A matrix algebra example: $\mathcal{C}_{11}$.

Prasad and Yeung give the name $\mathcal{C}_{11}$ to the following pair (k, ℓ):

Let $\ell = \mathbb{Q}(\zeta)$, where ζ is a primitive 12-th root of 1.

$\zeta^4 = \zeta^2 - 1$, so $[\ell : \mathbb{Q}] = 4$.

Let $k = \mathbb{Q}(r)$ for $r = \zeta + \zeta^{-1}$.

Then $r^2 = 3$ and $(\zeta^3)^2 = -1$. So $k = \mathbb{Q}(\sqrt{3})$ and $\ell = \mathbb{Q}(\sqrt{3}, i)$.

Let

$$F = \begin{pmatrix} -r-1 & 1 & 0 \\ 1 & 1-r & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Form the algebraic group G for which

$$G(k) = \{g \in M_{3\times 3}(\ell) : g^* F g = F \text{ and } \det(g) = 1\}.$$

So we are working with the involution

$$\iota(x) = F^{-1} x^* F,$$

as $\iota(x)x = 1$ iff $x^* F x = F$.

Two embeddings $k \hookrightarrow \mathbb{R}$, mapping r to $+\sqrt{3}$ and $-\sqrt{3}$, respectively.

For $r = +\sqrt{3}$, set

$$\Delta = \begin{pmatrix} r+1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{r+1} \end{pmatrix}.$$

Then $\Delta^* F_0 \Delta = -(r+1)F$, and so $g^* F g = F$ if and only if $\tilde{g} = \Delta g \Delta^{-1}$ satisfies $\tilde{g}^* F_0 \tilde{g} = F_0$.

So $g \mapsto \tilde{g}$ gives an isomorphism $G(k_v) \cong SU(2,1)$ for the archimedean place v of k corresponding to the first embedding.

Now let $r = -\sqrt{3}$ and

$$\Delta = \begin{pmatrix} r-1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{-r-1} \end{pmatrix}.$$

Then $\Delta^*\Delta = -(r+1)F$, and so $g^*Fg = F$ if and only if $\tilde{g} = \Delta g \Delta^{-1}$ satisfies $\tilde{g}^*\tilde{g} = I$.

So $g \mapsto \tilde{g}$ gives an isomorphism $G(k_v) \cong SU(3)$ for the archimedean place v of k corresponding to the second embedding.

In

$$3^{\alpha-1} d_{k,\ell} = [\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T}} e'(P_v), \qquad (*)$$

$\alpha = 1$ and $\mathcal{T}_0 = \emptyset$ (we're in a matrix algebra case), and $d_{k,\ell} = 864$. So

$$864 = [\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T}} e'(P_v).$$

If $v \in \mathcal{T}$, then

(a) $q_v^2 + q_v + 1$ divides $e'(P_v)$ if v splits in ℓ,

(b) $q_v^2 - q_v + 1$ divides $e'(P_v)$ if v does not split in ℓ.

If $q \geq 2$, then $q^2 + q + 1$ never divides 864 and $q^2 - q + 1$ divides 864 only for $q = 2$.

2 ramifies in k, as $2\mathfrak{o}_k = \mathfrak{p}^2$ for $\mathfrak{p} = (r+1)\mathfrak{o}_k$.

So there is only one 2-adic valuation v on k, and $q_v = 2$.

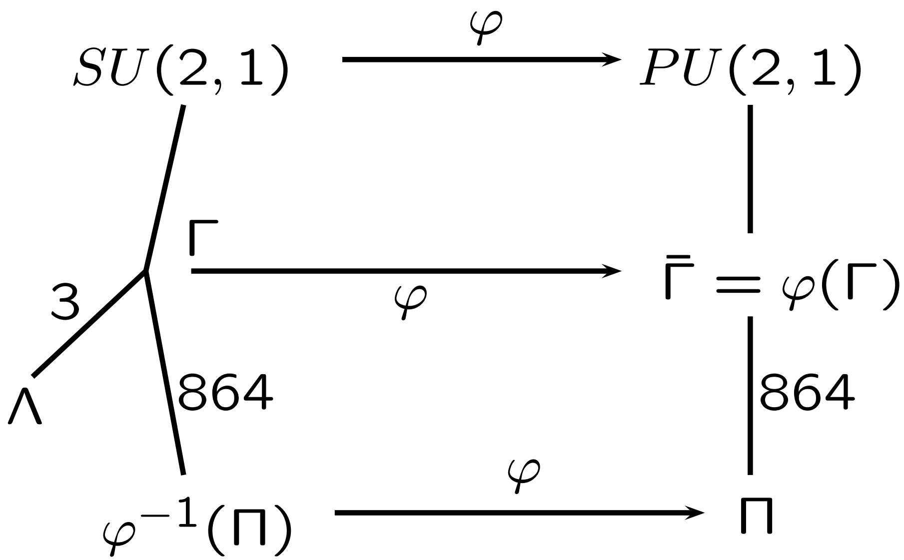
So $\mathcal{T} = \emptyset$ or $\mathcal{T} = \{v\}$ for this 2-adic v.

No $v \in V_f$ ramifies in ℓ.

We consider in detail the case $\mathcal{T} = \emptyset$. So equation $(*)$ is

$$864 = [\bar{\Gamma} : \Pi].$$

An earlier diagram is in this case



Here Π is the fundamental group of a hypothetical fake projective plane.

Our strategy is to

- concretely realize $\bar{\Gamma}$ and find a presentation for this group, then

- look for a subgroup Π of index 864 which is torsion-free and has finite abelianization.

It will turn out that there is, up to conjugation, just one torsion-free subgroup of index 864 in $\bar{\Gamma}$, but its abelianization is $\mathbb{Z}^2$. So the ball quotient

$$\Pi \backslash B(\mathbb{C}^2)$$

is NOT a fake projective plane.

When $v \in V_f$ splits in ℓ, $G(k_v) \cong SL(3, k_v)$, and we can choose as our parahoric subgroup

$$P_v = SL(3, \mathfrak{o}_v).$$

When v does not split in $\ell = k(s)$,

$$G(k_v) = \{g \in M_{3\times 3}(k_v(s)) : g^* F_v g = F_v \text{ and } \det(g) = 1\},$$

and we can choose

$$P_v = \{g \in M_{3\times 3}(\mathfrak{o}_{\tilde{v}}) : g^* F_v g = F_v \text{ and } \det(g) = 1\},$$

where $\tilde{v}$ is the unique extension of v to ℓ. Let

$$\Lambda = \{g \in G(k) \ : \ g_v \in P_v \text{ for all } v \in V_f\}.$$

Here g_v is the image of $g \in G(k)$ in $G(k_v)$.

If $g \in G(k)$ then g is a matrix with entries $a_{\alpha\beta} + ib_{\alpha\beta}$.

When v splits, then $g \in P_v$ iff $w(a_{\alpha\beta} + ib_{\alpha\beta}) \geq 0$ for both extensions w of v to ℓ and all α, β.

When v doesn't split, then $g \in P_v$ iff $\tilde{v}(a_{\alpha\beta} + ib_{\alpha\beta}) \geq 0$ for the unique extension $\tilde{v}$ of v to ℓ and all α, β.

An element t of ℓ is in $\mathfrak{o}_\ell = \mathbb{Z}[\zeta]$ if and only if $w(t) \geq 0$ for all non-archimedean valuations w on ℓ. So

$$\Lambda = \{g \in M_{3\times 3}(\mathbb{Z}[\zeta]) : g^* F g = F \text{ and } \det(g) = 1\}.$$

We also need to find matrices g with entries in $\mathbb{Z}[\zeta]$ so that $g^*Fg = F$, without the condition $\det(g) = 1$.

This will give us the normalizer Γ of Λ in $SU(2,1)$.

If $g \in M_{3\times 3}(\mathbb{Z}[\zeta])$ and $g^*Fg = F$, then $\det(g) \in \mathbb{Z}[\zeta]$ and $|\det(g)|^2 = 1$. So $\det(g) = \zeta^j$ for some $j \in \{0, \ldots, 11\}$.

Mapping ζ to $e^{2\pi i/12}$, the matrix $\zeta^{-j/3}\Delta g\Delta^{-1}$ is then in $SU(2,1)$ and normalizes the image $\{\Delta h\Delta^{-1} : h \in \Lambda\}$ of Λ in $SU(2,1)$. So it is in Γ.

Fact: You get all elements of Γ in this way.

Finding matrices $g \in M_{3\times 3}(\mathbb{Z}[\zeta])$ satisfying $g^*Fg = F$.

We use the action of $U(2,1)$ on $B(\mathbb{C}^2)$.

$$g.(z_1, z_2) = (z_1', z_2') \quad \text{means that } g \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} = c \begin{pmatrix} z_1' \\ z_2' \\ 1 \end{pmatrix} \quad \text{for some } c.$$

This action preserves the hyperbolic metric, which is given by

$$\cosh^2(d(z,w)) = \frac{|1 - \langle z, w\rangle|^2}{(1-|z|^2)(1-|w|^2)},$$

where $\langle z, w\rangle = z_1\bar{w}_1 + z_2\bar{w}_2$ and $|z|^2 = \langle z, z\rangle$.

Comparing $(3,3)$-entries on both sides of $g^* F_0 g = F_0$, we get

$$|g_{13}|^2 + |g_{23}|^2 = |g_{33}|^2 - 1, \qquad (\text{"column 3 condition"})$$

so $|g_{33}| \geq 1$ for any $g \in U(2,1)$.

If $g = (g_{jk}) \in U(2,1)$, then $g.(0,0) = (g_{13}/g_{33}, g_{23}/g_{33})$.
So column 3 condition $\Rightarrow$

$$\cosh^2(d(0, g.0)) = |g_{33}|^2.$$

Writing 0 in place of $(0,0)$.

$$g.0 = 0 \quad \Leftrightarrow \quad |g_{33}| = 1.$$

So $g.0 = 0$ implies that $g_{13} = 0 = g_{23}$. In fact, $g.0 = 0$ iff

$$g = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}.$$

The group of $g \in M_{3\times 3}(\mathbb{Z}[\zeta])$ such that $g^*Fg = F$ acts on $B(\mathbb{C}^2)$: for such g, $\tilde{g} = \Delta g \Delta^{-1}$ is in $U(2,1)$, and we set

$$g.z := \tilde{g}.z \quad \text{for } z \in B(\mathbb{C}^2).$$

The subgroup $\{\zeta^j I : j = 0, \ldots, 11\}$ acts trivially, and

$$\bar{\Gamma} \cong \{g \in M_{3\times 3}(\mathbb{Z}[\zeta]) : g^*Fg = F\}/\{\zeta^j I : j = 0, \ldots, 11\}.$$

Note that $\tilde{g}_{33} = g_{33}$, and so $\cosh^2(d(0, g.0)) = |g_{33}|^2$ is still valid.

The column 3 condition for g satisfying $g^*Fg = F$ is

$$|g_{13}|^2 + |g_{13} - (r-1)g_{23}|^2 = (r-1)(|g_{33}|^2 - 1).$$

So $g.0 = 0 \Leftrightarrow |g_{33}| = 1 \Leftrightarrow g_{13} = g_{23} = 0$. Again, $g.0 = 0$ iff

$$g = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}.$$

Since $g_{33} \in \mathbb{Z}[\zeta]$ and $|g_{33}|^2 = 1$, we have $g_{33} = \zeta^j$ for some j.

Routine calculations show that

$$u = \begin{pmatrix} \zeta^3+\zeta^2-\zeta & 1-\zeta & 0 \\ \zeta^3+\zeta^2-1 & \zeta-\zeta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } v = \begin{pmatrix} \zeta^3 & 0 & 0 \\ \zeta^3+\zeta^2-\zeta-1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which have entries in $\mathbb{Z}[\zeta]$, satisfy $u^*Fu = F = v^*Fv$ and

$$u^3 = I,\ v^4 = I,\ \text{and } (uv)^2 = (vu)^2.$$

They generate a group K of order 288 with this presentation.

Lemma. For the action of $\bar{\Gamma}$ on $B(\mathbb{C}^2)$, K is the stabilizer of the origin.

We next find a presentation for $\bar{\Gamma}$. Let

$$b = \begin{pmatrix} 1 & 0 & 0 \\ -2\zeta^3 - \zeta^2 + 2\zeta + 2 & \zeta^3 + \zeta^2 - \zeta - 1 & -\zeta^3 - \zeta^2 \\ \zeta^2 + \zeta & -\zeta^3 - 1 & -\zeta^3 + \zeta + 1 \end{pmatrix}.$$

This satisfies $b^* F b = F$ and $\det(b) = \zeta^4$.

Theorem. The elements u, v and b generate $\bar{\Gamma}$, and the relations

$$u^3 = v^4 = b^3 = 1,\ (uv)^2 = (vu)^2,\ vb = bv,\ (buv)^3 = (buvu)^2 v = 1.$$

give a presentation of $\bar{\Gamma}$.

Note that $(buv)^3 = (buvu)^2 v = \zeta^{-1} I$ as matrices, but we are working modulo $\{\zeta^j I : j = 0, \ldots, 11\}$.

Finding b, and showing that u, v and b generate $\bar{\Gamma}$.

For $g \in U(2,1)$,

$$g^* F_0 g = F_0 \;\Leftrightarrow\; F_0^{-1} g^* F_0 g = I \;\Leftrightarrow\; g F_0^{-1} g^* F_0 = I \;\Leftrightarrow\; g F_0^{-1} g^* = F_0^{-1}.$$

Also, $F_0^{-1} = F_0$. Comparing $(1,1)$ entries in $g F_0 g^* = F_0$, we get

$$|g_{11}|^2 + |g_{12}|^2 = |g_{13}|^2 + 1. \qquad \text{(“row 1 condition”)}$$

Lemma. If 5 complex numbers g_{11}, g_{12}, g_{13}, g_{23} and g_{33} are given satisfying the above column 3 and row 1 conditions, and if a $\theta \in \mathbb{C}$ is given with $|\theta| = 1$, there is a unique $g \in U(2,1)$ with the given 5 entries and with $\det(g) = \theta$.

Analogously, if $g \in M_{3\times 3}(\mathbb{Z}[\zeta])$ and $g^*Fg = F$, then

$$|g_{13}|^2 + |g_{13} - (r-1)g_{23}|^2 = (r-1)(|g_{33}|^2 - 1)$$

and

$$|g_{11}|^2 + |g_{11} + (r+1)g_{12}|^2 = (r+1)|g_{13}|^2 + 2.$$

Lemma. If 5 numbers g_{11}, g_{12}, g_{13}, g_{23} and g_{33} are given in $\ell = \mathbb{Q}(\zeta)$ satisfying the above modified column 3 and row 1 conditions, and if $\theta = \zeta^j$ is given, there is a unique $g \in M_{3\times 3}(\ell)$ with the given 5 entries and with $\det(g) = \theta$.

If the given 5 numbers g_{ij} are all in $\mathbb{Z}[\zeta]$, the numbers g_{21}, g_{22}, g_{31} and g_{32} are in $\ell = \mathbb{Q}(\zeta)$, but might not be in $\mathbb{Z}[\zeta]$.

Lemma. If $\alpha = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 \in \mathbb{Z}[\zeta]$, then

$$|\alpha|^2 = P(\alpha) + Q(\alpha)r,$$

where $P(\alpha)$ and $Q(\alpha)$ are integers, and $P(\alpha)$ is a positive definite quadratic form in $a_0, \ldots, a_3$, and $|Q(\alpha)| \le \frac{1}{r}P(\alpha)$.

In fact, $P(\alpha) \ge \frac{1}{2}\sum_j a_j^2$.

Proof. If $\alpha \in \mathfrak{o}_\ell$, then $|\alpha|^2 = \bar{\alpha}\alpha \in \mathfrak{o}_k = \{p + qr : p, q \in \mathbb{Z}\}$. The automorphism ψ of ℓ mapping ζ to ζ^5 maps r to $-r$ and commutes with conjugation. Apply ψ to both sides of $|\alpha|^2 = P(\alpha) + Q(\alpha)r$, and we get $|\psi(\alpha)|^2 = P(\alpha) - Q(\alpha)r$. Hence

$$P(\alpha) = \frac{1}{2}\Big(|\alpha|^2 + |\psi(\alpha)|^2\Big) \quad \text{and} \quad Q(\alpha) = \frac{1}{2r}\Big(|\alpha|^2 - |\psi(\alpha)|^2\Big) \le \frac{1}{r}P(\alpha).$$

The above column 3 condition for $g \in M_{3\times 3}(\mathbb{Z}[\zeta])$ with $g^*Fg = F$ has the form $|\alpha|^2 + |\beta|^2 = (r-1)(|\gamma|^2 - 1)$, with $\alpha, \beta, \gamma \in \mathbb{Z}[\zeta]$. Write $|\alpha|^2 = P(\alpha) + Q(\alpha)r$ and similarly for β and γ. Equating coefficients of r we get

$$P(\alpha) + P(\beta) + P(\gamma) = 3Q(\gamma) + 1,$$
$$Q(\alpha) + Q(\beta) + Q(\gamma) + 1 = P(\gamma).$$

So

$$P(\gamma) \leq \frac{1}{r}\Big(P(\alpha) + P(\beta) + P(\gamma)\Big) + 1 = \frac{1}{r}\Big(3Q(\gamma) + 1\Big) + 1$$

So

$$Q(\gamma) \leq \frac{1}{r}P(\gamma) \leq Q(\gamma) + \frac{r+1}{3}.$$

Now $d(0, g.0) \leq B$ implies that $|g_{33}|^2 \leq \cosh^2(B)$ and so

$$2P(g_{33}) \leq P(g_{33}) + rQ(g_{33}) + (r+1)/r = |g_{33}|^2 + (r+1)/r < \cosh^2(B) + 2.$$

So if $d(0, g.0) \leq B$ and if $g_{33} = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3$, then

$$\sum_j a_j^2 \leq 2P(g_{33}) < \cosh^2(B) + 2.$$

So, for moderate B, we can very quickly list the set of 5-tuples g_{11}, g_{12}, g_{13}, g_{23} and g_{33} in $\mathbb{Z}[\zeta]$ satisfying the column 3 and row 1 conditions, and also $|g_{33}|^2 \leq \cosh^2(B)$, and then for each of the 12 possible determinants $\theta = \zeta^j$ check whether the uniquely determined g satisfying $\det(g) = \theta$ and $g^*Fg = F$ has entries in $\mathbb{Z}[\zeta]$.

Let

$$d_0 = 0 < d_1 < d_2 < \cdots$$

be the distinct values taken by $d(0, g.0)$, $g \in \bar{\Gamma}$. So $\cosh^2(d_n) = p_n + q_n r$ for certain integers p_n and q_n. The first few $p_n + q_n r$'s are:

$$1,\ 2 + r,\ 4 + 2r,\ 6 + 3r,\ 7 + 4r,\ 11 + 6r,\ \ldots$$

The $(3, 3)$-entry of the matrix b is $-\zeta^3+\zeta+1$. We find that $|b_{33}|^2 = 2+r$. So $d(0, b.0) = d_1 \leq d(0, g.0)$ for all $g \in \bar{\Gamma} \setminus K$.

The set of $g \in \bar{\Gamma}$ such that $d(0, g.0) = d_1$ is the double coset KbK.

The set of $g \in \bar{\Gamma}$ such that $d(0, g.0) = d_2$ is the double coset $Kbu^{-1}bK$.

For the first few n, we can form

$$S_n = \{g \in \bar{\bar{\Gamma}} : d(0, g.0) \leq d_n\}.$$

Then

$$K = S_0 \subset S_1 \subset S_2 \subset S_3 \subset \cdots, \quad \text{and} \quad \bigcup_n S_n = \bar{\bar{\Gamma}}.$$

Now form

$$\mathcal{F}_n = \{z \in B(\mathbb{C}^2) : d(0, z) \leq d(g.0, z) \text{ for all } g \in S_n\}.$$

These satisfy

$$B(\mathbb{C}^2) = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \quad \text{and} \quad \bigcap_n \mathcal{F}_n = \mathcal{F}_{\bar{\bar{\Gamma}}}.$$

Let

$$r_n = \max\{d(0, z) : z \in \mathcal{F}_n\} \quad \text{and} \quad r_{\bar{\bar{\Gamma}}} = \max\{d(0, z) : z \in \mathcal{F}_{\bar{\bar{\Gamma}}}\}.$$

So

$$\infty = r_0 \geq r_1 \geq r_2 \geq \cdots$$

Lemma. If $d_n \geq r_n$, then S_n generates $\bar{\Gamma}$.

Proof. Suppose that $\langle S_n \rangle \subsetneqq \bar{\Gamma}$. Choose $h \in \bar{\Gamma} \setminus \langle S_n \rangle$ with $d(0, h.0)$ minimal. If $g \in S_n$, then $g^{-1}h \notin \langle S_n \rangle$, and so

$$d(0, h.0) \leq d(0, (g^{-1}h).0) = d(g.0, h.0) \quad \text{for all } g \in S_n.$$

Hence $h.0 \in \mathcal{F}_n$. But then $d(0, h.0) \leq r_n$, and by hypothesis $r_n \leq d_n$. Hence $h \in \{g \in \bar{\Gamma} : d(0, g.0) \leq d_n\} = S_n$, a contradiction.

Lemma. If $d_n \geq 2r_n$, then

(a) $\mathcal{F}_n = \mathcal{F}_{\bar{\Gamma}}$ and $r_n = r_{\bar{\Gamma}}$.

(b) the set S_n of generators, together with the relations $g_1 g_2 g_3 = 1$ which hold for $g_1, g_2, g_3 \in S_n$, form a presentation for $\bar{\Gamma}$.

Proof of (a): Suppose that $z \in \mathcal{F}_n \setminus \mathcal{F}_{\bar{\Gamma}}$. As $z \notin \mathcal{F}_{\bar{\Gamma}}$, there must exist a $g \in \bar{\Gamma}$ such that $d(g.0, z) < d(0, z)$. But using $d(0, z) \leq r_n$, we have

$$d(0, g.0) \leq d(0, z) + d(z, g.0) < 2d(0, z) \leq 2r_n \leq d_n,$$

so that $g \in S_n$. But then $d(g.0, z) < d(0, z)$ contradicts $z \in \mathcal{F}_n$.

(b) follows from a general result about group actions on topological spaces due to MacBeath. (see Theorem I.8.10 in Bridson & Häfliger's book).

Calculation. For this $(\mathcal{C}_{11}, \emptyset)$ case,

$$r_1 = r_2 = \cdots = \frac{1}{2} d_2 = \frac{1}{2} \cosh^{-1}(1 + \sqrt{3}),$$

so that we take $n = 2$ in the two lemmas.

So the set $S_2 = K \cup KbK \cup Kbu^{-1}bK$ generates $\bar{\Gamma}$, and the relations $g_1 g_2 g_3 = 1$, where the g_i's are in S_2, give a presentation for $\bar{\Gamma}$.

So u, v and b generate $\bar{\Gamma}$. The relations listed in the above theorem are these relations $g_1 g_2 g_3 = 1$, cleaned-up.

So: does there exist a $\Pi \leq \bar{\Gamma}$ with

$$\Pi \text{ torsion-free}, \quad [\bar{\Gamma} : \Pi] = 864 \quad \text{and} \quad \Pi/[\Pi,\Pi] \text{ finite?}$$

Magma's `LowIndexSubgroups`($\bar{\Gamma}$, 864) does not work—864 is too big.

We wrote a specialized `C` program to answer this.

Theorem. There is, up to conjugacy, just one torsion-free subgroup Π of index 864 in $\bar{\Gamma}$. It satisfies $\Pi/[\Pi,\Pi] \cong \mathbb{Z}^2$.

Corollary. There are no fake projective planes belonging to this class. However, $B(\mathbb{C}^2)/\Pi$ is a new compact surface with Euler characteristic 3.

The main idea behind the C program is this: Let $\Pi \leq \bar{\Gamma}$ be torsion-free, with $[\bar{\Gamma} : \Pi] = 864$.

Lemma. Consider the action of $\bar{\Gamma}$ acts on the coset space $\bar{\Gamma}/\Pi$. If $1 \neq g \in \bar{\Gamma}$ has finite order, then g's action has no fixed points.

Proof. If $g(h\Pi) = h\Pi$, then $h^{-1}gh \in \Pi$.

Lemma. Suppose that T is a finite set of size n, and that $\varphi : \bar{\Gamma} \to \text{Perm}(T)$ is a group homomorphism so that

- $(g, t) \mapsto \varphi(g)(t)$ gives a transitive action of $\bar{\Gamma}$ on T,
- for each $g \in \bar{\Gamma} \setminus \{1\}$ of finite order, the permutation $\varphi(g)$ has no fixed points.

Then for any $t_0 \in T$, $\Pi = \{g \in \bar{\Gamma} : \varphi(g)(t_0) = t_0\}$ is a torsion-free subgroup of $\bar{\Gamma}$ of index n.

Proof. If $T = \{t_0, \ldots, t_{n-1}\}$, for each i pick $g_i \in \bar{\Gamma}$ so that $\varphi(g_i)(t_0) = t_i$. Then $\bar{\Gamma} = \cup_i g_i \Pi$.

If Π exists, it has a transversal of the form $T = Kt_0 \cup Kt_1 \cup Kt_2$. We want to define $\varphi : \bar{\Gamma} \to \mathrm{Perm}(T)$, i.e., an action of $\bar{\Gamma}$ on T.

We may assume that the action of each $k \in K$ is : $k.(k' t_i) = (kk')t_i$.

In particular, the action of u and v gives known permutations U and V of T, and these satisfy $U^3 = V^4 = id$ and $(UV)^2 = (VU)^2$.

The action of the generator b gives a permutation B of T with no fixed points and satisfying

$$B^3 = id,\ BV = VB,\ (BUV)^3 = id \text{ and } (BUVU)^2 V = id.$$

A back-track search was run to find all possible B's. These were found, and corresponding Π's formed. Magma checked they were all conjugate, and that $\Pi/[\Pi,\Pi] \cong \mathbb{Z}^2$.

Theorem. Writing $j = (uv)^2$, the three elements

$$vubju^{-1}, \quad u^{-1}j^{-1}bj^2 \quad \text{and} \quad u^2vbuj^{-2}$$

of $\bar{\Gamma}$ generate a torsion-free subgroup Π of index 864, with $\Pi/[\Pi,\Pi] \cong \mathbb{Z}^2$.

We checked that Π is torsion-free as follows:

(1) $g \in \bar{\Gamma}$ of finite order $\Rightarrow \exists\, x \in B(\mathbb{C}^2)$ such that $g.x = x$.

(2) W.l.o.g. $x \in \mathcal{F}_{\bar{\Gamma}}$.

(3) $x \in \mathcal{F}_{\bar{\Gamma}} \;\Rightarrow\; d(0, g.0) \leq d(0, x) + d(x, g.x) + d(g.x, g.0) \leq 2r_{\bar{\Gamma}}$.

(4) $d(0, g.0) \leq 2r_{\bar{\Gamma}} \;\Rightarrow\; g \in K \,\cup\, KbK \,\cup\, Kbu^{-1}bK$.

We get a short list $g_1, \ldots, g_n$ of conjugacy class representatives of elements of finite order. Next we pick a transversal $t_1, \ldots, t_{864}$ for Π, e.g., $K \cup Kb \cup Kb^2$.

We need only check that $t_i g_j t_i^{-1} \notin \Pi$ for $i = 1, \ldots, 864$, and $j = 1, \ldots, n$.