When J. Ginibre met E. Schrödinger

Thomas J. Bothner

Department of Mathematics
King's College London

Joint with Jinho Baik, arXiv:1808.02419

CIRM - Integrability and Randomness
April 11th, 2019

Did they actually meet?

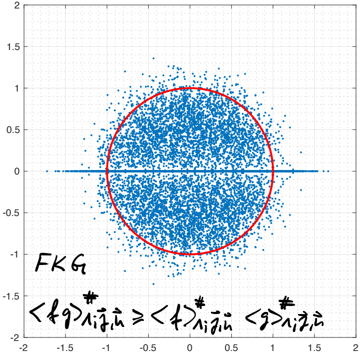


Figure 1: E.S.: 1887 - 1961 and J.G.: 1938 - ???

GOE to the comparison

Consider the Gaussian Orthogonal Ensemble (GOE), i.e. matrices

$$\mathbf{X} = \frac{1}{2}(\mathbf{Y} + \mathbf{Y}^T) \in \mathbb{R}^{n \times n}: \qquad Y_{jk} \overset{\text{iid}}{\sim} N(0,1). \quad \text{(Mehta 1960)}$$

GOE to the comparison

Consider the Gaussian Orthogonal Ensemble (GOE), i.e. matrices

$$\mathbf{X} = \frac{1}{2}(\mathbf{Y} + \mathbf{Y}^T) \in \mathbb{R}^{n\times n}: \qquad Y_{jk} \overset{\text{iid}}{\sim} N(0,1). \quad (\text{Mehta 1960})$$

Equivalently think of this setup as a log-gas system

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n, \qquad \lambda_j \in \mathbb{R},$$

GOE to the comparison

Consider the Gaussian Orthogonal Ensemble (GOE), i.e. matrices

$$\mathbf{X} = \frac{1}{2}(\mathbf{Y} + \mathbf{Y}^T) \in \mathbb{R}^{n \times n}: \qquad Y_{jk} \overset{\text{iid}}{\sim} N(0,1). \quad \text{(Mehta 1960)}$$

Equivalently think of this setup as a log-gas system

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n, \qquad \lambda_j \in \mathbb{R},$$

with joint pdf for the particles' locations equal to (Hsu 1939)

$$f(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |\lambda_k - \lambda_j| \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \lambda_j^2 \right).$$

GOE to the comparison

Consider the Gaussian Orthogonal Ensemble (GOE), i.e. matrices

$$\mathbf{X} = \frac{1}{2}(\mathbf{Y} + \mathbf{Y}^T) \in \mathbb{R}^{n \times n}: \qquad Y_{jk} \overset{\text{iid}}{\sim} N(0,1). \quad (\text{Mehta } 1960)$$

Equivalently think of this setup as a log-gas system

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n, \qquad \lambda_j \in \mathbb{R},$$

with joint pdf for the particles' locations equal to (Hsu 1939)

$$f(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \prod_{1 \le j < k \le n} |\lambda_k - \lambda_j| \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \lambda_j^2 \right).$$

Objective: What can we say about the underlying limit laws?

The eigenvalues $\{\lambda_j\}_{j=1}^n$ form a *Pfaffian point process* (Dyson 1970),

$$R_k(\lambda_1,\ldots,\lambda_k) := \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} f(\lambda_1,\ldots,\lambda_n)\prod_{j=k+1}^{n}\mathrm{d}\lambda_j = \mathrm{Pf}\big[\mathbf{K}_n(\lambda_i,\lambda_j)\big]_{i,j=1}^{k},$$

with a Hilbert-Schmidt class 2×2 matrix-valued kernel $\mathbf{K}_n$. Now analyze R_k asymptotically in different scaling regimes.

The eigenvalues $\{\lambda_j\}_{j=1}^n$ form a *Pfaffian point process* (Dyson 1970),

$$R_k(\lambda_1,\ldots,\lambda_k) := \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} f(\lambda_1,\ldots,\lambda_n)\prod_{j=k+1}^n \mathrm{d}\lambda_j = \mathrm{Pf}\big[\mathbf{K}_n(\lambda_i,\lambda_j)\big]_{i,j=1}^k,$$

with a Hilbert-Schmidt class 2×2 matrix-valued kernel $\mathbf{K}_n$. Now analyze R_k asymptotically in different scaling regimes.

(A) The global eigenvalue regime: define the ESD

$$\mu_{\mathbf{X}}(s) = \frac{1}{n}\#\{1\le j\le n,\ \lambda_j\le s\},\quad s\in\mathbb{R},$$

The eigenvalues $\{\lambda_j\}_{j=1}^{n}$ form a *Pfaffian point process* (Dyson 1970),

$$R_k(\lambda_1,\ldots,\lambda_k) := \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} f(\lambda_1,\ldots,\lambda_n)\prod_{j=k+1}^{n}\mathrm{d}\lambda_j = \mathrm{Pf}\big[\mathbf{K}_n(\lambda_i,\lambda_j)\big]_{i,j=1}^{k},$$

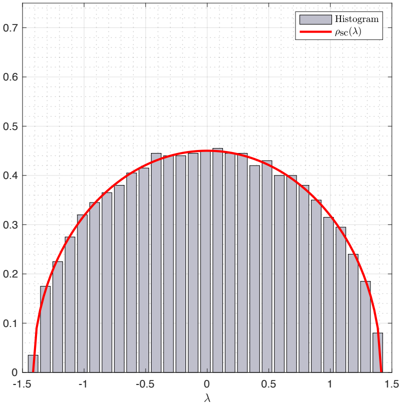
with a Hilbert-Schmidt class 2×2 matrix-valued kernel $\mathbf{K}_n$. Now analyze R_k asymptotically in different scaling regimes.

(A) The global eigenvalue regime: define the ESD

$$\mu_{\mathbf{X}}(s) = \frac{1}{n}\#\{1\leq j\leq n,\ \lambda_j\leq s\},\quad s\in\mathbb{R},$$

then, as $n\to\infty$, the random measure $\mu_{\mathbf{X}/\sqrt{n}}$ converges almost surely to the Wigner semi-circular distribution (Wigner 1955)

$$\rho_{\mathrm{sc}}(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}_+\,\mathrm{d}\lambda.$$



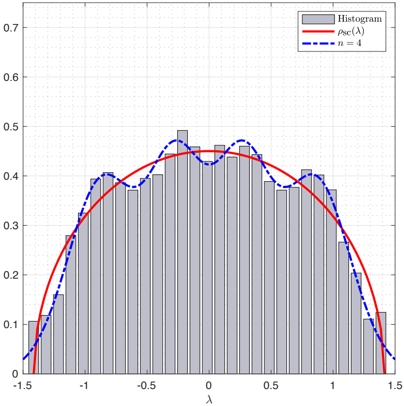


Figure 2: Wigner's law for one (rescaled) 2000×2000 GOE matrix on the left, plotted is the rescaled histogram of the 2000 eigenvalues and the semicircular density $\rho_{\text{sc}}(\lambda)$. On the right we compare Wigner's law to the exact eigenvalue density for $n = 4$ and the associated eigenvalue histogram (sampled 4000 times).

Universality I

Wigner's law is a universal limiting law (Arnold 1967, ...), it holds true for any (properly centered and scaled) symmetric or Hermitian *Wigner matrix* $\mathbf{X} = (X_{jk})_{j,k=1}^{n}$ with $\mathbb{E}|X_{jk}|^2 < \infty$ where $X_{jk}, j < k$ are iid real or complex variables and X_{jj} iid real variables independent of the upper triangular ones.

Universality I

Wigner's law is a universal limiting law (Arnold 1967, ...), it holds true for any (properly centered and scaled) symmetric or Hermitian *Wigner matrix* $\mathbf{X} = (X_{jk})_{j,k=1}^{n}$ with $\mathbb{E}|X_{jk}|^2 < \infty$ where $X_{jk}, j < k$ are iid real or complex variables and X_{jj} iid real variables independent of the upper triangular ones.

(B) The local eigenvalue regime: We shall zoom in on the right edge point $\lambda_0 = \sqrt{2n}$ and let n be even (Forrester, Nagao, Honner 1999),

$$\frac{1}{\sqrt{2}n^{\frac{1}{6}}}\mathbf{K}_n\left(\sqrt{2n} + \frac{x}{\sqrt{2}n^{\frac{1}{6}}}, \sqrt{2n} + \frac{y}{\sqrt{2}n^{\frac{1}{6}}}\right) \to \mathbf{Q}_{\mathrm{Ai}}(x, y),$$

as $n \to \infty$ uniformly in $x, y \in \mathbb{R}$ chosen from compact subsets.

Here, $\mathbf{Q}_{\mathrm{Ai}}$ is Hilbert-Schmidt on $L^2(s_0, \infty)$ with kernel entries

$$Q_{11}(x, y) = Q_{22}(y, x) = K_{\mathrm{Ai}}(x, y) + \frac{1}{2}\,\mathrm{Ai}(x) \int_{-\infty}^{y} \mathrm{Ai}(t)\,\mathrm{d}t$$

$$Q_{12}(x, y) = -\frac{\partial}{\partial y} K_{\mathrm{Ai}}(x, y) - \frac{1}{2}\,\mathrm{Ai}(x)\mathrm{Ai}(y),$$

and

$$Q_{21}(x, y) = -\int_{x}^{\infty} K_{\mathrm{Ai}}(t, y)\,\mathrm{d}t - \frac{1}{2}\int_{x}^{y} \mathrm{Ai}(t)\,\mathrm{d}t + \frac{1}{2}\int_{x}^{\infty} \mathrm{Ai}(t)\,\mathrm{d}t \cdot \int_{y}^{\infty} \mathrm{Ai}(t)\,\mathrm{d}t - \frac{1}{2}\,\mathrm{sgn}(x - y),$$

Here, $\mathbf{Q}_{\mathrm{Ai}}$ is Hilbert-Schmidt on $L^2(s_0,\infty)$ with kernel entries

$$Q_{11}(x,y) = Q_{22}(y,x) = K_{\mathrm{Ai}}(x,y) + \frac{1}{2}\,\mathrm{Ai}(x)\int_{-\infty}^{y}\mathrm{Ai}(t)\,\mathrm{d}t$$

$$Q_{12}(x,y) = -\frac{\partial}{\partial y}K_{\mathrm{Ai}}(x,y) - \frac{1}{2}\,\mathrm{Ai}(x)\mathrm{Ai}(y),$$

and

$$Q_{21}(x,y) = -\int_x^{\infty}K_{\mathrm{Ai}}(t,y)\,\mathrm{d}t - \frac{1}{2}\int_x^{y}\mathrm{Ai}(t)\,\mathrm{d}t + \frac{1}{2}\int_x^{\infty}\mathrm{Ai}(t)\,\mathrm{d}t\cdot\int_y^{\infty}\mathrm{Ai}(t)\,\mathrm{d}t - \frac{1}{2}\,\mathrm{sgn}(x-y),$$

where we use the trace-class kernel (on $L^2(s_0,\infty)$)

$$K_{\mathrm{Ai}}(x,y) = \int_0^{\infty}\mathrm{Ai}(x+s)\mathrm{Ai}(y+s)\,\mathrm{d}s.$$

Universality II

The limiting kernel $\mathbf{Q}_{\mathrm{Ai}}(x, y)$ is once more universal (Soshnikov 1999), it governs the soft edge scaling limits of the k-point correlation functions for any (properly centered and scaled) real *Wigner matrix* $\mathbf{X}$ (modulo some decay constraints).

Universality II

The limiting kernel $\mathbf{Q}_{\mathrm{Ai}}(x, y)$ is once more universal (Soshnikov 1999), it governs the soft edge scaling limits of the k-point correlation functions for any (properly centered and scaled) real *Wigner matrix* $\mathbf{X}$ (modulo some decay constraints).

The above kernel allows us to formulate the following central limit theorem for the largest eigenvalue $\lambda_{\max}$ in the GOE, as $n \to \infty$,

$$\lambda_{\max}(\mathbf{X}) \Rightarrow \sqrt{2n} + \frac{1}{\sqrt{2}n^{\frac{1}{6}}} F_1$$

Universality II

The limiting kernel $\mathbf{Q}_{\mathrm{Ai}}(x,y)$ is once more universal (Soshnikov 1999), it governs the soft edge scaling limits of the k-point correlation functions for any (properly centered and scaled) real *Wigner matrix* $\mathbf{X}$ (modulo some decay constraints).

The above kernel allows us to formulate the following central limit theorem for the largest eigenvalue $\lambda_{\max}$ in the GOE, as $n \to \infty$,

$$\lambda_{\max}(\mathbf{X}) \Rightarrow \sqrt{2n} + \frac{1}{\sqrt{2}n^{\frac{1}{6}}} F_1$$

where the cdf of F_1 equals (Tracy, Widom 2005)

$$\big(\mathbb{P}(F_1 \leq s)\big)^2 = \det_2\big(1 - \mathbf{G}\mathbf{Q}_{\mathrm{Ai}}\mathbf{G}^{-1} \restriction_{L^2(s,\infty)\oplus L^2(s,\infty)}\big).$$

with $\mathbf{G} = \mathrm{diag}(g, g^{-1})$ and $g(x) = \sqrt{1+x^2}$.

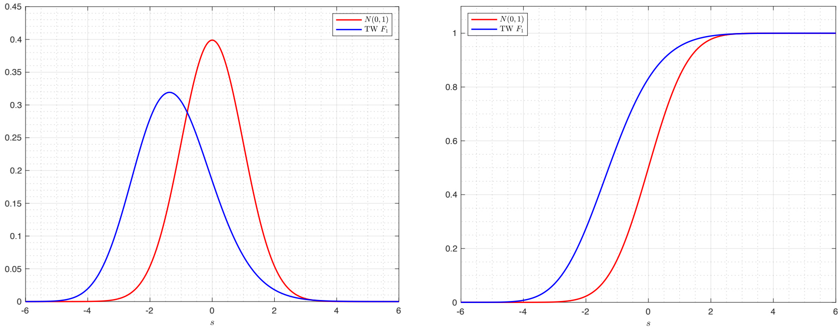


Figure 3: Tracy-Widom distribution F_1 (blue) versus $N(0,1)$ (red).

	mean	variance	skewness	kurtosis
$N(0,1)$	0	1	0	0
F_1	-1.20653	1.60778	0.29346	0.16524

There are other explicit formulæ for the cdf of F_1:

1. Airy determinant and resolvent formula (Forrester 2006)

$$\big(\mathbb{P}(F_1 \leq s)\big)^2 = \det(1 - (K_{\mathrm{Ai}} + U \otimes V) \restriction_{L^2(s,\infty)})$$

where $(U \otimes V)(x,y) = \mathrm{Ai}(x)A(y)$ and $A(x) = \int_{-\infty}^{x} \mathrm{Ai}(y)\,\mathrm{d}y$.

There are other explicit formulæ for the cdf of F_1:

1. Airy determinant and resolvent formula (Forrester 2006)

$$\big(\mathbb{P}(F_1 \leq s)\big)^2 = \det(1 - (K_{\mathrm{Ai}} + U \otimes V) \restriction_{L^2(s,\infty)})$$

   where $(U \otimes V)(x,y) = \mathrm{Ai}(x)A(y)$ and $A(x) = \int_{-\infty}^{x} \mathrm{Ai}(y)\,\mathrm{d}y$.

2. Expression in terms of Painlevé-II (Tracy, Widom 1996)

$$\big(\mathbb{P}(F_1 \leq s)\big)^2 = \exp\left(-\int_s^\infty (x-s)q^2(x)\mathrm{d}x - \int_s^\infty q(x)\mathrm{d}x\right)$$

   where q solves $\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} = xq + 2q^3$ with $q(x) \sim \mathrm{Ai}(x),\ x \to +\infty$.

There are other explicit formulæ for the cdf of F_1:

1. Airy determinant and resolvent formula (Forrester 2006)

$$\big(\mathbb{P}(F_1 \leq s)\big)^2 = \det(1 - (K_{\mathrm{Ai}} + U \otimes V) \restriction_{L^2(s,\infty)})$$

   where $(U \otimes V)(x, y) = \mathrm{Ai}(x)A(y)$ and $A(x) = \int_{-\infty}^{x} \mathrm{Ai}(y)\, \mathrm{d}y$.

2. Expression in terms of Painlevé-II (Tracy, Widom 1996)

$$\big(\mathbb{P}(F_1 \leq s)\big)^2 = \exp\left(-\int_s^\infty (x-s)q^2(x)\mathrm{d}x - \int_s^\infty q(x)\mathrm{d}x\right)$$

   where q solves $\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} = xq + 2q^3$ with $q(x) \sim \mathrm{Ai}(x),\ x \to +\infty$.

3. Single determinantal formula (Ferrari, Spohn 2005)

$$\mathbb{P}(F_1 \leq s) = \det(1 - F \restriction_{L^2(s,\infty)}), \qquad K_{\mathrm{Ai}} = F^2.$$

Somewhat GOE but non-symmetric

We now consider the Real Ginibre ensemble (GinOE), i.e. matrices

$$\mathbf{X} \in \mathbb{R}^{n \times n}: \qquad X_{jk} \overset{\text{iid}}{\sim} N(0,1). \quad (\text{Ginibre 1965})$$

Somewhat GOE but non-symmetric

We now consider the Real Ginibre ensemble (GinOE), i.e. matrices

$$\mathbf{X} \in \mathbb{R}^{n \times n}: \qquad X_{jk} \overset{\text{iid}}{\sim} N(0,1). \quad \text{(Ginibre 1965)}$$

Equivalently think of this setup as a log-gas system

$$\underbrace{\lambda_1 < \lambda_2 < \ldots < \lambda_L}_{\to \vec{\alpha}=(\lambda_1,\ldots,\lambda_L)}; \quad \underbrace{x_1 < \ldots < x_M; \quad y_1, \ldots, y_M > 0}_{\to \vec{\beta}=(x_1+\mathrm{i}y_1,\ldots,x_M+\mathrm{i}y_M)}: \;\; L + 2M = n$$

Somewhat GOE but non-symmetric

We now consider the Real Ginibre ensemble (GinOE), i.e. matrices

$$\mathbf{X} \in \mathbb{R}^{n\times n}: \qquad X_{jk} \overset{\text{iid}}{\sim} N(0,1). \quad \text{(Ginibre 1965)}$$

Equivalently think of this setup as a log-gas system

$$\underbrace{\lambda_1 < \lambda_2 < \ldots < \lambda_L}_{\to \vec{\alpha}=(\lambda_1,\ldots,\lambda_L)}; \quad \underbrace{x_1 < \ldots < x_M; \quad y_1, \ldots, y_M > 0}_{\to \vec{\beta}=(x_1+\mathrm{i}y_1,\ldots,x_M+\mathrm{i}y_M)}: \quad L + 2M = n$$

with (L, M)-*partial* joint pdf (Lehmann, Sommers 1991)

$$f_{L,M}(\vec{\alpha},\vec{\beta}) = \frac{1}{Z_{n,L,M}} \prod_{1\leq j<k\leq n} |z_k - z_j| \exp\left(-\frac{1}{2}\sum_{j=1}^{n} z_j^2\right)$$
$$\times \prod_{j=1}^{n} \sqrt{\operatorname{erfc}(\sqrt{2}\,|\Im z_j|)}, \qquad \vec{z} \equiv (\vec{\alpha},\vec{\beta},\vec{\beta}^*) \in \mathbb{C}^n.$$

The saturn effect

The above pdf is *not* absolutely continuous, in particular

$$\mathbb{P}(z_1,\ldots,z_n\in\mathbb{R})=2^{-\frac{n}{4}(n-1)}\qquad(\text{Edelman 1997})$$

The saturn effect

The above pdf is *not* absolutely continuous, in particular

$$\mathbb{P}(z_1, \ldots, z_n \in \mathbb{R}) = 2^{-\frac{n}{4}(n-1)} \qquad \text{(Edelman 1997)}$$

Objective: What can we say about the underlying limit laws?

The saturn effect

The above pdf is *not* absolutely continuous, in particular

$$\mathbb{P}(z_1,\ldots,z_n\in\mathbb{R})=2^{-\frac{n}{4}(n-1)}\qquad \text{(Edelman 1997)}$$

Objective: What can we say about the underlying limit laws?

The eigenvalues $\{z_j\}_{j=1}^n$ form a *Pfaffian point field* (Borodin, Sinclair 2009), with $\vec{\mu}\in\mathbb{R}^\ell,\vec{\nu}\in\mathbb{C}^m$, and $\vec{\mu}\vee\vec{\alpha}=(\mu_1,\ldots,\mu_\ell,\alpha_1,\ldots,\alpha_{L-\ell})$,

$$R_{\ell,m}(\vec{\mu},\vec{\nu}):=\sum_{\substack{(L,M)\\ L\geq\ell,M\geq m}}\frac{1}{(L-\ell)!(M-m)!}\int_{\mathbb{R}^{L-\ell}}\int_{\mathbb{C}^{M-m}}f_{L,M}(\vec{\mu}\vee\vec{\alpha},\vec{\nu}\vee\vec{\beta})$$

$$\times\prod_{j=1}^{L-\ell}\mathrm{d}\alpha_j\prod_{k=1}^{M-m}\mathrm{d}^2\beta_k=\mathrm{Pf}\begin{bmatrix}[\mathbf{K}_n^{\mathbb{R},\mathbb{R}}(\mu_j,\mu_k)]_{j,k=1}^{\ell\times\ell} & [\mathbf{K}_n^{\mathbb{R},\mathbb{C}}(\mu_j,\nu_k)]_{j,k=1}^{\ell\times m}\\ [\mathbf{K}_n^{\mathbb{C},\mathbb{R}}(\nu_j,\mu_k)]_{j,k=1}^{m\times\ell} & [\mathbf{K}_n^{\mathbb{C},\mathbb{C}}(\nu_j,\nu_k)]_{j,k=1}^{m\times m}\end{bmatrix}$$

using four 2×2 matrix-valued Hilbert-Schmidt kernels $\mathbf{K}_n^{\#,\#}$. Now analyze $R_{\ell,m}$ asymptotically in different scaling regimes.

using four 2×2 matrix-valued Hilbert-Schmidt kernels $\mathbf{K}_n^{\#,\#}$. Now analyze $R_{\ell,m}$ asymptotically in different scaling regimes.

(A) The global eigenvalue regime: define the ESD

$$\mu_{\mathbf{X}}(s,t) = \frac{1}{n}\#\{1 \leq j \leq n,\ \Re z_j \leq s,\ \Im z_j \leq t\}, \qquad s,t \in \mathbb{R},$$

then, as $n \to \infty$, the random measure $\mu_{\mathbf{X}/\sqrt{n}}$ converges almost surely to the uniform distribution on the unit disk (Ginibre 1965)

$$\rho_c(z) = \frac{1}{\pi}\chi_{|z|<1}(z)\,\mathrm{d}^2 z$$

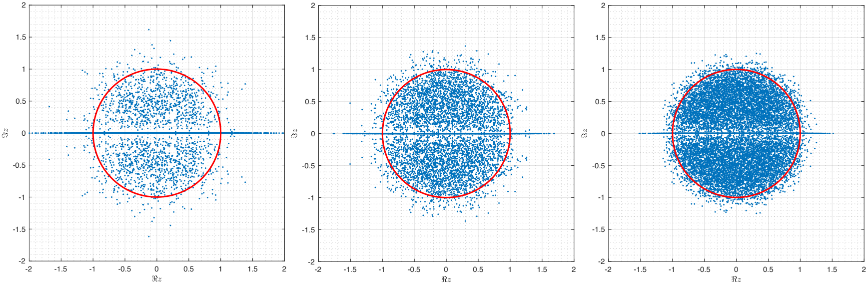


Figure 4: The circular law for 1000 real (rescaled) Ginibre matrices of varying dimensions $n \times n$ in comparison with the unit circle boundary. We plot $n = 4, 8, 16$ from left to right. A saturn effect is clearly visible on the real line.

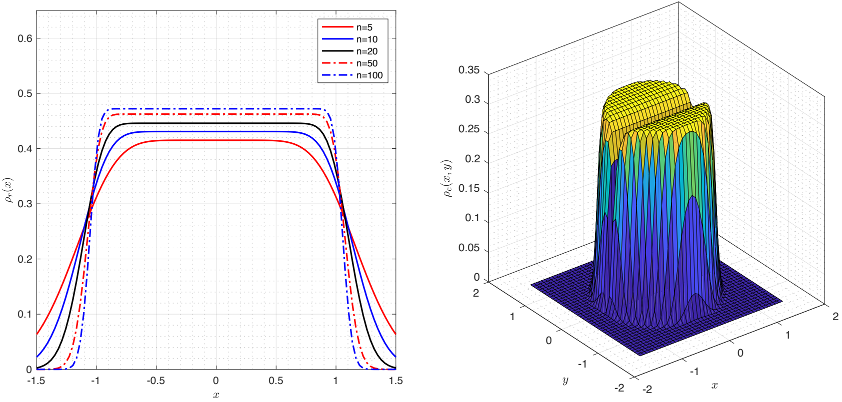


Figure 5: Densities of normalized real (left) and complex (right) eigenvalues for $n = 5, 10, 20, 50, 100$ (left) and $n = 100$ (right). The larger n, the better their approach to the uniform density on $[-1, 1]$ (left) and $x^2 + y^2 \leq 1$ (right).

Universality III

The circular law is a universal limiting law (Tao, Vu 2010), it holds true for any (properly centered and scaled) non-Hermitian real or complex matrix $\mathbf{X} = (X_{jk})_{j,k=1}^{n}$ with $\mathbb{E}|X_{jk}|^2 < \infty$ where X_{jk} are iid real or complex variables.

Universality III

The circular law is a universal limiting law (Tao, Vu 2010), it holds true for any (properly centered and scaled) non-Hermitian real or complex matrix $\mathbf{X} = (X_{jk})_{j,k=1}^{n}$ with $\mathbb{E}|X_{jk}|^2 < \infty$ where X_{jk} are iid real or complex variables.

(B) The local eigenvalue regime: We shall zoom in on $\sqrt{n}$ for n even and discuss "only" the scaling limit of $\mathbf{K}_n^{\mathbb{R},\mathbb{R}}(\cdot,\cdot)$ (Borodin, Sinclair 2009; Poplavskyi, Tribe, Zaboronski 2016):

$$\mathbf{K}_n^{\mathbb{R},\mathbb{R}}(\sqrt{n}+x,\sqrt{n}+y) \to \mathbf{P}_{\mathfrak{e}}(x,y)$$

as $n \to \infty$ uniformly in $x, y \in \mathbb{R}$ chosen from compact subsets.

Here, $\mathbf{P}_{\mathfrak{e}}$ is Hilbert-Schmidt on $L^2(s_0,\infty)$ with kernel entries

$$P_{11}(x,y)=P_{22}(y,x)=K_{\mathfrak{e}}(x,y)+\frac{1}{2}\mathfrak{e}(x)\int_{-\infty}^{y}\mathfrak{e}(t)\,\mathrm{d}t$$

$$P_{12}(x,y)=-\frac{\partial}{\partial y}K_{\mathfrak{e}}(x,y)-\frac{1}{2}\mathfrak{e}(x)\mathfrak{e}(y),$$

and

$$P_{21}(x,y)=-\int_x^{\infty}K_{\mathfrak{e}}(t,y)\,\mathrm{d}t-\frac{1}{2}\int_x^{y}\mathfrak{e}(t)\,\mathrm{d}t$$

$$+\frac{1}{2}\int_x^{\infty}\mathfrak{e}(t)\,\mathrm{d}t\cdot\int_y^{\infty}\mathfrak{e}(t)\,\mathrm{d}t-\frac{1}{2}\,\mathrm{sgn}(x-y)$$

Here, $\mathbf{P}_{\mathfrak{e}}$ is Hilbert-Schmidt on $L^2(s_0,\infty)$ with kernel entries

$$P_{11}(x,y)=P_{22}(y,x)=K_{\mathfrak{e}}(x,y)+\frac{1}{2}\mathfrak{e}(x)\int_{-\infty}^{y}\mathfrak{e}(t)\,\mathrm{d}t$$

$$P_{12}(x,y)=-\frac{\partial}{\partial y}K_{\mathfrak{e}}(x,y)-\frac{1}{2}\mathfrak{e}(x)\mathfrak{e}(y),$$

and

$$P_{21}(x,y)=-\int_{x}^{\infty}K_{\mathfrak{e}}(t,y)\,\mathrm{d}t-\frac{1}{2}\int_{x}^{y}\mathfrak{e}(t)\,\mathrm{d}t$$

$$+\frac{1}{2}\int_{x}^{\infty}\mathfrak{e}(t)\,\mathrm{d}t\cdot\int_{y}^{\infty}\mathfrak{e}(t)\,\mathrm{d}t-\frac{1}{2}\,\mathrm{sgn}(x-y)$$

where we use the trace-class kernel (on $L^2(s_0,\infty)$)

$$K_{\mathfrak{e}}(x,y)=\int_{0}^{\infty}\mathfrak{e}(x+s)\mathfrak{e}(y+s)\,\mathrm{d}s;\qquad \mathfrak{e}(x)=\frac{1}{\sqrt{\pi}}\mathrm{e}^{-x^2}.$$

Universality IV

It is an open question whether $\mathbf{P}_{\mathfrak{e}}(x, y)$ is universal, i.e. whether it governs the scaling limits of the real-real (ℓ, m)-point correlation functions for any (properly centered and scaled) non-Hermitian real matrix $\mathbf{X}$ with iid entries (modulo some decay constraints).

Universality IV

It is an open question whether $\mathbf{P}_{\mathfrak{e}}(x, y)$ is universal, i.e. whether it governs the scaling limits of the real-real (ℓ, m)-point correlation functions for any (properly centered and scaled) non-Hermitian real matrix $\mathbf{X}$ with iid entries (modulo some decay constraints).

The above kernel allows us to formulate the following central limit theorem for the largest real eigenvalue $\lambda_{\max} \in \mathbb{R}$ in the GinOE, as $n \to \infty$,

$$\lambda_{\max}(\mathbf{X}) \Rightarrow \sqrt{n} + G$$

Universality IV

It is an open question whether $\mathbf{P}_{\mathfrak{e}}(x,y)$ is universal, i.e. whether it governs the scaling limits of the real-real (ℓ,m)-point correlation functions for any (properly centered and scaled) non-Hermitian real matrix $\mathbf{X}$ with iid entries (modulo some decay constraints).

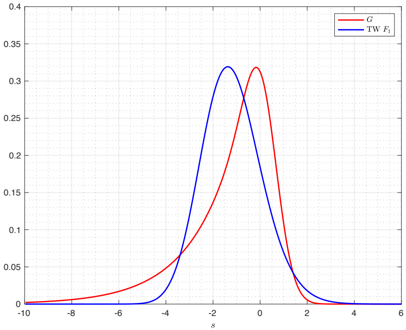
The above kernel allows us to formulate the following central limit theorem for the largest real eigenvalue $\lambda_{\max}\in\mathbb{R}$ in the GinOE, as $n\to\infty$,

$$\lambda_{\max}(\mathbf{X})\Rightarrow\sqrt{n}+G$$

where the cdf of G equals (Rider, Sinclair 2014; Poplavskyi, Tribe, Zaboronski 2017)

$$\big(\mathbb{P}(G\leq s)\big)^2=\det_2\big(1-\mathbf{G}\mathbf{P}_{\mathfrak{e}}\mathbf{G}^{-1}\restriction_{L^2(s,\infty)\oplus L^2(s,\infty)}\big)$$

with $\mathbf{G}=\mathrm{diag}(g,g^{-1})$ and $g(x)=\sqrt{1+x^2}$.



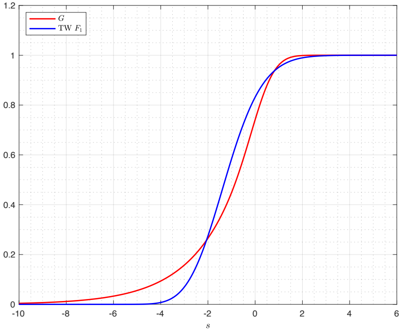


Figure 6: Tracy-Widom distribution F_1 (blue) versus G (red).

	mean	variance	skewness	kurtosis
G	-1.30319	3.97536	-1.76969	5.14560
F_1	-1.20653	1.60778	0.29346	0.16524

What other explicit formulæ for the cdf of G are available?

1. Exponential determinant and resolvent formula (Rider, Sinclair 2014; Poplavskyi, Tribe, Zaboronski 2017)

$$\big(\mathbb{P}(G \leq s)\big)^2 = \det(1 - (K_{\mathfrak{e}} + U_{\mathfrak{e}} \otimes V_{\mathfrak{e}}) \restriction_{L^2(s,\infty)})$$

where $(U_{\mathfrak{e}} \otimes V_{\mathfrak{e}})(x, y) = \mathfrak{e}(x)\mathfrak{E}(y)$ and $\mathfrak{E}(y) = \int_{-\infty}^{x} \mathfrak{e}(y)\,\mathrm{d}y$.

What other explicit formulæ for the cdf of G are available?

1. Exponential determinant and resolvent formula (Rider, Sinclair 2014; Poplavskyi, Tribe, Zaboronski 2017)

$$\big(\mathbb{P}(G \leq s)\big)^2 = \det(1 - (K_{\mathfrak{e}} + U_{\mathfrak{e}} \otimes V_{\mathfrak{e}}) \restriction_{L^2(s,\infty)})$$

   where $(U_{\mathfrak{e}} \otimes V_{\mathfrak{e}})(x,y) = \mathfrak{e}(x)\mathfrak{E}(y)$ and $\mathfrak{E}(y) = \int_{-\infty}^{x} \mathfrak{e}(y)\,\mathrm{d}y$.
2. Any Tracy-Widom type closed form expression?

What other explicit formulæ for the cdf of G are available?

1. Exponential determinant and resolvent formula (Rider, Sinclair 2014; Poplavskyi, Tribe, Zaboronski 2017)

$$\big(\mathbb{P}(G \leq s)\big)^2 = \det(1 - (K_{\mathfrak{e}} + U_{\mathfrak{e}} \otimes V_{\mathfrak{e}}) \restriction_{L^2(s,\infty)})$$

   where $(U_{\mathfrak{e}} \otimes V_{\mathfrak{e}})(x, y) = \mathfrak{e}(x)\mathfrak{E}(y)$ and $\mathfrak{E}(y) = \int_{-\infty}^{x} \mathfrak{e}(y)\,\mathrm{d}y$.
2. Any Tracy-Widom type closed form expression?
3. Any Ferrari-Spohn type determinantal formula?

What other explicit formulæ for the cdf of G are available?

1. Exponential determinant and resolvent formula (Rider, Sinclair 2014; Poplavskyi, Tribe, Zaboronski 2017)

$$\big(\mathbb{P}(G \leq s)\big)^2 = \det(1 - (K_{\mathfrak{e}} + U_{\mathfrak{e}} \otimes V_{\mathfrak{e}}) \restriction_{L^2(s,\infty)})$$

   where $(U_{\mathfrak{e}} \otimes V_{\mathfrak{e}})(x,y) = \mathfrak{e}(x)\mathfrak{E}(y)$ and $\mathfrak{E}(y) = \int_{-\infty}^{x} \mathfrak{e}(y)\,\mathrm{d}y$.
2. Any Tracy-Widom type closed form expression?
3. Any Ferrari-Spohn type determinantal formula?

Roadblock: $K_{\mathrm{Ai}}(x,y)$ has a Christoffel-Darboux type structure, i.e.

$$K_{\mathrm{Ai}}(x,y) = \frac{\phi(x)\psi(y) - \psi(x)\phi(y)}{x-y}, \quad \phi(z) = \mathrm{Ai}(z), \;\; \psi(z) = \mathrm{Ai}'(z),$$

but this is *not* true for $K_{\mathfrak{e}}(x,y)$!

What other explicit formulæ for the cdf of G are available?

1. Exponential determinant and resolvent formula (Rider, Sinclair 2014; Poplavskyi, Tribe, Zaboronski 2017)

$$\big(\mathbb{P}(G \leq s)\big)^2 = \det(1 - (K_{\mathfrak{e}} + U_{\mathfrak{e}} \otimes V_{\mathfrak{e}}) \restriction_{L^2(s,\infty)})$$

   where $(U_{\mathfrak{e}} \otimes V_{\mathfrak{e}})(x,y) = \mathfrak{e}(x)\mathfrak{E}(y)$ and $\mathfrak{E}(y) = \int_{-\infty}^{x} \mathfrak{e}(y)\,\mathrm{d}y$.
2. Any Tracy-Widom type closed form expression?
3. Any Ferrari-Spohn type determinantal formula?

Roadblock: $K_{\mathrm{Ai}}(x,y)$ has a Christoffel-Darboux type structure, i.e.

$$K_{\mathrm{Ai}}(x,y) = \frac{\phi(x)\psi(y) - \psi(x)\phi(y)}{x-y}, \qquad \phi(z) = \mathrm{Ai}(z), \quad \psi(z) = \mathrm{Ai}'(z),$$

but this is *not* true for $K_{\mathfrak{e}}(x,y)$! $\longrightarrow$ Do remember N. Wiener and $\mathcal{F}$!

And finally they meet

Define

$$F(t;\gamma) := \sqrt{\det(1-\gamma(K_{\mathfrak{e}} + U_{\mathfrak{e}} \otimes V_{\mathfrak{e}}) \restriction_{L^2(t,\infty)})}, \quad t \in \mathbb{R},\ \gamma \in [0,1].$$

and consider the following Riemann-Hilbert problem (RHP):

And finally they meet

Define

$$F(t;\gamma) := \sqrt{\det(1-\gamma(K_{\mathfrak{e}} + U_{\mathfrak{e}} \otimes V_{\mathfrak{e}}) \restriction_{L^2(t,\infty)})}, \quad t \in \mathbb{R},\ \gamma \in [0,1].$$

and consider the following Riemann-Hilbert problem (RHP):

Zakharov-Shabat (ZS) RHP

For any $(x,\gamma) \in \mathbb{R} \times [0,1]$ determine $\mathbf{X}(z) = \mathbf{X}(z;x,\gamma) \in \mathbb{C}^{2\times 2}$ such that

(1) $\mathbf{X}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ and continuous on $\overline{\mathbb{C}_{\pm}}$.

And finally they meet

Define

$$F(t;\gamma) := \sqrt{\det(1-\gamma(K_{\mathfrak{e}} + U_{\mathfrak{e}} \otimes V_{\mathfrak{e}}) \restriction_{L^2(t,\infty)})}, \quad t \in \mathbb{R},\ \gamma \in [0,1].$$

and consider the following Riemann-Hilbert problem (RHP):

Zakharov-Shabat (ZS) RHP

For any $(x,\gamma) \in \mathbb{R} \times [0,1]$ determine $\mathbf{X}(z) = \mathbf{X}(z;x,\gamma) \in \mathbb{C}^{2\times 2}$ such that

(1) $\mathbf{X}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ and continuous on $\overline{\mathbb{C}_{\pm}}$.

(2) The limits $\mathbf{X}_{\pm}(z) := \lim_{\epsilon\downarrow 0} \mathbf{X}(z \pm \mathrm{i}\epsilon), z \in \mathbb{R}$ satisfy

$$\mathbf{X}_+(z) = \mathbf{X}_-(z) \begin{bmatrix} 1-|r(z)|^2 & -\bar{r}(z)\mathrm{e}^{-2\mathrm{i}xz} \\ r(z)\mathrm{e}^{2\mathrm{i}xz} & 1 \end{bmatrix}; \quad r(z) = r(z;\gamma) = -\mathrm{i}\sqrt{\gamma}\mathrm{e}^{-\frac{1}{4}z^2}$$

And finally they meet

Define

$$F(t;\gamma) := \sqrt{\det(1-\gamma(K_{\mathfrak{e}} + U_{\mathfrak{e}} \otimes V_{\mathfrak{e}}) \restriction_{L^2(t,\infty)})}, \quad t \in \mathbb{R},\ \gamma \in [0,1].$$

and consider the following Riemann-Hilbert problem (RHP):

Zakharov-Shabat (ZS) RHP

For any $(x,\gamma) \in \mathbb{R} \times [0,1]$ determine $\mathbf{X}(z) = \mathbf{X}(z;x,\gamma) \in \mathbb{C}^{2\times 2}$ such that

(1) $\mathbf{X}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ and continuous on $\overline{\mathbb{C}_\pm}$.

(2) The limits $\mathbf{X}_\pm(z) := \lim_{\epsilon\downarrow 0} \mathbf{X}(z \pm \mathrm{i}\epsilon), z \in \mathbb{R}$ satisfy

$$\mathbf{X}_+(z) = \mathbf{X}_-(z) \begin{bmatrix} 1-|r(z)|^2 & -\bar{r}(z)\mathrm{e}^{-2\mathrm{i}xz} \\ r(z)\mathrm{e}^{2\mathrm{i}xz} & 1 \end{bmatrix}; \quad r(z) = r(z;\gamma) = -\mathrm{i}\sqrt{\gamma}\mathrm{e}^{-\frac{1}{4}z^2}$$

(3) As $z \to \infty$, we require $\mathbf{X}(z) = \mathbb{I} + \mathbf{X}_1 z^{-1} + \mathcal{O}\left(z^{-2}\right)$, $\mathbf{X}_i = \mathbf{X}_i(x,\gamma)$.

Baik-B 2018

The ZS-RHP is uniquely solvable for any $(x, \gamma) \in \mathbb{R} \times [0,1]$. Also, $X_1^{12}(\cdot, \gamma) \in \mathbb{R}$ is continuous for any $\gamma \in [0,1]$

Baik-B 2018

The ZS-RHP is uniquely solvable for any $(x,\gamma)\in\mathbb{R}\times[0,1]$. Also, $X_1^{12}(\cdot,\gamma)\in\mathbb{R}$ is continuous for any $\gamma\in[0,1]$ and

$$\big(F(t;\gamma)\big)^2=\exp\left(-\frac{1}{4}\int_t^\infty(x-t)\left|y\left(\frac{x}{2};\gamma\right)\right|^2\mathrm{d}x\right)\times\left\{\cosh\mu(t;\gamma)-\sqrt{\gamma}\sinh\mu(t;\gamma)\right\},\qquad(1)$$

using the abbreviations

$$\mu(t;\gamma):=-\frac{\mathrm{i}}{2}\int_t^\infty y\left(\frac{x}{2};\gamma\right)\mathrm{d}x,$$

and $y(x;\gamma):=2\mathrm{i}X_1^{12}(x,\gamma)$.

Identity (1) mirrors a corresponding γ-deformed TW identity in the superimposed GOE (Forrester 2006).

But where is Prof. Schrödinger?

Set

$$\mathbf{\Psi}(z) := \mathbf{X}(z)\mathrm{e}^{-\mathrm{i}xz\sigma_3}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

then $\frac{\partial \mathbf{\Psi}}{\partial x}\mathbf{\Psi}^{-1}$ is entire,

But where is Prof. Schrödinger?

Set

$$\mathbf{\Psi}(z) := \mathbf{X}(z)\mathrm{e}^{-\mathrm{i}xz\sigma_3}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

then $\frac{\partial \mathbf{\Psi}}{\partial x}\mathbf{\Psi}^{-1}$ is entire, in fact by (3) above and Liouville's theorem

$$\frac{\partial \mathbf{\Psi}}{\partial x} = \left\{-\mathrm{i}z\sigma_3 + 2\mathrm{i}\begin{bmatrix} 0 & y \\ \bar{y} & 0 \end{bmatrix}\right\}\mathbf{\Psi}, \tag{2}$$

which is the famous Zakharov-Shabat system of 1972.

But where is Prof. Schrödinger?

Set

$$\boldsymbol{\Psi}(z) := \mathbf{X}(z)\mathrm{e}^{-\mathrm{i}xz\sigma_3}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

then $\frac{\partial \boldsymbol{\Psi}}{\partial x}\boldsymbol{\Psi}^{-1}$ is entire, in fact by (3) above and Liouville's theorem

$$\frac{\partial \boldsymbol{\Psi}}{\partial x} = \left\{-\mathrm{i}z\sigma_3 + 2\mathrm{i}\begin{bmatrix} 0 & y \\ \bar{y} & 0 \end{bmatrix}\right\}\boldsymbol{\Psi}, \tag{2}$$

which is the famous Zakharov-Shabat system of 1972. It is directly related to several of the most interesting nonlinear evolution equations in $1+1$ dimensions which are solvable by the IST method.

For instance, in order to solve the Cauchy problem for the defocusing nonlinear Schrödinger equation,

$$\mathrm{i}y_t + y_{xx} - 2|y|^2 y = 0, \quad y(x,0) = y_0(x) \in \mathcal{S}(\mathbb{R}); \quad y = y(x,t) : \mathbb{R}^2 \to \mathbb{C},$$

one first computes the reflection coefficient $r(z) \in \mathcal{S}(\mathbb{R})$ associated to y_0 through the *direct scattering transform*. Note that $y_0 \to r$ is a bijection from $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R}) \cap \{r : \|r\|_\infty < 1\}$ (Beals, Coifman 1984).

For instance, in order to solve the Cauchy problem for the defocusing nonlinear Schrödinger equation,

$$\mathrm{i}y_t + y_{xx} - 2|y|^2 y = 0, \quad y(x,0) = y_0(x) \in \mathcal{S}(\mathbb{R}); \;\; y = y(x,t) : \mathbb{R}^2 \to \mathbb{C},$$

one first computes the reflection coefficient $r(z) \in \mathcal{S}(\mathbb{R})$ associated to y_0 through the *direct scattering transform*. Note that $y_0 \to r$ is a bijection from $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R}) \cap \{r : \|r\|_\infty < 1\}$ (Beals, Coifman 1984). Second, one considers the ZS-RHP above subject to the replacement

$$\mathrm{e}^{2\mathrm{i}xz} \to \mathrm{e}^{2\mathrm{i}(2tz^2 + xz)}, \quad t \in \mathbb{R},$$

and provided this problem is solvable, its (unique) solution solves dNLS with $y(x,0) = y_0(x)$ via $y(x,t) = 2\mathrm{i}X_1^{12}(x,t)$.

Some corollaries

First, tail estimates for G.

Baik-B 2018

Let $\gamma \in [0,1]$, then as $t \to +\infty$,

$$F(t;\gamma) = 1 - \frac{\gamma}{4}\mathrm{erfc}(t) + \mathcal{O}\big(\gamma^{\frac{3}{2}} t^{-1} \mathrm{e}^{-2t^2}\big).$$

Some corollaries

First, tail estimates for G.

Baik-B 2018

Let $\gamma \in [0,1]$, then as $t \to +\infty$,

$$F(t;\gamma) = 1 - \frac{\gamma}{4}\mathrm{erfc}(t) + \mathcal{O}\big(\gamma^{\frac{3}{2}} t^{-1} \mathrm{e}^{-2t^2}\big).$$

On the other hand, as $t \to -\infty$,

$$F(t;\gamma) = \mathrm{e}^{\eta_1(\gamma)t}\eta_0(\gamma)\big(1 + o(1)\big), \quad \eta_1(\gamma) = \frac{1}{2\sqrt{2\pi}}\mathrm{Li}_{\frac{3}{2}}(\gamma)$$

in terms of the polylogarithm $\mathrm{Li}_s(z)$ and with a t-independent positive factor $\eta_0(\gamma)$. Also, $\eta_0(1) = 0.75277069$.

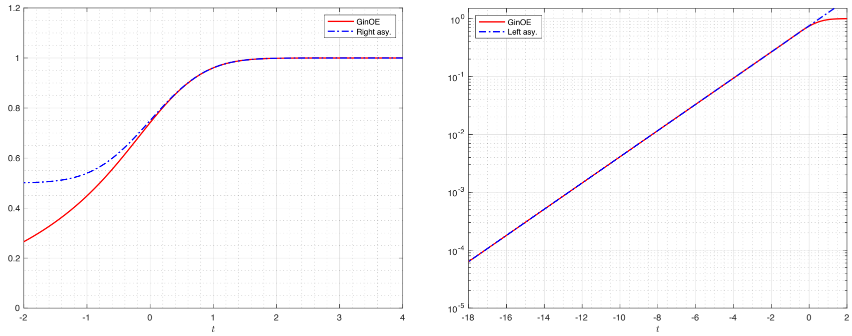


Figure 7: We doublecheck our tail estimates (blue) against the numerically computed values of $F(t;1)$ (red).

Second, the analogue of the Ferrari-Spohn formula for G:

Baik-B 2018

We have

$$F(t;1) = \det(1 - S \restriction_{L^2(t,\infty)}) \tag{3}$$

where $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the integral operator with kernel

$$S(x,y) = \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-\frac{1}{4}(x+y)^2}$$

Baik-B 2018

We have

$$F(t;1) = \det(1 - S \restriction_{L^2(t,\infty)}) \tag{3}$$

where $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the integral operator with kernel

$$S(x,y) = \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-\frac{1}{4}(x+y)^2}$$

Next objectives: What is the probabilistic interpretation of $F(t;\gamma)$ for all $\gamma \in [0,1]$? What is the exact value of $\eta_0(\gamma)$? Can we generalize (3) for $F(t;\gamma), \gamma \in [0,1]$?

Some proof ideas

Recall that

$$K_{\mathfrak{e}}(x,y)=\int_0^{\infty}\mathfrak{e}(x+s)\mathfrak{e}(y+s)\,\mathrm{d}s;\qquad \mathfrak{e}(x)=\frac{1}{\sqrt{\pi}}\mathrm{e}^{-x^2} \tag{4}$$

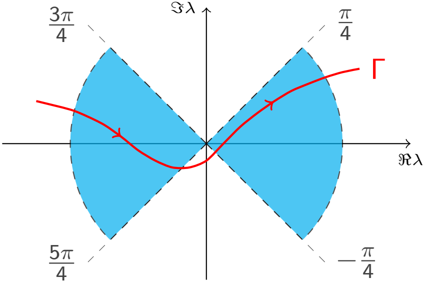
Some proof ideas

Recall that

$$K_{\mathfrak{e}}(x,y) = \int_0^\infty \mathfrak{e}(x+s)\mathfrak{e}(y+s)\,\mathrm{d}s; \qquad \mathfrak{e}(x) = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-x^2} \tag{4}$$

and use

$$\mathrm{e}^{-x^2} = \frac{1}{2\sqrt{\pi}}\int_\Gamma \mathrm{e}^{-\frac{1}{4}\lambda^2 \pm \mathrm{i}x\lambda}\,\mathrm{d}\lambda, \quad x \in \mathbb{R}.$$



After substitution into (4),

$$K_{\mathfrak{e}}(x,y)=\frac{1}{(2\pi)^2}\int_{\Gamma_\lambda}\int_{\Gamma_w}\mathrm{e}^{-\frac{1}{4}(\lambda^2+w^2)}\mathrm{e}^{-\mathrm{i}(x\lambda-yw)}\left[\int_0^\infty \mathrm{e}^{-\mathrm{i}u(\lambda-w)}\,\mathrm{d}u\right]\mathrm{d}w\,\mathrm{d}\lambda$$

we choose $(\lambda,w)\in\Gamma_\lambda\times\Gamma_w$ such that $\Im w>\Im\lambda$ and obtain

$$K_{\mathfrak{e}}(x,y)=\frac{1}{(2\pi)^2}\int_{\Gamma_\lambda}\int_{\Gamma_w}\frac{\mathrm{e}^{-\frac{1}{4}(\lambda^2+w^2)-\mathrm{i}(x\lambda-yw)}}{\mathrm{i}(\lambda-w)}\,\mathrm{d}w\,\mathrm{d}\lambda.$$

After substitution into (4),

$$K_{\mathfrak{e}}(x,y)=\frac{1}{(2\pi)^2}\int_{\Gamma_\lambda}\int_{\Gamma_w}\mathrm{e}^{-\frac{1}{4}(\lambda^2+w^2)}\mathrm{e}^{-\mathrm{i}(x\lambda-yw)}\left[\int_0^\infty\mathrm{e}^{-\mathrm{i}u(\lambda-w)}\,\mathrm{d}u\right]\mathrm{d}w\,\mathrm{d}\lambda$$

we choose $(\lambda,w)\in\Gamma_\lambda\times\Gamma_w$ such that $\Im w>\Im\lambda$ and obtain

$$K_{\mathfrak{e}}(x,y)=\frac{1}{(2\pi)^2}\int_{\Gamma_\lambda}\int_{\Gamma_w}\frac{\mathrm{e}^{-\frac{1}{4}(\lambda^2+w^2)-\mathrm{i}(x\lambda-yw)}}{\mathrm{i}(\lambda-w)}\,\mathrm{d}w\,\mathrm{d}\lambda.$$

Use residue theorem

Suppose $w\in\Gamma_w$ satisfies $\Im w>0$. Then for any $y,t\in\mathbb{R}:y\neq t$,

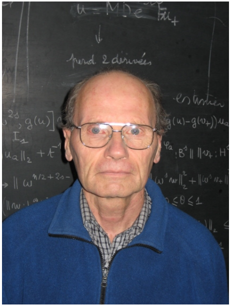
$$\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\mu(y-t)}\frac{\mathrm{d}\mu}{\mu-w}=\mathrm{e}^{\mathrm{i}w(y-t)}\chi_{(t,\infty)}(y).$$

Now combine ($\Gamma_\lambda = \mathbb{R}, \Gamma_w \equiv \Gamma$),

$$K_{\mathfrak{e}}(x,y)\chi_{(t,\infty)}(y) = \iint_{\mathbb{R}^2} \frac{\mathrm{e}^{-\mathrm{i}x\lambda}}{\sqrt{2\pi}} \underbrace{\left[\frac{1}{(2\pi)^2}\int_\Gamma \frac{\mathrm{e}^{-\frac{1}{4}(\lambda^2+w^2)-\mathrm{i}t(\mu-w)}}{(\lambda-w)(w-\mu)}\,\mathrm{d}w\right]}_{=:E(\lambda,\mu)} \frac{\mathrm{e}^{\mathrm{i}y\mu}}{\sqrt{2\pi}}\,\mathrm{d}\mu\,\mathrm{d}\lambda$$

Thus, $K_{\mathfrak{e}}\chi_{(t,\infty)}$ on $L^2(\mathbb{R})$ is simply the operator composition $\mathcal{F}E\mathcal{F}^{-1}$. After properly modifying the function spaces, E is trace-class and can be massaged into the aforementioned Christoffel-Darboux structure.

Thank you very much for your attention!!!



References I

L. Arnold, On the asymptotic distribution of the eigenvalues of random matrices, *J. Math. Anal. Appl.* **20** (1967), 262-268.

J. Baik, T. Bothner, The largest real eigenvalue in the real Ginibre ensemble and its relation to the Zakharov-Shabat system, preprint arXiv:1808.02419

R. Beals, R. Coifman, Scattering and inverse scattering for first order systems, *Comm. Pure Appl. Math.* **37** (1984), no.1, 39-90.

A. Borodin, C. Sinclair, The Ginibre ensemble of real random matrices and its scaling limits, *Comm. Math. Phys.* **291** (2009), no. 1, 177-224.

References II

- A. Borodin, M. Poplavskyi, C. Sinclair, R. Tribe, O. Zaboronski, Erratum to: The Ginibre ensemble of real random matrices and its scaling limits, *Commun. Math. Phys.* **346** (2016), 1051-1055.
- F. Dyson, A note no correlations between eigenvalues of a random matrix, *Commun. Math. Phys.* **19** (1970), 235-250.
- A. Edelman, The probability that a random real Gaussian matrix has k real eigenvalues, related distributions, and the circular law, *J. Multivariate Anal.* **60** (1997), no. 2, 203-232.
- P. Ferrari, H. Spohn, A determinantal formula for the GOE Tracy-Widom distribution, *J. Phys. A: Math. Gen.* **38** (2005), L557.

References III

- P. Forrester, T. Nagao, G. Honner, Correlations for the orthogonal-unitary and symplectic-unitary transitions at the hard and soft edges, *Nuclear Physics* B 553 (1999), 601-643.
- P. Forrester, Hard and soft edge spacing distributions for random matrix ensembles with orthogonal and symplectic symmetry, *Nonlinearity* **19** (2006), 2989-3002.
- J. Ginibre, Statistical ensembles of complex, quaternion, and real matrices, *Journal of Math. Phys.* **6**, 440 (1965).
- P. Hsu, On the distribution of the roots of certain determinantal equations, *Ann. Eugen.* **9** (1939), 250-258.

References IV

- N. Lehmann, H-J. Sommers, Eigenvalue statistics of random real matrices, *Physical Review Letters* **67**, no. 8 (1991).
- M. Mehta, On the statistical properties of the level-spacings in nuclear spectra, *Nuclear Physics* **18** (1960), 395-419.
- M. Poplavskyi, R. Tribe, O. Zaboronski, On the distribution of the largest real eigenvalue for the real Ginibre ensemble, *Ann. Appl. Probab.* **27** (2017), no. 3, 1395-1413.
- B. Rider, C. Sinclair, Extremal laws for the real Ginibre ensemble, *Ann. Appl. Probab.* **24** (2014), no. 4, 1621-1651.
- A. Soshnikov, Universality at the edge of the spectrum in Wigner random matrices, *Commun. Math. Phys.* **207** (1999), 697-733.

References V

- T. Tao, V. Vu, Random matrices: universality of ESDs and the circular law, *Ann. Probab.* **38** (2010), no. 5, 2023-2065.
- C. Tracy, H. Widom, On orthogonal and symplectic matrix ensembles, *Commun. Math. Phys.* **177** (1996), 727-754.
- C. Tracy, H. Widom, Matrix kernels for the Gaussian orthogonal and symplectic ensembles, *Ann. Inst. Fourier (Grenoble)* **55** (2005), no. 6, 2197-2207.
- E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, *Ann. of Math.* (2) **62** (1955), 548-564.

References VI

V. Zakharov, A. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Soviet Physics JETP*, **34** (1972), 62-69.