_2, \qquad \text{weak convergence},$$

  where m_1 and m_2 are probability measures. For example, $\lambda_1 = \cdots = \lambda_{[N/2]} = N$, $\lambda_{[N/2]+1} = \cdots = \lambda_N = 0$, or $\lambda_i = N - i$, for $i = 1, 2, \ldots, N$.
- We are interested in the **asymptotic behaviour** of the decomposition of the tensor product into irreducibles, i.e., we are interested in the asymptotic behaviour of the random probability measure $m[\rho^{\pi^\lambda \otimes \pi^\mu}]$.

Limit results for tensor products

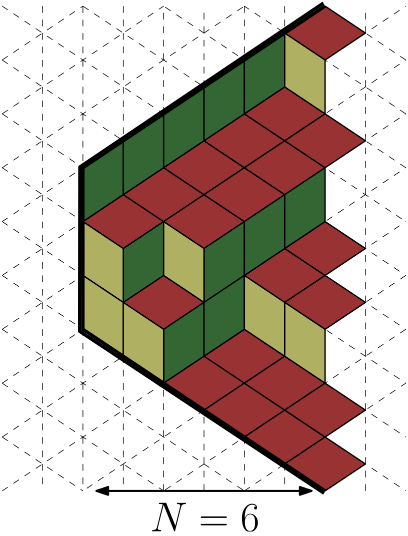
Under assumptions above, we have (Bufetov-Gorin'13):

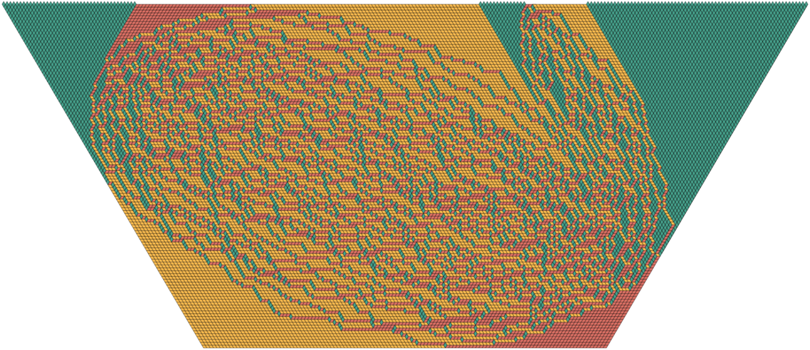
- Law of Large Numbers:

$$\lim_{N\to\infty} m[\rho^{\pi^\lambda \otimes \pi^\mu}] = m_1 \otimes m_2,$$

  where $m_1 \otimes m_2$ is a deterministic measure on $\mathbb{R}$.
  We call $m_1 \otimes m_2$ the *quantized free convolution* of measures m_1 and m_2.
- Similar results for symmetric group were obtained by Biane'98.
- Central Limit Theorem: Bufetov-Gorin'16.

Lozenge tilings



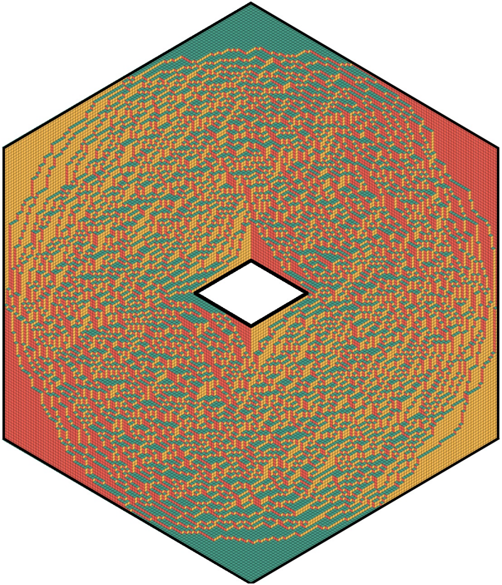


Petrov'12, Bufetov-Gorin'16: LLN+CLT.

Lozenge tilings

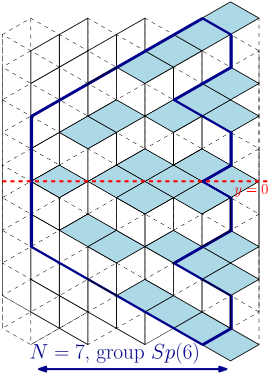
Kenyon-Okounkov conjecture for fluctuations:

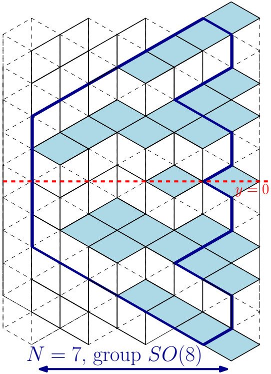
- Kenyon (2004) : a class of domains with no frozen regions.
- Borodin-Ferrari (2008): Some infinite domains with frozen regions.
- Petrov (2012), Bufetov-Gorin (2016): A class of simply-connected domains with arbitrary boundary conditions on one side.
- (Boutillier-de Tilière (2009), Dubedat (2011)),Berestycki-Laslier-Ray-16) : Some non-planar domains.
- Bufetov-Gorin (2017): Some domains with holes.



Bufetov-Gorin'17: LLN+CLT (with the use of Borodin-Gorin-Guionnet'15).

Projections for *Sp* and *SO*





Bufetov-Gorin'13: limit shapes for these tilings; connection with free probability.

Asymptotics of a normalised Schur function

An important role in all these applications is played by the following asymptotics.

r is fixed, $N \to \infty$.

The following two relations are equivalent (Guionnet-Maida'04, and also Gorin-Panova'13, Bufetov-Gorin'13).

$$\frac{1}{N} \log \frac{s_\lambda(x_1, \dots, x_r, 1^{N-r})}{s_\lambda(1^N)} \to F_1(x_1) + \cdots + F_1(x_r)$$

$$\iff \frac{1}{N} \sum_{i=1}^{n} \delta\left(\frac{\lambda_i + N - i}{N}\right) \to \mu_1$$

Notation: $1^N := (1, 1, \dots, 1)$ – N-tuple of 1's.

Extreme characters of the infinite-dimensional unitary group

Consider the tower of embedded unitary groups

$$U(1) \subset U(2) \subset \cdots \subset U(N) \subset U(N+1) \subset \ldots .$$

The infinite–dimensional unitary group $U(\infty)$ is the union of these groups.

Character of $U(\infty)$ is a positive-definite class function $\chi : U(\infty) \to \mathbb{C}$, normalised at unity: $\chi(e) = 1$.

We consider characters instead of representations. **Extreme characters** serve as an analogue of irreducible representations.

Characters of $U(\infty)$ are completely determined by their values on diagonal matrices $diag(u_1, u_2, \dots)$. Let us denote these values by $\chi(u_1, u_2, \dots)$.

The classification of the extreme characters of $U(\infty)$ is given by Edrei-Voiculescu theorem (Edrei'53, Voiculescu'76, Vershik-Kerov'82, Boyer'83, Okounkov-Olshanski'98).

Extreme characters have a multiplicative form

$$\chi^{ext}(u_1, u_2, \dots) = \Phi(u_1)\Phi(u_2)\dots.$$

$$\Phi(u) := \exp\left(\gamma^+(u-1) + \gamma^-\left(u^{-1}-1\right)\right) \times \prod_{i=1}^{\infty}\left(\frac{(1+\beta_i^+(u-1))(1+\beta_i^-(u^{-1}-1))}{(1-\alpha_i^+(u-1))(1-\alpha_i^-(u^{-1}-1))}\right).$$

$$\alpha^\pm = \alpha_1^\pm \geq \alpha_2^\pm \geq \dots \geq 0, \qquad \beta^\pm = \beta_1^\pm \geq \beta_2^\pm \geq \dots \geq 0,$$

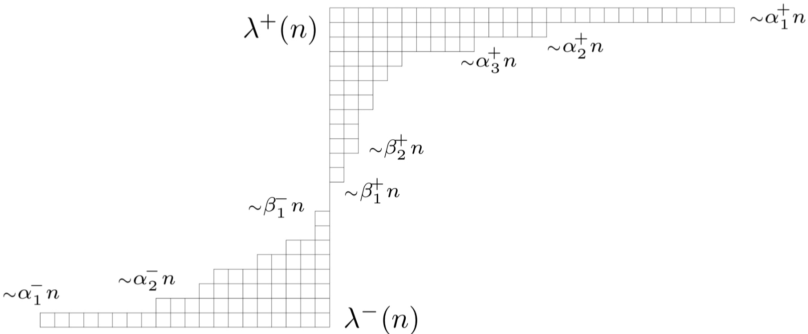
$$\gamma^\pm \geq 0, \qquad \beta_1^+ + \beta_1^- \leq 1.$$

Asymptotic approach: Vershik-Kerov'82, Okounkov-Olshanski'98:

r is fixed, $N \to \infty$.

$$\frac{s_{\lambda(N)}(x_1, \dots, x_r, 1^{N-r})}{s_{\lambda(N)}(1^N)} \to \Phi(x_1)\Phi(x_2)\cdots\Phi(x_r)$$

is equivalent to a certain condition on growth of signatures $\lambda(N)$, which, in particular, encodes the parameters of Φ.



Random matrix counterpart: Ergodic unitarily invariant measures on infinite Hermitian matrices.
Vershik'74, Pickrell'91, Olshanski-Vershik'96

$diag(a_1, \ldots, a_N)$ – diagonal matrix with eigenvalues $a_1, \ldots, a_N$.

$$HC(a_1, \ldots, a_N; b_1, \ldots, b_N) := \int_{U(N)} \exp\left(Tr\left(diag(a_1, \ldots, a_N) U_N diag(b_1, \ldots, b_N) U_N^*\right)\right) dU_N$$

$$HC(a_1, \ldots, a_N; b_1, \ldots, b_N) = \frac{\det\left(\exp(a_i b_j)\right)_{i,j=1}^N}{\prod_{i<j}(a_i - a_j) \prod_{i<j}(b_i - b_j)}$$

r is fixed, $N \to \infty$
Olshanski-Vershik'96

$$\log HC(x_1, \dots, x_r, 0, \dots, 0; \lambda_1, \dots, \lambda_N) \to \Phi_0(x_1) + \dots + \Phi_0(x_r)$$

$$\iff \sum_{i=1}^{N} \delta\left(\frac{\lambda_i}{N}\right) \to \mu_0, \qquad \text{convergence of all} \geq 1 \text{ moments}$$

In other words,

$$\frac{\lambda_1}{N} \to \alpha_1, \dots, \frac{\lambda_i}{N} \to \alpha_i, \dots,$$
$$\frac{\lambda_N}{N} \to \alpha_{-1}, \dots, \frac{\lambda_{N-i+1}}{N} \to \alpha_{-i}, \dots$$

and there willl be two more parameters γ_1, γ_2 related to 0.

The two key facts are similar.
r is fixed, $N \to \infty$.

$$\log HC(x_1, \ldots, x_r, 0, \ldots, 0; \lambda_1, \ldots, \lambda_N) \to \Phi_0(x_1) + \cdots + \Phi_0(x_r)$$
$$\iff \sum_{i=1}^{N} \delta\left(\frac{\lambda_i}{N}\right) \to \mu_0$$

$$\frac{1}{N} \log HC(x_1, \ldots, x_r, 0, \ldots, 0; \lambda_1, \ldots, \lambda_N) \to \Phi_1(x_1) + \cdots + \Phi_1(x_r)$$
$$\iff \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i}{N}\right) \to \mu_1$$

Intermediate regime: Matrices

Bufetov'19+: Let $0 \leq \theta \leq 1$.
r is fixed, $N \to \infty$.
We have

$$\frac{1}{N^\theta} \log HC(x_1, \ldots, x_r, 0, \ldots, 0; \lambda_1, \ldots, \lambda_N) \to \Phi_\theta(x_1) + \cdots + \Phi_\theta(x_r)$$

$$\iff \frac{1}{N^\theta} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i}{N}\right) \to \mu_\theta \qquad .$$

$\Phi_\theta \longleftrightarrow \mu_\theta$ — bijection between possible limits.

$U(\infty)$ growth

The two key facts are similar.
r is fixed, $N \to \infty$.
The following three relations are equivalent.

$$\log \frac{s_\lambda(x_1, \dots, x_r, 1^{N-r})}{s_\lambda(1^N)} \to F_0(x_1) + \cdots + F_0(x_r)$$

$$\sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right) - \sum_{i=1}^{N} \delta\left(\frac{N-i}{N}\right) \to \nu_0$$

$$\sum_{i=1}^{N} \left(\prod_{j \neq i} \frac{(\lambda_i - i) - (\lambda_j - j) - 1}{(\lambda_i - i) - (\lambda_j - j)}\right) \delta\left(\frac{\lambda_i + N - i}{N}\right) \to \hat{\nu}_0;$$

The two key facts are similar.

r is fixed, $N \to \infty$.

$$\frac{1}{N} \log \frac{s_\lambda(x_1, \ldots, x_r, 1^{N-r})}{s_\lambda(1^N)} \to F_1(x_1) + \cdots + F_1(x_r)$$

$$\frac{1}{N} \sum_{i=1}^{n} \delta\left(\frac{\lambda_i + N - i}{N}\right) - \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{N - i}{N}\right) \to \nu_1$$

$$\frac{1}{N} \sum_{i=1}^{N} \left(\prod_{j \neq i} \frac{(\lambda_i - i) - (\lambda_j - j) - 1}{(\lambda_i - i) - (\lambda_j - j)} \right) \delta\left(\frac{\lambda_i + N - i}{N}\right) \to \hat{\nu}_1.$$

Intermediate regime: Representations

Bufetov'19+: Let $0 \leq \theta \leq 1$.
r is fixed, $N \to \infty$.
The following three relations are equivalent.

$$\frac{1}{N^\theta} \log \frac{s_\lambda(x_1, \ldots, x_r, 1^{N-r})}{s_\lambda(1^N)} \to F_\theta(x_1) + \cdots + F_\theta(x_r)$$

$$\frac{1}{N^\theta} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right) - \frac{1}{N^\theta} \sum_{i=1}^{N} \delta\left(\frac{N-i}{N}\right) \to \nu_\theta$$

$$\frac{1}{N^\theta} \sum_{i=1}^{N} \left(\prod_{j \neq i} \frac{(\lambda_i - i) - (\lambda_j - j) - 1}{(\lambda_i - i) - (\lambda_j - j)} \right) \delta\left(\frac{\lambda_i + N - i}{N}\right) \to \hat{\nu}_\theta.$$

Perelomov-Popov measures

For a signature λ we set

$$m_{PP}[\lambda] := \frac{1}{N}\sum_{i=1}^{N}\left(\prod_{j\neq i}\frac{(\lambda_i - i) - (\lambda_j - j) - 1}{(\lambda_i - i) - (\lambda_j - j)}\right)\delta\left(\frac{\lambda_i + N - i}{N}\right).$$

This definition is inspired by the theorem of Perelomov and Popov (1968).

For any representation π we define the random probability measure $m_{PP}[\rho^\pi]$ as the pushforward of ρ^π with respect to the map $\lambda \to m_{PP}[\lambda]$.

Law of Large Numbers

Consider two sequences of signatures $\lambda = \lambda(N)$ and $\mu = \mu(N)$ which satisfy

$$m_{PP}[\lambda] \xrightarrow[N\to\infty]{} m_1, \qquad m_{PP}[\mu] \xrightarrow[N\to\infty]{} m_2, \qquad \text{weak convergence},$$

where m_1 and m_2 are probability measures.
We are interested in the **asymptotic behaviour** of the random probability measure $m_{PP}[\rho^{\pi^\lambda \otimes \pi^\mu}]$.

Theorem (Bufetov-Gorin, 2013)

As $N \to \infty$, random measures $m_{PP}[\rho^{\pi^\lambda \otimes \pi^\mu}]$ converge in the sense of moments, in probability to a deterministic measure $m_1 \boxplus m_2$ which is the free convolution of m_1 and m_2.

Universal enveloping algebra

- Let $\mathcal{U}(\mathfrak{gl}_N)$ denote the complexified universal enveloping algebra of $U(N)$. This algebra is spanned by generators E_{ij} subject to the relations

$$[E_{ij}, E_{kl}] = \delta^k_j E_{il} - \delta^l_i E_{kj}.$$

- Let $E(N) \in \mathcal{U}(\mathfrak{gl}_N) \otimes \mathrm{Mat}_{N\times N}$ denote the following $N \times N$ matrix, whose matrix elements belong to $\mathcal{U}(\mathfrak{gl}_N)$:

$$E(N) = \begin{pmatrix} E_{11} & E_{12} & \dots & E_{1N} \\ E_{21} & \ddots & & E_{2N} \\ \vdots & & & \vdots \\ E_{N1} & E_{N2} & \dots & E_{NN} \end{pmatrix}$$

Let $\mathcal{Z}(\mathfrak{gl}_N)$ denote the center of $\mathcal{U}(\mathfrak{gl}_N)$.

Theorem (Perelomov–Popov, 1968)

For $p = 0, 1, 2, \ldots$ consider the element

$$X_p = \mathrm{Trace}\,(E^p) = \sum_{i_1,\ldots,i_p=1}^{N} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_p i_1} \in \mathcal{U}(\mathfrak{gl}_N).$$

Then $X_p \in \mathcal{Z}(\mathfrak{gl}_N)$. Moreover, in the irreducible representation π^λ the element X_p acts as scalar $C_p[\lambda]$

$$C_p[\lambda] = \sum_{i=1}^{N} \left(\prod_{j \neq i} \frac{(\lambda_i - i) - (\lambda_j - j) - 1}{(\lambda_i - i) - (\lambda_j - j)} \right) (\lambda_i + N - i)^p \,.$$