EXAMPLE: The case $(k, \ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{-7}))$.

Let $m = \mathbb{Q}(\zeta)$, where ζ is a primitive 7-th root of 1. Then m is a Galois extension of $\mathbb{Q}$, with cyclic Galois group generated by $\chi : \zeta \mapsto \zeta^3$.

$$s^2 = -7 \quad \text{for} \quad s = 1 + 2\zeta + 2\zeta^2 + 2\zeta^4.$$

So m contains $\ell = \mathbb{Q}(s)$, and $\mathrm{Gal}(m/\ell) = \langle \varphi \rangle$, where $\varphi(\zeta) = \zeta^2$. Form

$$\mathcal{D} = \{a + b\sigma + c\sigma^2 \ : \ a, b, c \in m\},$$

where $\sigma a \sigma^{-1} = \varphi(a)$ for all $a \in m$, and where

$$\sigma^3 = D = \frac{3 + s}{4}.$$

Note that $\bar{D}D = 1$.

We have seen that $\mathcal{D}$ is a division algebra. The proof uses the 2-adic valuation on $\mathbb{Q}$, and in fact shows that $\mathcal{D} \otimes_\ell \mathbb{Q}_2$ is a division algebra.

Recall: algebra homomorphism $\Psi : \mathcal{D} \to M_{3\times 3}(m)$ so that

$$\Psi(a + b\sigma + c\sigma^2) = \begin{pmatrix} a & b & c \\ D\varphi(c) & \varphi(a) & \varphi(b) \\ D\varphi^2(b) & D\varphi^2(c) & \varphi^2(a) \end{pmatrix}.$$

Recall: involution ι_0 of second kind on $\mathcal{D}$ so that $\iota_0(\sigma) = \sigma^{-1}$ and $\iota(a) = \bar{a}$ for $a \in m$. We modify this:

Set $\iota(\xi) = W^{-1}\iota_0(\xi)W$. Then $\Psi(\iota(\xi)) = F^{-1}\Psi(\xi)^* F$ for

$$F = \begin{pmatrix} W & 0 & 0 \\ 0 & \varphi(W) & 0 \\ 0 & 0 & \varphi^2(W) \end{pmatrix},$$

where $W = \zeta + \zeta^{-1}$.

Note $W^3 + W^2 - 2W - 1 = 0$, $\varphi(W) = W^2 - 2$ and $\varphi^2(W) = 1 - W - W^2$.

We form the algebraic group G, with

$$G(\mathbb{Q}) = \{\xi \in \mathcal{D} : \iota(\xi)\xi = 1 \ \& \ \mathrm{Nrd}(\xi) = 1\}.$$

Recall reason for choice of ι: when we embed m in $\mathbb{C}$, mapping ζ to $e^{2\pi i/7}$, the images of W, $\varphi(W)$ and $\varphi^2(W)$ are > 0, < 0 and < 0. So if

$$\Delta = \begin{pmatrix} 0 & 0 & |\varphi^2(W)|^{1/2} \\ 0 & |\varphi(W)|^{1/2} & 0 \\ |W|^{1/2} & 0 & 0 \end{pmatrix},$$

then $\Delta^* F_0 \Delta = -F$.

So $g^* F g = F$ iff $\tilde{g} = \Delta g \Delta^{-1}$ satisfies $\tilde{g}^* F_0 \tilde{g} = F_0$.

So

$$\mathcal{D} \xrightarrow{\Psi} M_{3\times 3}(m) \hookrightarrow M_{3\times 3}(\mathbb{C}) \xrightarrow{\Delta \cdot \Delta^{-1}} M_{3\times 3}(\mathbb{C})$$

maps $G(k)$ in $G(k_v) \cong SU(2,1)$ for the one archimedean valuation v on $k = \mathbb{Q}$.

If $\Pi \subset PU(2,1)$ is the fundamental group of an fpp, commensurable with $\Lambda = \{g \in G(k) \;:\; g_v \in P_v \text{ for all } v \in V_f\}$ for this G. Then

$$3^{\alpha-1} d_{k,\ell} = [\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T}} e'(P_v).$$

and $\alpha = 2$ (since we are in a division algebra case) and $d_{k,\ell} = 21$.

$k = \mathbb{Q}$, so V_f = set of prime numbers. So

$$3^2 \times 7 = [\bar{\Gamma} : \Pi] \prod_{q \in \mathcal{T}} e'(P_q),$$

and $\mathcal{T}$ is a finite set of primes. If $q \in \mathcal{T}_0$, then $e'(P_q) = (q-1)^2(q+1)$. So q must be 2, and $e'(P_q) = 3$. So

$$3 \times 7 = [\bar{\Gamma} : \Pi] \prod_{q \in \mathcal{T},\, q \neq 2} e'(P_q).$$

If $2 \neq q \in \mathcal{T}$ splits in ℓ then $q^2 + q + 1$ divides $e'(P_q)$ and so divides 21. This can't happen.

If $7 \neq q \in \mathcal{T}$ does not split in ℓ, then $q^2 - q + 1$ divides $e'(P_q)$ and so $q^2 - q + 1$ divides 21. So $q = 3$ or $q = 5$. Note 3, 5 can't both be in $\mathcal{T}$.

As $\mathcal{T}_0 = \{2\}$ and $\mathcal{T}_0 \subset \mathcal{T}$, 2 must be in $\mathcal{T}$. So

$$\mathcal{T} = \{2\}, \quad \{2,3\} \quad \text{or} \quad \{2,5\}.$$

If $q = 3$ or 5, $q \in \mathcal{T}$ means that P_q is the stabilizer of a type 2 vertex of the building X_q.

The prime 7 ramifies in ℓ, and the building X_7 is a homogeneous tree, in which each vertex has $7+1=8$ neighbours.

7 is not in $\mathcal{T}$, but P_7 can be the stabilizer of a type 1 vertex of X_7, or the stabilizer of a type 2 vertex of X_7. These two P_7's are *not conjugate*. Set

$$\boldsymbol{\mathcal{T}_1} = \{q \in V_f \ : \ \begin{array}{l} \bullet\ q \text{ doesn't split in } \ell, \text{ and} \\ \bullet\ P_q \text{ is the stabilizer of a type 2 vertex of } X_q\}. \end{array}$$

The possibilities for $\mathcal{T}_1$ in this case are

$$\emptyset, \quad \{7\}, \quad \{3\}, \quad \{3,7\}, \quad \{5\}, \quad \text{and} \quad \{5,7\}.$$

If $q \neq 2$ splits in ℓ, then $G(\mathbb{Q}_q) \cong SL(3, \mathbb{Q}_q)$.

We choose $P_q = SL(3, \mathbb{Z}_q)$ for all these q's. This is the stabilizer of the lattice class $[\mathbb{Z}_q^3]$ (a vertex of the building of $G(\mathbb{Q}_q)$).

If q does not split in ℓ, then $G(\mathbb{Q}_q) \cong \{g \in SL(3, \mathbb{Q}_q(s)) \ : g^* F'_q g = F'_q\}$ for an Hermitian F'_q.

We can arrange the isomorphism so that $F'_q \in GL(3, \mathfrak{o}_{\tilde{q}})$. Here $\tilde{q}$ is the unique extension of q to ℓ, and $\mathfrak{o}_{\tilde{q}}$ is the valuation ring in $\ell_{\tilde{q}} = \mathbb{Q}_q(s)$.

This implies that the $\mathfrak{o}_{\tilde{q}}$-lattice $\mathcal{L} = \mathfrak{o}_{\tilde{q}}^3$ in $\mathbb{Q}_p(s)^3$ is self-dual. That is:

$$\mathcal{L}' = \{\boldsymbol{y} \in \mathbb{Q}_q(s)^3 : \boldsymbol{y}^* F'_q \boldsymbol{x} \in \mathfrak{o}_{\tilde{q}} \text{ for all } \boldsymbol{x} \in \mathcal{L}\}$$

is equal to $\mathcal{L}$.

So we can choose as our parahoric

$$P_q = \{g \in SL(3, \mathfrak{o}_{\tilde{q}}) \ : g^* F_q' g = F_q'\}.$$

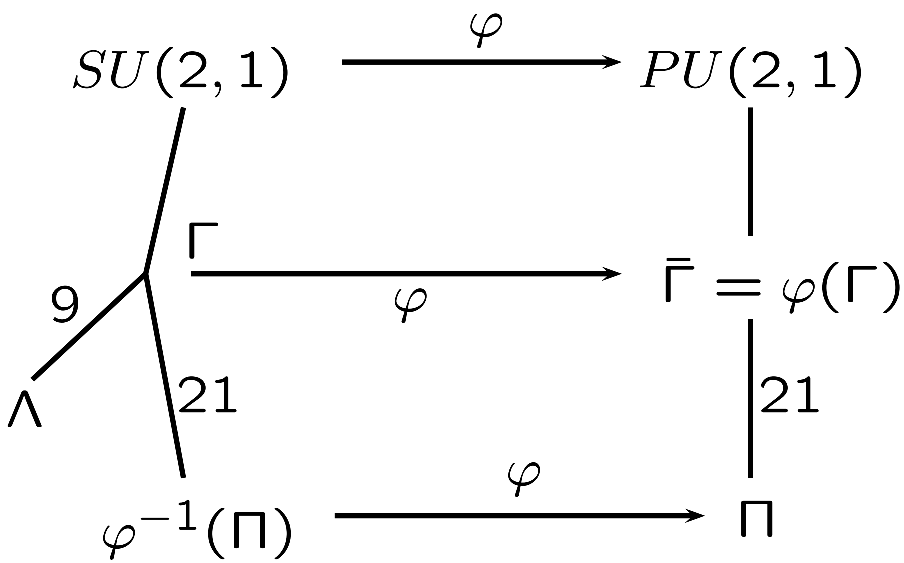
This is the stabilizer in $G(\mathbb{Q}_q)$ of the "type 1" vertex $\mathfrak{o}_{\tilde{q}}^3$ of the building of $G(\mathbb{Q}_q)$ (which is a tree).

We set $P_2 = G(\mathbb{Q}_2)$. This is compact. Then

$$\Lambda = \{g \in G(\mathbb{Q}) \ : g_q \in P_q \text{ for all primes } q\}$$

is a principal arithmetic subgroup with "$\mathcal{T}_1$" equal to $\emptyset$.

An earlier diagram is in this case



Here Π is the fundamental group of a hypothetical fake projective plane.

We want to find enough elements of Λ and its normalizer Γ to get a presentation of $\bar{\Gamma}$.

Let

$$\xi = \sum_{j=1}^{6} \sum_{k=0}^{2} a_{jk} \zeta^{j-1} \sigma^k \in \mathcal{D}$$

be in $G(\mathbb{Q})$. What are the conditions on the coefficients a_{jk} in order that $\xi \in \Lambda$?

As we need the normalizer Γ of Λ, so also we need to look at $\xi \in \mathcal{D}$ satisfying $\iota(\xi)\xi = 1$, but not necessarily $\mathrm{Nrd}(\xi) = 1$.

Then $\xi_q \in GL(3, \mathbb{Q}_q)$ if $q > 2$ splits. When is it in $GL(3, \mathbb{Z}_q)$?

Also $\xi_q \in GL(3, \mathbb{Q}_q(s))$ and $\xi_q^* F_q' \xi_q = F_q'$ if q doesn't split. When is it in $GL(3, \mathfrak{o}_{\tilde{q}})$?

It turns out that if $q \neq 2, 7$, we simply need that the a_{jk}'s have no q's in their denominators. That is, $a_{jk} \in \mathbb{Q} \cap \mathbb{Z}_q$.

As $G(\mathbb{Q}_q)$ is compact for $q = 2$, we get no 2-adic condition. So 2's are allowed in the denominators of the a_{jk}'s.

The situation when $q = 7$ is more complicated.

Let's look at the case $q = 7$. It turns out that $m = \mathbb{Q}(\zeta)$ does not embed in $\mathbb{Q}_7(s)$.

We can find an $\eta \in \mathbb{Q}_7(s)$ so that $\bar{\eta}\eta = 1$ and $N_{\mathbb{Q}_7(\zeta)/\mathbb{Q}_7(s)}(\eta) = D$ $(= (3+s)/4)$.

As the one 7-adic valuation v on $\ell = \mathbb{Q}(s)$ ramifies in m, we can't use the norm theorem from class field theory.

We can take $\eta = c-(8c^2-3c-4)s/7$, where $c = 6+0\times 7+1\times 7^2+4\times 7^3+\cdots$ is the solution in $\mathbb{Q}_7$ of $16c^3 - 12c - 3 = 0$. Then $\bar{\eta}\eta = 1$ and

$$N_{\mathbb{Q}_7(\zeta)/\mathbb{Q}_7(s)}(\eta) = \eta^3 = D.$$

The isomorphism $G(\mathbb{Q}_7) \cong \{g \in SL(3, \mathbb{Q}_7(s)) : g^* F' g = F'\}$ is seen using Ψ and the isomorphism $\mathcal{D} \otimes_\ell \mathbb{Q}_7(s) \cong M_{3\times 3}(\mathbb{Q}_7(s))$:

$$\mathcal{D} \stackrel{\Psi}{\to} M_{3\times 3}(m) \hookrightarrow M_{3\times 3}(\mathbb{Q}_7(\zeta)) \ \stackrel{J\cdot J^{-1}}{\longrightarrow} \ M_{3\times 3}(\mathbb{Q}_7(\zeta)),$$

where $J = \Theta C$, and

$$C = \begin{pmatrix} \eta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\varphi(\eta) \end{pmatrix} \quad \text{and} \quad \Theta = \begin{pmatrix} \theta_0 & \varphi(\theta_0) & \varphi^2(\theta_0) \\ \theta_1 & \varphi(\theta_1) & \varphi^2(\theta_1) \\ \theta_2 & \varphi(\theta_2) & \varphi^2(\theta_2) \end{pmatrix},$$

where $\theta_0, \theta_1, \theta_2$ is a basis of m over ℓ. We calculate that if $\xi \in \mathcal{D}$ and $\iota(\xi)\xi = 1$, its image in $M_{3\times 3}(\mathbb{Q}_7(s))$ is unitary with respect to

$$F_7' = J^{*-1} F J^{-1} = \Theta^{*-1} C^{*-1} F C^{-1} \Theta^{-1}.$$

Because $\bar{\eta}\eta = 1$, we find that $C^* = C^{-1}$. As C is diagonal, it commutes with F. So $C^{*-1}FC^{-1} = F$.

Adroitly choosing $\theta_0 = s$, $\theta_1 = s(\zeta - 1)$ and $\theta_2 = (\zeta - 1)^2$, we find that $7J^{*-1}FJ^{-1}$ equals

$$F' = \begin{pmatrix} 3 & 3 & s \\ 3 & 2 & (1+s)/2 \\ -s & (1-s)/2 & 0 \end{pmatrix},$$

which has entries in $\mathfrak{o}_\ell$ and determinant 1.

Because F' is in $GL(3, \mathfrak{o}_{\tilde{7}})$, the lattice $\mathfrak{o}_{\tilde{7}}^3$ in $\mathbb{Q}_7(s)^3$ is self-dual.

Now suppose that $\xi \in \mathcal{D}$ and $\iota(\xi)\xi = 1$. Look at its image ξ_7, which is a matrix with entries in $\mathbb{Q}_7(s)$ unitary with respect to F'.

As $\xi_7 = J\Psi(\xi)J^{-1}$, where $J = \Theta C$, is a matrix with entries in $\mathbb{Q}_7(s)$, we can write

$$\xi_7 = \begin{pmatrix} x_{11} + y_{11}s & x_{12} + y_{12}s & x_{13} + y_{13}s \\ x_{21} + y_{21}s & x_{22} + y_{22}s & x_{23} + y_{23}s \\ x_{31} + y_{31}s & x_{32} + y_{32}s & x_{33} + y_{33}s \end{pmatrix},$$

where $x_{ij}, y_{ij} \in \mathbb{Q}_7$ for each i, j. Each of these is a linear combination of the coefficients a_{ij}'s of ξ. So we can write

$$\boldsymbol{x} = M\boldsymbol{a},$$

where $\boldsymbol{a}$ and $\boldsymbol{x}$ are column vectors of length 18, made from the coefficients a_{ij} and from the numbers x_{ij} and y_{ij}, and where M is an 18×18 matrix with entries in $\mathbb{Q}_7$. In this case, the entries of M are explicit polynomials in the $c \in \mathbb{Q}_7$ used in solving $\mathrm{N}_{\mathbb{Q}_7(\zeta)/\mathbb{Q}_7(s)}(\eta) = D$.

$$\xi_7(\mathfrak{o}_{\tilde{7}}^3) = \mathfrak{o}_{\tilde{7}}^3 \quad \Leftrightarrow \quad \xi_7(\mathfrak{o}_{\tilde{7}}^3) \subset \mathfrak{o}_{\tilde{7}}^3 \quad \Leftrightarrow \quad \xi_7 \text{ has entries in } \mathfrak{o}_{\tilde{7}}.$$

In this case, $\mathfrak{o}_{\tilde{7}} = \{x + ys : x, y \in \mathbb{Z}_7\} \subset \mathbb{Q}_7(s)$, and so

$$\xi_7(\mathfrak{o}_{\tilde{7}}^3) = \mathfrak{o}_{\tilde{7}}^3 \quad \Leftrightarrow \quad x_{ij}, y_{ij} \in \mathbb{Z}_7 \text{ for all } i, j \quad \Leftrightarrow \quad \boldsymbol{x} = M\boldsymbol{a} \text{ has entries in } \mathbb{Z}_7.$$

If $L \in GL(18, \mathbb{Z}_7)$, then

$$M\boldsymbol{a} \text{ has entries in } \mathbb{Z}_7 \quad \Leftrightarrow \quad LM\boldsymbol{a} \text{ has entries in } \mathbb{Z}_7.$$

We can choose $L \in GL(18, \mathbb{Z}_7)$ so that $LM = \mathcal{E}$ is in "reduced row echelon form". Then

$$\xi_7(\mathfrak{o}_{\tilde{7}}^3) = \mathfrak{o}_{\tilde{7}}^3 \quad \Leftrightarrow \quad \mathcal{E}\boldsymbol{a} \text{ has entries in } \mathbb{Z}_7.$$

We only need a 7-adic approximation M_7 (mod 49 is enough) to M to get $\mathcal{E}$. The following Magma commands give us $\mathcal{E}$.

M_7:=`Matrix(IntegerRing(49),18,18,[...]);`

$\mathcal{E}$:=`EchelonForm`(M_7);

We used the order

$$a_{10}, \ldots, a_{60}, a_{11}, \ldots, a_{61}, a_{12}, \ldots, a_{62}$$

for the coefficients of ξ.

We get:

$$\mathcal{E} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 3 & 6 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 6 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 3 & 5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 6 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 5 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 6 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7
\end{pmatrix}.$$

To summarize:

For each prime $q \neq 2$ which splits in ℓ, let $P_q = SL(3, \mathbb{Z}_q)$.

For each prime q which does not split in ℓ, let $P_q = \{g \in SL(3, \mathfrak{o}_{\tilde{q}}) : g^* F'_q g = F'_q\}$.

Let $\Lambda = \{\xi \in G(\mathbb{Q}) : \xi_q \in P_q$ for all $q \neq 2\}$.

The elements ξ of Λ are the

$$\xi = \sum_{j=1}^{6} \sum_{k=0}^{2} a_{jk} \zeta^{j-1} \sigma^k \in \mathcal{D}$$

such that $\iota(\xi)\xi = 1$, $\mathrm{Nrd}(\xi) = 1$, $a_{j,k} \in \mathbb{Z}[1/2, 1/7]$ for all j, k, and such that $\mathcal{E}\boldsymbol{a}$ has entries in $\mathbb{Z}_7$.

The image $\bar{\Gamma}$ of the normalizer Γ of Λ is isomorphic to the group of elements $\xi\mathcal{Z}$, where $\mathcal{Z} = \{t1 : t \in \ell \ \& \ \bar{t}t = 1\}$, where ξ has the form

$$\xi = \sum_{j=1}^{6} \sum_{k=0}^{2} a_{jk} \zeta^{j-1} \sigma^{k} \in \mathcal{D}$$

such that $\iota(\xi)\xi = 1$, $a_{j,k} \in \mathbb{Z}[1/2, 1/7]$ for all j, k, and such that $\mathcal{E}\boldsymbol{a}$ has entries in $\mathbb{Z}_7$.

We find elements of Λ and $\bar{\Gamma}$ using the Cayley transform and a computer search.

If $\eta \in \mathcal{D}$ and $\iota(\eta) = -\eta$, then $\eta \neq 1$, so that $1 - \eta$ is invertible. Let $\xi = (1+\eta)(1-\eta)^{-1} \in \mathcal{D}$. Then $\iota(\xi)\xi = 1$. Conversely, if $\iota(\xi)\xi = 1$ and $\xi \neq -1$, then $\eta = (\xi - 1)(\xi + 1)^{-1}$ satisfies $\iota(\eta) = -\eta$.

For

$$\eta = \sum_{j=1}^{6} \sum_{k=0}^{2} s_{jk} \zeta^{j-1} \sigma^k \in \mathcal{D},$$

the condition $\iota(\eta) = -\eta$ imposes 9 linear conditions on the 18 rational numbers s_{jk}. This allows us to eliminate 9 of these variables.

In order to work with integer coefficients, we look at $(d1 + \eta)(d1 - \eta)^{-1}$, where $d \geq 1$ is an integer.

Looking for elements ξ of Λ, we need $\mathsf{Nrd}(\xi) = 1$, and so $\mathsf{Nrd}(d1 + \eta) = \mathsf{Nrd}(d1 - \eta)$. This imposes a condition $\mathsf{cond}_a = 0$ on the s_{jk}'s, where cond_a is a cubic polynomial in d and the 9 non-eliminated s_{jk}'s.

The elements ξ of Γ satisfy $\mathsf{Nrd}(\xi) = \pm D^n$ for some integer n, and it is enough to look for elements satisfying $\mathsf{Nrd}(\xi) = D$. So we want $\mathsf{Nrd}(d1 + \eta) = D\,\mathsf{Nrd}(d1 - \eta)$. This imposes a condition $\mathsf{cond}_b = 0$ on the s_{jk}'s, where cond_b is another cubic polynomial in d and the 9 non-eliminated s_{jk}'s.

Whenever we find s_{jk}'s satisfying cond_a or cond_b, we calculate the coefficients a_{jk} of $\xi = (d1 + \eta)(d1 - \eta)^{-1}$, looking for a_{jk}'s satisfying the above arithmetic conditions.

The cubic equations $\mathrm{cond}_a = 0$ or $\mathrm{cond}_b = 0$ are time consuming to check. Fortunately, there are strong necessary conditions on the s_{jk}'s which can be checked after a quadratic polynomial (the bitrace of $\Psi(\eta)$) in the s_{jk}'s has been calculated. So cond_a and cond_b are calculated only for s_{jk}'s in a tiny proportion of the search space.

This is especially important in cases when $k \neq \mathbb{Q}$, and we have 36 rational coefficients to start with, and still have 18 variables after the condition $\iota(\eta) = -\eta$ is imposed.

Running, overnight say, two programs, one imposing $\text{cond}_a = 0$ and the other $\text{cond}_b = 0$, we get a few hundred elements of $\bar{\Gamma}$. One may verify that each element found is a word in z and b, where

$$z = \zeta + 0\sigma + 0\sigma^2,$$

$$b = \alpha + \beta\sigma + \gamma\sigma^2$$

for

$$\alpha = \frac{1}{7}\Big(0 + 4\zeta + 5\zeta^2 + 3\zeta^3 - 2\zeta^4 - 3\zeta^5\Big)$$

$$\beta = \frac{1}{7}\Big(7 + 5\zeta - 6\zeta^2 + 2\zeta^3 + 8\zeta^4 - 2\zeta^5\Big)$$

$$\gamma = \frac{1}{7}\Big(7 + \ \ \zeta + 3\zeta^2 + 6\zeta^3 + 3\zeta^4 - 6\zeta^5\Big).$$

(b equals $(d1 + \eta)(d1 - \eta)^{-1}$ with $d = 7$ and $|s_{ij}| \leq 12$ for all i, j.)

So it *seems* that $\bar{\Gamma}$ is generated by z and b (more exactly, by $z\mathcal{Z}$ and $b\mathcal{Z}$). We prove this (and more) as follows.

To see that $\bar{\Gamma}$ is generated by z and b as follows.

1) Starting from the outcome of our search programs, and a number d_1 (to be chosen later) we form a set S of elements g of $\bar{\Gamma}$ satisfying $d(g.0,0) \leq d_1$ which we arrange in order of increasing $d(g.0,0)$.

2) We enlarge S if necessary, so that

- whenever $g \in S$, also $g^{-1} \in S$, and

- whenever $g, g' \in S$ and $d((gg').0,0) \leq d_1$ also gg' is in S.

3) We form

$$\mathcal{F}_S = \{z \in B(\mathbb{C}^2) : d(z, 0) \leq d(z, g.0) \text{ for all } g \in S\}.$$

Note that $\mathcal{F}_S \supset \mathcal{F}_{\bar{\Gamma}}$, where

$$\mathcal{F}_{\bar{\Gamma}} = \{z \in B(\mathbb{C}^2) : d(z, 0) \leq d(z, g.0) \text{ for all } g \in \bar{\Gamma}\}.$$

4) We numerically calculated the normalized hyperbolic volume vol($\mathcal{F}_S$). We compare this with the known value of vol($\mathcal{F}_{\bar{\Gamma}}$) ($= 1/864$ in this case).

Lemma. If vol($\mathcal{F}_S$) $<$ 2vol($\mathcal{F}_{\bar{\Gamma}}$), then $\bar{\Gamma}$ is generated by S.

Now suppose that $d(z, 0)$ is bounded on $\mathcal{F}_S$, and we can calculate

$$r_0(S) = \sup\{d(z, 0) : z \in \mathcal{F}_S\}.$$

Theorem. Suppose that S satisfies 1) and 2) above, and that S generates $\bar{\Gamma}$. If $\boldsymbol{d_1 > 2r_0(S)}$, then S is the set of **all** $g \in \bar{\Gamma}$ such that $d(g.0, 0) \leq d_1$.

Corollary 1. Under the hypotheses of the theorem, the set $\mathcal{F}_S$ is equal to the Dirichlet fundamental domain $\mathcal{F}_{\bar{\Gamma}}$ of $\bar{\Gamma}$.

Corollary 2. Under the hypotheses of the theorem, the set S, together with all the relations $g_1 g_2 g_3 = 1$, where each g_i is in S, forms a presentation of the group $\bar{\Gamma}$.

In this case, the elements of $\bar{\Gamma}$ can be divided into double cosets KgK, where $K = \langle z \rangle$. We form

$$S = \bigcup_{i=0}^{8} Ka_iK \ \cup \ \bigcup_{i=1}^{6} Kb_jK \ \cup \ \bigcup_{i=1}^{6} Kb_j^{-1}K,$$

where $\mathrm{Nrd}(a_i) = 1$ and $\mathrm{Nrd}(b_j) = D$ for each i, j, and $a_0 = id$. So S has $7 + 8 \times 49 + 6 \times 49 + 6 \times 49 = 987$ elements.

Here $a_1, \ldots, a_8$ are

$$\bar{D}b^3,\ Db^{-3},\ bz^2b^{-1},\ bz^{-2}b^{-1},\ \text{and}$$
$$\bar{D}bz^{-1}b^2,\ Db^{-2}zb^{-1},\ bzb^{-2}zb,\ b^2z^5b^{-1}zb^{-1},$$

and $b_1, \ldots, b_6$ are

$$Db^{-2},\ b,\ b^2zb^{-1},\ \bar{D}b^3z^5b,\ \bar{D}bz^5b^2z^5b,\ \text{and } Db^{-3}zb.$$

We apply the lemma and the theorem to confirm that $\bar{\Gamma}$ is generated by z and b, and get the presentation

$$
\begin{aligned}
\bar{\Gamma} = \langle z, b \mid \\
& z^7, \\
& (b^{-2}z^1)^3, \\
& (b^2z^{-2}b^2z^2)^3, \\
& (b^2z^{-2}b^2z^4)^3, \\
& b^3z^{-2}b^{-1}z^2b^{-2}z^1, \\
& b^3z^1b^3z^3bz^2b^{-1}z^{-1}, \\
& b^3z^2b^2z^{-2}b^{-1}z^{-1}b^{-3}z^1b^{-1}z^{-1}\rangle.
\end{aligned}
$$

Magma shows via the command

```
LowIndexSubgroups(G,<21,21>);
```

that up to conjugacy there are 3 subgroups of index 21.

$$\Pi_a = \langle b^3,\ zbz^{-1}b^{-1}z,\ (bz^{-1})^3,\ zb^{-1}z^{-2}b\rangle,$$
$$\Pi_b = \langle b^3,\ z^2b^{-1}zb,\ zbz^2b^{-1}\rangle,$$
$$\Pi_c = \langle b,\ z^{-1}b^{-2}z^{-1},\ zbzbz^{-1}\rangle.$$

The abelianization $\Pi/[\Pi,\Pi]$ is finite in each case, equalling

$$C_2^4, \quad C_{14}, \quad \text{and} \quad C_2 \times C_{42}, \quad \text{respectively.}$$

Check that each Π is torsion free is easy using;

$$g \in \bar{\Gamma} \text{ torsion} \quad \Rightarrow \quad g \text{ is conjugate to an element of } S.$$

What happens if we replace $P_7 = \{g \in SL(3, \mathfrak{o}_{\underset{\sim}{7}}) : g^* F' g = F'\}$ (which is the stabilizer in $G(\mathbb{Q}_7)$ of the type 1 vertex $\mathcal{L} = \mathfrak{o}_{\underset{\sim}{7}}^3$ of the building X_7), by the stabilizer of a type 2 vertex?

Let

$$c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathcal{M} = c(\mathfrak{o}_{\underset{\sim}{7}}^3)$ is again a lattice in $\mathbb{Q}_7(s)^3$. Now $\det(c) = s$, and

$$c^* F' c,\ s(c^* F' c)^{-1},\ c, \text{ and } sc^{-1} \text{ have entries in } \mathbb{Z}[s] \subset \mathbb{Z}_7[s].$$

This means that

$$s\mathcal{M}' \subsetneqq \mathcal{M} \subsetneqq \mathcal{M}' \quad \text{and} \quad s\mathcal{L} \subsetneqq \mathcal{M} \subsetneqq \mathcal{L}.$$

So the pair $(\mathcal{M}, \mathcal{M}')$ is a type 2 vertex of the building X_7, and is adjacent to the type 1 vertex $\mathcal{L}$.

If $g^* F' g = F'$, then $g(\mathcal{M}) = \mathcal{M}$ iff $g(\mathcal{M}') = \mathcal{M}'$. Form

$$P_7' = \{g \in G(\mathbb{Q}_7) : g(\mathcal{M}) = \mathcal{M}\}.$$

Form the principal arithmetic subgroup Λ' which is the same as before, except that P_7 is replaced by P_7'. Let

$$\xi = \sum_{j=1}^{6} \sum_{k=0}^{2} a_{jk} \zeta^{j-1} \sigma^k \in \mathcal{D}$$

satisfy $\iota(\xi)\xi = 1$. Let ξ_7 be its image in $\{g \in GL(3, \mathbb{Q}_7(s)) : g^* F' g = F'\}$. When does $\xi_7(\mathcal{M}) = \mathcal{M}$?

We find that this holds iff $\mathcal{E}' \boldsymbol{a}$ has entries in $\mathbb{Z}_7$, where $\mathcal{E}'$ is an 18×18 matrix, a little different from $\mathcal{E}$.

and

$$\mathcal{E}' = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 5 & 3 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 3 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 5 & 6 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 5 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 5 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 4 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7
\end{pmatrix}.$$

The same method shows that $\bar{\Gamma}$ is again generated by z (as before) and an element b (different from previous b), and has presentation

$$
\begin{aligned}
\bar{\Gamma} = \langle\ z,\ b \mid\ & \\
& b^3 = 1, \\
& z^7 = 1, \\
& (bz^{-2}bz^{-1})^3 = 1, \\
& b^{-1}zbz^2bz^2b^{-1}z^{-1}bz^2b^{-1}z = 1, \\
& bz^2b^{-1}z^{-1}bz^{-1}bz^2b^{-1}z^{-1}bz^{-1}bz^{-3} = 1, \\
& bz^2bzbz^{-2}b^{-1}zbz^{-1}bz^{-2}b^{-1}z^2 = 1\rangle.
\end{aligned}
$$

The relations here hold modulo scalars. For example, $b^3 = D1$.

One can show that Λ is generated by z and bzb^{-1}.

Magma gives 4 index 21 subgroups of $\bar{\Gamma}$:

$$\begin{aligned}
\Pi_a &= \langle bzb^{-1}z^{-2},\ bz^{-1}b^{-1}z^{2},\ zbz^{3}b^{-1}\rangle,\\
\Pi_b &= \langle zbz^{-1}b^{-1},\ z^{3}bzb^{-1},\ z^{2}b^{-1}z^{-1}b,\ zb^{-1}zbz\rangle,\\
\Pi_c &= \langle zbz^{-1}b^{-1},\ z^{2}bz^{2}b^{-1},\ zb^{-1}zbz\rangle,\\
\Pi_d &= \langle zb^{-1},\ z^{-3}b,\ b^{-1}zbzb\rangle.
\end{aligned}$$

all are torsion-free with finite abelianization.

So there are 4 fake projective planes in the class $(a = 7, p = 2, \{7\})$.

$\Pi = \Pi_d$ is the fundamental group of Mumford's original example, since

- $N_{\bar{\Gamma}}(\Pi) = \Pi$, which means that $\mathrm{Aut}(B(\mathbb{C}^2)/\Pi)$ is trivial, and

- there is a surjective homomorphism $\bar{\Gamma} \to PSL(2, \mathbb{F}_7)$.