Spherical Sherrington-Kirkpatrick model and random matrix

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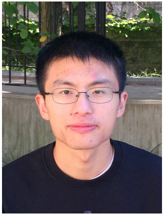

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The largest eigenvalue of a symmetric matrix satisfies

$$\lambda_1 = \max_{\|x\|=1} \langle x, Mx \rangle$$

Consider its finite temperature version

$$\frac{1}{\beta} \log \left[\int_{\|x\|=1} e^{\beta \langle x, Mx \rangle} \, d\Omega(x) \right], \qquad \beta = \frac{1}{T}$$

This is known as the free energy of the **Spherical Sherrington-Kirkpatrick (SSK)** model

In general, a **spherical spin glass** model is defined by a random symmetric polynomial $H(\sigma)$. Its **random** Gibbs measure is defined by

$$p(\sigma) = \frac{1}{Z_N} e^{\beta H(\sigma)} \quad \text{for } \sigma \in \mathbb{R}^N \text{ with } \|\sigma\| = \sqrt{N}$$

and its free energy by

$$F_N = \frac{1}{N\beta} \log Z_N = \frac{1}{N\beta} \log \left[\int_{\|\sigma\|=\sqrt{N}} e^{\beta H(\sigma)} \mathrm{d}\Omega(\sigma) \right]$$

One may consider models in other manifold or graph. The case with $\{-1,1\}^N$ is especially important and this is the usual spin glass model.

Goal

We consider the spherical spin glass model as $N \to \infty$ for

$$(1)\ \text{SSK} \qquad H(\sigma) = \frac{1}{2}\sigma^T M\sigma$$

$$(2)\ \text{SSK} + \text{CW} \qquad H(\sigma) = \frac{1}{2}\sigma^T M\sigma + \frac{m}{2N}\sigma^T\sigma$$

$$(3)\ \text{SSK} + \text{external field} \qquad H(\sigma) = \frac{1}{2}\sigma^T M\sigma + h\sigma^T g$$

We use random matrix theory to study **fluctuations** of the free energy and the spin distribution. Assume that the semicircle law has support $[-2, 2]$. For (1), RMT tells us that

$$F_N \stackrel{\mathcal{D}}{\simeq} 1 + \frac{\mathrm{TW}_1}{2N^{2/3}} \quad \text{at } T = 0$$

For $T > 0$? For (2) and (3)?

Outline

(1) **SSK model** $H(\sigma) = \frac{1}{2}\sigma^T M\sigma$

- (i) Fluctuation results
- (ii) History
- (iii) Random single integral formula
- (iv) Linear statistics vs largest eigenvalue

(2) SSK+CW

(3) SSK+external field

Fluctuations of free energy

Theorem [Baik and Lee 2016]

- For $T < 1$,

$$F_N \overset{\mathcal{D}}{\simeq} \left(1 - \frac{3T}{4} + \frac{T \log T}{2}\right) + \frac{1-T}{2N^{2/3}} \mathrm{TW}_1$$

- For $T > 1$,

$$F_N \overset{\mathcal{D}}{\simeq} \frac{1}{4T} + \frac{T}{2N} \mathcal{N}(-\alpha, 4\alpha)$$

  where $\alpha = -\frac{1}{2}\log(1 - T^{-2})$

$T = 1$ is an open problem

History: limiting free energy

- SSK: Kosterlitz, Thouless, Jones (1976), Guionnet and Maïda (2005), Panchenko and Talagrand (2007)
- General spin glass: **Parisi formula** (1980), Crisanti and Sommers formula (1992)
- Guerra (2003), Talagrand (2006), Panchenko (2014)

- SK for **high temperature**, $T > 1$: Gaussian, N^{-1} [Aizenmann, Lebowitz, Ruelle 1987], [Fröhlich and Zegarliński 1987], [Comets and Neveu 1995]
- pure p-spin spin glass **high temperature**: Gaussian, $N^{-p/2}$ [Bovier, Kurkova, and Löwe 2002]
- pure p-spin Spherical spin glass with $p \geq 3$ **zero temperature**, $T = 0$: Gumbel N^{-1} [Subag and Zeitouni 2017]
- (spherical) spin glass with external field for **all temperatures**, $T > 0$: Gaussian $N^{-1/2}$ [Chen, Dey, Parchenko 2017] No phase transition!

Random single integral formula

Lemma [Kosterlitz, Thouless, Jones 1976]

$$Z_N = C_N \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{N}{2}G(z)} dz, \quad G(z) = \beta z - \frac{1}{N}\sum_{k=1}^{N} \log(z-\lambda_k) \quad \text{with } \gamma > \lambda_1$$

Proof: By definition, $Z_N = \int_{\|\sigma\|=\sqrt{N}} e^{\beta\sigma^T M\sigma} \mathrm{d}\Omega(\sigma) = \int_{\|u\|=\sqrt{N}} e^{\beta\sum_i \lambda_i u_i^2} \mathrm{d}\Omega(u)$

- Let $f(r) = r^{N/2-1} \int_{\|u\|=1} e^{r\sum_i \lambda_i u_i^2} \mathrm{d}\omega(u)$
- **Laplace transform** $L(z) = \int_0^\infty e^{-zr} f(r)\mathrm{d}r = \int_{\mathbb{R}^N} e^{-z\sum y_i^2 + \sum \lambda_i y_i^2} \mathrm{d}^N y$
- By Gaussian integral, $L(z) = \prod_{i=1}^N \sqrt{\frac{\pi}{z-\lambda_i}}$
- **Inverse Laplace transform** $f(r) = \frac{1}{2\pi \mathrm{i}} \int e^{rz} L(z)\mathrm{d}z$

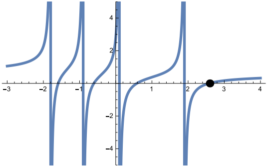
Does the method of steepest-descent apply to random integrals? Yes, thanks to

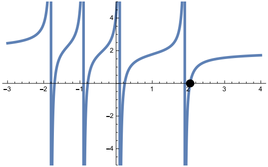
Rigidity of eigenvalues [Erdös, Yau and Yin (2012)]

$$|\lambda_k - \gamma_k| \le \hat{k}^{-1/3} N^{-2/3+\epsilon} \quad \text{uniformly for } 1 \le k \le N \text{ with high probability}$$

where $\hat{k} = \min\{k, N+1-k\}$ and γ_k is the classical location (i.e. quantile of the semicircle law), $\int_{\gamma_k}^{2} \frac{\sqrt{4-x^2}}{2\pi} \mathrm{d}x = \frac{k}{N}$

$$G'(z) = \beta - \frac{1}{N}\sum_{k=1}^{N} \frac{1}{z-\lambda_k}, \qquad \mathrm{Re}(z) > \lambda_1$$





- For $\beta < 1$, $\beta - \int \frac{\mathrm{d}\sigma_{\mathrm{sc}}(x)}{z_c - x} \simeq 0$ implies that $z_c \simeq \beta + \frac{1}{\beta}$
- For $\beta > 1$, $z_c = \lambda_1 + O(N^{-1+\epsilon})$ with high probability

High temperature regime $\beta < 1$

We have, with $z_c = \beta + \frac{1}{\beta}$,

$$G(z_c) = \beta z_c - \frac{1}{N}\sum_{k=1}^{N} \log(z_c - \lambda_k)$$

For $T > 1$, a linear statistic gives fluctuations:

$$F_N = \frac{1}{4T} + \frac{T}{2N}\left(\log(1 - T^{-2}) - L_N\right) + O(N^{-2+\epsilon})$$

with high probability where

$$L_N = \sum_{i=1}^{N} g(\lambda_i) - N\int_{-2}^{2} g(x)\mathrm{d}\sigma_{sc}(x)$$

with $g(x) = \frac{1}{2}\log\left(T + T^{-1} - x\right)$

- The critical point is close to a branch point
- It still holds that $\log\left[\int e^{\frac{N}{2}G(z)}dz\right] \simeq \frac{N}{2}G(z_c)$
- Using $z_c = \lambda_1 + O(N^{-1+\epsilon})$ and noting $\lambda_1 = 2 + O(N^{-2/3+\epsilon})$

$$\begin{aligned} G(z_c) &= \beta z_c - \frac{1}{N}\sum_{i=2}^{N}\log(z_c - \lambda_i) - \frac{1}{N}\log(z_c - \lambda_1) \\ &\simeq \beta\lambda_1 - \frac{1}{N}\sum_{i=2}^{N}\left[\log(2-\lambda_i) + \frac{1}{2-\lambda_i}(\lambda_1 - 2)\right] \\ &\simeq \beta\lambda_1 - \int_{-2}^{2}\log(2-s)\mathrm{d}\sigma_{sc}(s) - (\lambda_1 - 2)\int_{-2}^{2}\frac{\mathrm{d}\sigma_{sc}(s)}{2-s} \end{aligned}$$

For $T < 1$, the largest eigenvalue gives the fluctuations:

$$F_N = \left(1 - \frac{3T}{4} + \frac{T\log T}{2}\right) + \frac{1-T}{2}(\lambda_1 - 2) + O(N^{-1+\epsilon})$$

with high probability

Outline

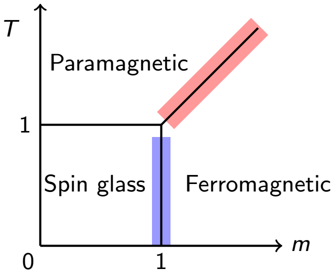
(1) SSK model

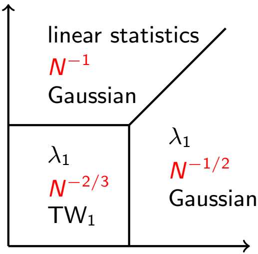
(2) **SSK+CW (Curie-Weiss)**

$$H(\sigma) = \frac{1}{2}\sigma^T M\sigma + \frac{m}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j = \frac{1}{2}\sigma^T\left(M + \frac{m}{N}11^T\right)\sigma$$

(3) SSK+external field

- Random symmetric matrix with non-zero mean (spiked random matrix)
- Limiting free energy was obtained by [Kosterlitz-Thouless-Jones 1976]
- Fluctuations including **Spin–Ferro** ($m = 1 + aN^{-2/3}$) [Baik-Lee 2017]
- **Para–Ferro** $m = T + bN^{-1/2}$ [Baik-Lee-Wu 2018]





(1) SSK model

(2) SSK + CW

(3) **SSK + external field**

   (a) Free energy

   (b) Spin distribution

SSK + external field

- $H(\sigma) = \frac{1}{2}\sigma^T M\sigma + h\sigma^T g$
- $g = (g_1, \cdots, g_N)$ is a standard normal vector
- h is a coupling constant (strength of the external field)
- (Chen, Dey, Panchenko 2017) Gaussian $N^{-1/2}$ **for all** $T > 0$ **if** $h > 0$
- On the other hand, if $h = 0$, there is a transition at $T = 1$
- (Fyodorov and le Doussal 2014) For $T = 0$, the number of local max/min of $H(\sigma)$ has a transition when $h = O(N^{-1/6})$
- Goal: Recover [CDP] result and study the free energy when $h = HN^{-1/6}$

Let u_i be a unit **eigenvector** associated to λ_i. We have the random integral formula with

$$G(z) = \beta z - \frac{1}{N}\sum_{i=1}^{N}\log(z-\lambda_i) + \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{z-\lambda_i}, \qquad n_i = u_i^T g$$

For $h > 0$ and every $\beta > 0$,

$$G'(z) \simeq \beta - \int \frac{\mathrm{d}\sigma_{\mathrm{sc}}(x)}{z-x} - h^2\beta\int\frac{\mathrm{d}\sigma_{\mathrm{sc}}(x)}{(z-x)^2}$$

has the unique root $z_c > 2$ since $G'(2) = -\infty$ and $G'(\infty) = \beta > 0$. Insert this z_c to $G(z)$ and consider

$$\sum_{i=1}^{N}\frac{n_i^2}{z_c-\lambda_i} = \sum_{i=1}^{N}\frac{1}{z_c-\lambda_i} + \sum_{i=1}^{N}\frac{n_i^2-1}{z_c-\lambda_i}$$

The first sum has fluctuations of $O(1)$ from **linear statistics**. The second sum has fluctuations of $O(\sqrt{N})$ by **usual CLT**. We recover [CDP] result.

Conjecture [Baik, le Doussal, Wu 2019] For $T < 1$ and $h = HN^{-1/6}$,

$$F_N \simeq \left(1 - \frac{3T}{4} + \frac{T\log T}{2} + \frac{h^2}{2}\right) + \frac{\mathcal{F}}{N^{2/3}}$$

with high probability

Let $\{\alpha_i\}$ be a GOE Airy point process ($\alpha_i \sim -(3\pi i/2)^{2/3}$ as $i \to \infty$) and let $\{\nu_i\}$ be independent standard normal random variables. Let $s > 0$ be the solution of the equation

$$\frac{1-T}{H^2} = \sum_{i=1}^{\infty} \frac{\nu_i^2}{s + \alpha_1 - \alpha_i}$$

Set (cf. [Landon and Sosoe 2019])

$$\mathcal{E}(s) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{\nu_i^2}{s + \alpha_1 - \alpha_i} - \int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}} \frac{\mathrm{d}x}{\sqrt{x}} \right)$$

Then,

$$\mathcal{F} \stackrel{\mathcal{D}}{=} \frac{(1-T)(s+\alpha_1) + H^2\mathcal{E}(s)}{2}$$

Gibbs moment generating function of $\mathfrak{O}^2$ is

$$\langle e^{\beta\xi N\mathfrak{O}^2}\rangle = \frac{1}{Z_N}\int e^{\beta\xi N\mathfrak{O}^2} e^{\beta H(\sigma)}\mathrm{d}\Omega(\sigma) = \frac{1}{Z_N}\int e^{\xi\sigma^T u_1 u_1^T\sigma+\beta(\frac{1}{2}\sigma^T M\sigma+h\sigma^T g)}\mathrm{d}\Omega(\sigma)$$

Since

$$N\mathfrak{O}^2 = \sigma^T u_1 u_1^T \sigma$$

we have

$$\xi N\mathfrak{O}^2 + H(\sigma) = \sigma^T\left(\frac{1}{2}M + \xi u_1 u_1^T\right)\sigma + h\sigma^T g$$

Thus,

$$\langle e^{\xi\mathfrak{O}^2}\rangle = \frac{Z_N|_{\lambda_1\mapsto\lambda_1+2\xi}}{Z_N}$$

Conjecture [Baik, le Doussal, Wu 2019] For $T < 1$,

$$(\hat{\sigma}^T u_1)^2 \to \begin{cases} 0 & \text{for } h > 0 \\ 1 - T - H^2 \sum_{i=2}^{\infty} \frac{\nu_i^2}{(s+\alpha_1-\alpha_i)^2} & \text{for } h = HN^{-1/6} \\ 1 - T & \text{for } h = 0 \text{ (This is well-known)} \end{cases}$$

Here, for $h = HN^{-1/6}$, $s > 0$ is the solution of $\frac{1-T}{H^2} = \sum_{i=1}^{\infty} \frac{\nu_i^2}{(s+\alpha_1-\alpha_i)^2}$

We can also compute the next order term. For example,

$$1 - T - H^2 \sum_{i=2}^{\infty} \frac{\nu_i^2}{(s+\alpha_1-\alpha_i)^2} + \frac{2H\sqrt{T}}{N^{1/6}} \left[\sum_{i=2}^{N} \frac{n_i^2}{(a_1 - a_i)^3} \right]^{1/2} \mathfrak{N}(0,1)$$

The case of $h = 0$ is related to the work of [Sosoe and Vu 2018, Landon and Sosoe 2019]

Summary

1. Spherical spin glass is defined by random Gibbs measure on a sphere
2. Three Hamiltonians were considered: (1) SSK, (2) SSK + CW, (3) SSK + external field
3. There is a random integral formula (single-variable!) for the partition function to which the method of steepest-descent is applicable using the rigidity of the eigenvaues
4. The fluctuations of the free energy were obtained. There are interesting transitional behaviors.
5. Spin distributions were also studied.

Thank you for attention!