The building X_v on which $G(k_v)$ and $\bar{G}(k_v)$ act, when v splits in ℓ.

$K :=$ nonarchimedean local field, with valuation v.

$\mathfrak{o}_K := \{x \in K : v(x) \geq 0\}$.

$\{x \in K : v(x) > 0\}$ equals $\pi\mathfrak{o}_K$.

$q := |\mathfrak{o}_K/\pi\mathfrak{o}_K|$.

When $K = k_v$, write $\mathfrak{o}_v$ for $\mathfrak{o}_{k_v}$, q_v for q.

Any basis $\{v_1, v_2, v_3\}$ of K gives a **lattice** in K^3:

$$\mathcal{L} = \{a_1 v_1 + a_2 v_2 + a_3 v_3 : a_1, a_2, a_3 \in \mathfrak{o}_K\}. \tag{1}$$

E.g. $\{v_1, v_2, v_3\} = \{e_1, e_2, e_3\}$ gives $\mathcal{L}_0 := \mathfrak{o}_K^3$.

$\mathsf{Lat}_K :=$ set of lattices in K^3.

$g \in GL(3, K)$ & $\mathcal{L} \in \mathsf{Lat}_K \;\Rightarrow\; g(\mathcal{L}) \in \mathsf{Lat}_K$.

$GL(3, K)$ acts transitively on Lat_K.

$GL(3, \mathfrak{o}_K) := \{g \in GL(3, K) \;:\; g$ & g^{-1} have entries in $\mathfrak{o}_K\}$.

$GL(3, \mathfrak{o}_K)$ equals $\{g \in GL(3, K) \;:\; g(\mathcal{L}_0) = \mathcal{L}_0\}$.

$GL(3, \mathfrak{o}_K) = \{g \in M_{3\times 3}(\mathfrak{o}_K) \;:\; v(\det(g)) = 0\}$.

$\mathcal{L}_1, \mathcal{L}_2 \in \mathsf{Lat}_K$ **equivalent** if $\mathcal{L}_2 = t\mathcal{L}_1$, some $t \in K^\times$.

$[\mathcal{L}] :=$ equivalence class of $\mathcal{L}$.

$X_K :=$ set of equivalence classes.

For $g \in GL(3, K)$, $g.[\mathcal{L}] := [g(\mathcal{L})]$.

$GL(3, K)$ acts transitively on X_K.

$g = tI \Rightarrow g.[\mathcal{L}] = [\mathcal{L}]$ for all $\mathcal{L} \in \mathsf{Lat}_K$.

$PGL(3, K)$ acts transitively on X_K.

For $i \in \{0, 1, 2\}$, $[g(\mathcal{L}_0)] \in X_K$ has **type $\boldsymbol{i}$** if $v(\det(g)) \equiv i \pmod 3$.

$SL(3, K)$ acts transitively on $\{[\mathcal{L}] \in X_K \ : \ \mathrm{type}([\mathcal{L}]) = i\}$.

$[\mathcal{L}_1]$ is **adjacent** to $[\mathcal{L}_2]$ if there are representatives $\mathcal{L}_j$ of $[\mathcal{L}_j]$ for $j = 1, 2$ so that

$$\pi\mathcal{L}_1 \subsetneqq \mathcal{L}_2 \subsetneqq \mathcal{L}_1.$$

This implies

$$\pi\mathcal{L}_2 \subsetneqq \pi\mathcal{L}_1 \subsetneqq \mathcal{L}_2,$$

so adjacency is a symmetric relation. Adjacent lattice classes have different types.

Fact: Given $[\mathcal{L}]$ with type$([\mathcal{L}]) = i$, and $j \neq i$,

$$\sharp\{[\mathcal{M}] \in X_K \ : \ [\mathcal{M}] \text{ adjacent to } [\mathcal{L}] \ \& \ \text{type}([\mathcal{M}]) = j\} \ = \ q^2 + q + 1.$$

Proof: $\mathcal{L}/\pi\mathcal{L}$ is a vector space of dimension 3 over the residual field $\mathfrak{o}_K/\pi\mathfrak{o}_K$. For $\nu = 1, 2$, $\pi\mathcal{L} \subset \mathcal{M} \subset \mathcal{L}$ and type$([\mathcal{M}]) = i + \nu \pmod 3$ iff $\mathcal{M}/\pi\mathcal{L}$ is a ν-dimensional subspace.

$[\mathcal{L}_1], [\mathcal{L}_2], [\mathcal{L}_3]$ form a **chamber** if there are representatives $\mathcal{L}_j$ of $[\mathcal{L}_j]$ for $j = 1, 2, 3$ so that

$$\pi\mathcal{L}_1 \subsetneqq \mathcal{L}_3 \subsetneqq \mathcal{L}_2 \subsetneqq \mathcal{L}_1.$$

Each chamber contains one lattice class of each type.

Any pair of adjacent lattice classes lies in $q + 1$ distinct chambers.

Any lattice class belongs to $(q^2 + q + 1)(q + 1)$ distinct chambers.

X_K is a simplicial complex.

For $g \in SL(3, K)$ and $\mathcal{L} \in \mathsf{Lat}_K$, $g.[\mathcal{L}] = [\mathcal{L}]$ iff $g(\mathcal{L}) = \mathcal{L}$.

For $\mathcal{L} \in \mathsf{Lat}_K$, $\{g \in SL(3, K) : g(\mathcal{L}) = \mathcal{L}\}$ is a maximal compact subgroup of $SL(3, K)$.

Any maximal compact subgroup of $SL(3, K)$ has this form.

There are three conjugacy classes of maximal compact subgroups of $SL(3, K)$, corresponding to the three types.

Any two maximal compact subgroups of $SL(3, K)$ are conjugate by an element of $GL(3, K)$.

For $i = 1, 2$, $SL(3, \mathfrak{o}_K)$ acts transitively on

$$\{[\mathcal{L}] \in X_K \ : \ [\mathcal{L}] \text{ adjacent to } [\mathcal{L}_0] \ \& \ \text{type}([\mathcal{L}]) = i\}.$$

For any edge containing $[\mathcal{L}_0]$, the stabilizer in $SL(3, K)$ of that edge has index $q^2 + q + 1$ in $SL(3, \mathfrak{o}_K)$.

$SL(3, \mathfrak{o}_K)$ acts transitively on the set of chambers containing $[\mathcal{L}_0]$.

For any chamber containing $[\mathcal{L}_0]$, the stabilizer in $SL(3, K)$ of that chamber has index $(q^2 + q + 1)(q + 1)$ in $SL(3, \mathfrak{o}_K)$.

We have seen that when $v \in V_f \setminus \mathcal{T}_0$ splits in ℓ, then $G(k_v) \cong SL(3, k_v)$. The *parahoric* subgroups of $G(k_v)$ are its subgroups corresponding to the stabilizers of vertices, edges and chambers of $X_v := X_{k_v}$.

In particular,

$$P_v = \{g \in SL(3, k_v) : g(\mathfrak{o}_v^3) = \mathfrak{o}_v^3\} = SL(3, \mathfrak{o}_v).$$

is a maximal parahoric subgroup of $G(k_v)$.

Given $\xi \in \mathcal{D}$ satisfying $\iota(\xi)\xi = 1$, we need to understand when the image ξ_v of ξ in $GL(3, k_v)$ under the map

$$\mathcal{D} \to \mathcal{D} \otimes_\ell k_v \cong M_{3\times 3}(k_v)$$

fixes the lattice class $[\mathfrak{o}_v^3]$. In particular, if $\xi \in G(k)$ when does ξ_v fix $\mathfrak{o}_v^3$?

$b_1, \ldots, b_6 :=$ basis of m over k.

$$\xi \in \mathcal{D} \quad \Rightarrow \quad \xi = \sum_{i,j} a_{ij} b_i \sigma^j.$$

Form matrix $B = (\psi_i(b_j))$, where $\mathrm{Gal}(m/k) = \{\psi_1, \ldots, \psi_6\}$.

Fact: $\det(B)^2 \in \ell$.

Lemma. $b_1, \ldots, b_6$ can be chosen so that $v(\det(B)^2) = 0$ for all $v \in V_f \setminus \mathcal{T}_0$ which split in ℓ.

Example. In $(a = 7, p = 2)$ case, $m = \mathbb{Q}[\zeta]$, where $\zeta = \zeta_7$. Choose $b_1, \ldots, b_6 = 1, \zeta, \ldots, \zeta^5$. Then $\det(B) = 7^2 s$ for $s = 1 + 2\zeta + 2\zeta^2 + 2\zeta^4$, so $\det(B)^2 = -7^5$. So $v(\det(B)^2) = 0$ unless $v = u_7$.

Proposition. If $\iota(\xi)\xi = 1$, and $v \in V_f \setminus \mathcal{T}_0$ splits in ℓ, then

$$\xi_v(\mathfrak{o}_v^3) = \mathfrak{o}_v^3 \quad \Leftrightarrow \quad a_{ij} \in k \cap \mathfrak{o}_v \text{ for all } i, j.$$

When $m \hookrightarrow k_v$, the calculations only involve the matrix B.

When $m \not\hookrightarrow k_v$, we also need the following:

Lemma. If $v \in V_f \setminus \mathcal{T}_0$ splits in ℓ, but $m \not\hookrightarrow k_v$, there is an $\eta_v \in k_v(Z)$ such that $N_{k_v(Z)/k_v}(\eta_v) = D$. Moreover, $\tilde{v}(\eta_v) = 0$.

Reasons: Embed ℓ in k_v, then extend to $m = \ell(Z) \hookrightarrow k_v(Z)$. This gives valuation $\tilde{v}$ on $k_v(Z)$ extending v. The image of $D \in \ell$ in k_v satisfies $v(D) = 0$. Case by case we see $k_v(Z)$ is an unramified extension of k_v. Now apply norm theorem from local class field theory.

When $v \in V_f \setminus \mathcal{T}_0$ splits in ℓ but $m \not\hookrightarrow k_v$, the isomorphism $\mathcal{D} \otimes_\ell k_v \cong M_{3\times 3}(k_v)$ involves conjugation by $J_v = \Theta C_v$. Here C_v is diagonal matrix with diagonal entries η_v, 1 and $1/\varphi(\eta_v)$, and $\Theta = (\varphi^i(\theta_j))$ for some basis $\theta_0, \theta_1, \theta_2$ of m over ℓ.

Lemma. $\theta_0, \theta_1, \theta_2$ can be chosen in $\mathfrak{o}_m$ and so that $\tilde{v}(\det(\Theta)) = 0$ for all $v \in V_f \setminus \mathcal{T}_0$ which split in ℓ.

Example. In $(a = 7, p = 2)$ case, $m = \mathbb{Q}[\zeta]$, where $\zeta = \zeta_7$. Choose $\theta_0, \theta_1, \theta_2 = 1, \zeta, \zeta^2$. Then $\det(\Theta) = -s$. So $\tilde{v}(\Theta) = 0$ unless $v = u_7$.

The building X_v when v does not split in ℓ.

$K :=$ non-archimedean local field, with valuation v, as before.

$L := K(s)$, a separable quadratic extension of K.

The automorphism $a + bs \mapsto a - bs$ of L is denoted $x \mapsto \bar{x}$.

$\tilde{v} :=$ unique extension to L of v.

$\mathfrak{o}_L := \{x \in L : \tilde{v}(x) \geq 0\}$, and $\{x \in L : \tilde{v}(x) > 0\}$ equals $\pi_L \mathfrak{o}_L$.

When $K = k_v$, where $v \in V_f$ does not split in ℓ, $L = k_v(s)$ is the completion $\ell_{\tilde{v}}$ of ℓ with respect to the unique extension $\tilde{v}$ of v to ℓ, and we write $\mathfrak{o}_{\tilde{v}}$ for $\mathfrak{o}_L$.

$f : L^3 \times L^3 \to L$: a nondegenerate sesquilinear form on L^3.

Then $f \leftrightarrow$ a nonsingular Hermitian matrix F:

$$f(x, y) = \boldsymbol{y}^* F \boldsymbol{x},$$

where $x = x_1 e_1 + x_2 e_2 + x_3 e_3$ and $y = y_1 e_1 + y_2 e_2 + y_3 e_3$.

The **unitary group** of F is

$$U_F = \{g : L^3 \to L^3 \ : \ f(gx, gy) = f(x, y) \text{ for all } x, y \in L^3\}.$$

$$U_F \cong \{g \in M_{3\times 3}(L) : g^* F g = F\}.$$

$$SU_F \cong \{g \in M_{3\times 3}(L) : g^* F g = F \text{ and } \det(g) = 1\}.$$

For $\mathcal{L} \in \mathsf{Lat}_L$,

$$\mathcal{L}' = \{x \in L^3 : f(x,y) \in \mathfrak{o}_L \text{ for all } y \in \mathcal{L}\} \tag{2}$$

is again a lattice, called the **dual lattice** of $\mathcal{L}$ with respect to f.

Then

$$(\mathcal{L}')' = \mathcal{L} \quad \text{and} \quad \mathcal{L}_1 \subset \mathcal{L}_2 \iff \mathcal{L}_2' \subset \mathcal{L}_1'.$$

For $\mathcal{L}_0 = \mathfrak{o}_L^3$ and $g \in GL(3, L)$,

$$(g(\mathcal{L}_0))' = (g^* F)^{-1}(\mathcal{L}_0). \tag{3}$$

Let

$$\mathsf{Lat}_1 = \{\mathcal{L} \in \mathsf{Lat}_L : \mathcal{L}' = \mathcal{L}\},$$

and

$$\mathsf{Lat}_2 = \{(\mathcal{M}, \mathcal{M}') : \mathcal{M} \in \mathsf{Lat}_L \text{ and } \pi\mathcal{M}' \subsetneqq \mathcal{M} \subsetneqq \mathcal{M}'\}.$$

$(\mathcal{M}, \mathcal{M}') \in \mathsf{Lat}_2$ iff the lattice classes $[\mathcal{M}]$ and $[\mathcal{M}']$ are adjacent in the building X_L of $SL(3, L)$.

$\mathcal{L} \in \mathsf{Lat}_1$ and $(\mathcal{M}, \mathcal{M}') \in \mathsf{Lat}_2$ are called **adjacent** if

$$\pi\mathcal{L} \subsetneqq \mathcal{M} \subsetneqq \mathcal{L}.$$

This means that

$$\pi\mathcal{L} \subsetneqq \pi\mathcal{M}' \subsetneqq \mathcal{M} \subsetneqq \mathcal{L},$$

so that $[\mathcal{L}]$, $[\mathcal{M}]$ and $[\mathcal{M}']$ form a chamber in X_L.

Using (3), we see that

$$\mathcal{L}_0 \in \mathsf{Lat}_1 \;\Leftrightarrow\; F \in GL(3, \mathfrak{o}_L).$$

If $g \in U_F$ and $\mathcal{L} \in \mathsf{Lat}_L$, then

- $(g(\mathcal{L}))' = g(\mathcal{L}')$,
- g fixes $\mathcal{L}$ iff g fixes $\mathcal{L}'$,
- if $\mathcal{L} \in \mathsf{Lat}_1$, then $g(\mathcal{L}) \in \mathsf{Lat}_1$, and
- if $(\mathcal{M}, \mathcal{M}') \in \mathsf{Lat}_2$, then $(g(\mathcal{M}), g(\mathcal{M}')) \in \mathsf{Lat}_2$.

So U_F acts on each of the sets Lat_1 and Lat_2.

Lemma. Suppose that $F \in GL(3, \mathfrak{o}_L)$ so that $\mathcal{L}_0 = \mathfrak{o}_L^3$ is in Lat_1. Let $g \in GL(3, K)$, and $\mathcal{M} = g(\mathcal{L}_0)$. Then $(\mathcal{M}, \mathcal{M}')$ is in Lat_2, and is adjacent to $\mathcal{L}_0$ if and only if

(a) If g has entries in $\mathfrak{o}_L$,

(b) If πg^{-1} has entries in $\mathfrak{o}_L$,

(c) If $g^* F g$ has entries in $\mathfrak{o}_L$,

(d) If $\pi (g^* F g)^{-1}$ has entries in $\mathfrak{o}_L$,

(e) If $\tilde{v}(\det(g)) = \tilde{v}(\pi)$.

Lemma. Let $g \in U_F$. Then

(a) If $\mathcal{L} \in \mathsf{Lat}_1$, then $g(\mathcal{L}) = \mathcal{L}$ if and only if $g(\mathcal{L}) \subset \mathcal{L}$.

(b) If $(\mathcal{M}, \mathcal{M}') \in \mathsf{Lat}_2$, then $g(\mathcal{M}) = \mathcal{M}$ if and only if $g(\mathcal{M}) \subset \mathcal{M}$.

(c) If $(\mathcal{M}, \mathcal{M}') \in \mathsf{Lat}_2$, then $g(\mathcal{M}') = \mathcal{M}'$ if and only if $g(\mathcal{M}') \subset \mathcal{M}'$.

There is a building $\mathcal{B}$ associated with SU_F. This is a very special case of results of Bruhat and Tits (see §10.1 in Bruhat-Tits "Groupes Réductifs sur un corps local I: Données radicielles valuées", *Publ. Math. I.H.E.S.* **41** (1972), 5–251, and §1.15 in Tits, "Reductive groups over local fields", *Proc. Amer. Math. Soc. Symp. Pure Math.* **33** (1979), 29–69. In this case, the building $\mathcal{B}$ is a tree.

Theorem. With the above notation, the set $\text{Lat}_1 \cup \text{Lat}_2$, together with the above adjacency relation, forms a tree T. This tree is homogeneous of degree $q+1$ when L is a ramified extension of K, and is bihomogeneous when L is an unramified extension of K, each $v \in \text{Lat}_1$ having q^3+1 neighbors, and each $v \in \text{Lat}_2$ having $q+1$ neighbors. It is isomorphic to the Bruhat Tits building $\mathcal{B}$ associated with SU_F.

Elements of Lat_i are called vertices of **type** $\boldsymbol{i}$ of this tree.

Assume now $K = k_v$ and $L = k_v(s)$, where $v \in V_f$ does not split in ℓ.

We have seen that

$$G(k_v) \cong \{g \in SL(3, k_v(s)) \ : \ g^* F'_v g = F'_v\}$$

for some invertible Hermitian F'_v.

Lemma. In each case, we can arrange that $F'_v \in GL(3, \mathfrak{o}_{\tilde{v}})$, so that $\mathfrak{o}_{\tilde{v}}^3$ is a type 1 vertex of X_v.

Recall: $\Psi(\iota(\xi)) = F^{-1}\Psi(\xi)^* F$ for either

$$F = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{OR} \quad F = \begin{pmatrix} T & 0 & 0 \\ 0 & \varphi(T) & 0 \\ 0 & 0 & \varphi^2(T) \end{pmatrix}$$

for $\mathcal{D}$ defined using m so that Gal$(m/\mathbb{Q})$ is non-abelian and σ satisfying $\sigma^3 = p$ OR for $\mathcal{D}$ defined using m so that Gal$(m/\mathbb{Q})$ is abelian and σ satisfying $\sigma^3 = D$, respectively.

Recall: $\tilde{v} :=$ unique extension of v to ℓ.

$\mathfrak{o}_{\tilde{v}} := \{x \in \ell_{\tilde{v}} = k_v(s) : \tilde{v}(x) \geq 0\}$.

F has entries in m.

When $v \in V_f$ does not split in ℓ, and $m \hookrightarrow k_v(s) = \ell_{\tilde{v}}$, F'_v is just the image of F under the embedding $M_{3\times 3}(m) \hookrightarrow M_{3\times 3}(k_v(s))$.

Example: $(a = 7, p = 2)$. F has diagonal entries T, $\varphi(T)$, $\varphi^2(T)$, where $T = \zeta + \zeta^{-1} \in \mathfrak{o}_m \subset \mathfrak{o}_{\tilde{v}}$, and $\det(F) = N_{m/\ell}(T) = 1$.

When $v \in V_f$ does not split in ℓ, and $m \not\leftrightarrow k_v(s) = \ell_{\tilde{v}}$, F'_v is $c_v J_v^{*-1} F_v J_v^{-1}$, where F_v is the image of F in $M_{3\times 3}(k_v(s, Z))$, and $J_v = \Theta C_v$, as before.

Lemma. We can choose $\eta_v \in k_v(s, Z)$ so that $N_{k_v(s,Z)/k_v(s)}(\eta_v) = D$ so that $\bar{\eta}_v = \eta_v$ when $\bar{D} = D$, and so that $\bar{\eta}_v \eta_v = 1$ when $\bar{D}D = 1$.

When the extension $\tilde{v}$ of v to v ramifies in m, we need to make case by case calculations.

Example: $(a = 7, p = 2)$. The 7-adic valuation on $k = \mathbb{Q}$ does not split in $\ell = \mathbb{Q}(s)$ (where $s^2 = -7$) — it ramifies there. Its unique extension to ℓ ($\leftrightarrow s\mathfrak{o}_\ell$) ramifies in $m = \mathbb{Q}(\zeta)$, where $\zeta = \zeta_7$, because $N_{m/\ell}(\zeta - 1) = s$.

We cannot use the norm theorem to show that $D = (3+s)/4$ is a norm $N_{k_v(s,Z)/k_v(s)}(\eta_v)$ when v is the 7-adic valuation.

However:

Hensel's Lemma shows that $16c^3 - 12c - 3 = 0$ holds for a unique $c \in \mathbb{Q}_7$. Then $\eta = c + (8c^2 - 3c - 4)s/7$ is in $\mathbb{Q}_7(s) \subset \mathbb{Q}_7(s,Z) = k_v(s,Z)$ and satisfies $\bar{\eta}\eta = 1$ and $N_{\ell_{\tilde{v}}(Z)/\ell_{\tilde{v}}}(\eta) = \eta^3 = D$.

In cases like this when $\bar{D}D = 1$ and $\bar{\eta}_v\eta_v = 1$, the matrix C_v has inverse C_v^* and so

$$J_v^{*-1}F_vJ_v^{-1} = \Theta^{*-1}C_v^{*-1}F_vC_v^{-1}\Theta^{-1} \text{ equals } \Theta^{*-1}F_v\Theta^{-1}$$

The choice of the basis $\theta_0, \theta_1, \theta_2$ of m over ℓ used to define Θ can be made independent of v. If c_v is constant, the matrix $F'_v = c_vJ_v^{*-1}F_vJ_v^{-1}$ has entries in ℓ and is independent of v.

Example: $(a = 7, p = 2)$. If we choose $\theta_0, \theta_1, \theta_2 = 1, \zeta, \zeta^2$ as before, the $\Theta^{*-1} F_v \Theta^{-1}$ is not in $GL(3, \mathfrak{o}_{\tilde{v}})$ for the 7-adic valuation v. We choose instead $\theta_0 = s$, $\theta_1 = s(\zeta - 1)$ and $\theta_2 = (\zeta - 1)^2$, and find that

$$7 J_v^{*-1} F_v J_v^{-1} = 7\Theta^{*-1} F_v \Theta^{-1} = \begin{pmatrix} 3 & 3 & s \\ 3 & 2 & (1+s)/2 \\ -s & (1-s)/2 & 0 \end{pmatrix},$$

which has entries in $\mathfrak{o}_\ell \subset \mathfrak{o}_{\tilde{v}}$ and determinant 1.

So for the case $(a = 7, p = 2)$, whenever $v \in V_f$ does not split in ℓ, and m does not embed in $k_v(s)$,

$$G(k_v) \cong \{g \in SL(3, k_v(s)) : g^* F' g = F'\}$$

for the above $F' \in SL(3, \mathfrak{o}_\ell)$, and the lattice $\mathfrak{o}_{\tilde{v}}^3 \subset k_v(s)^3$ is a type 1 vertex of the building X_v.

Let $\xi \in \mathcal{D}$ satisfy $\iota(\xi)\xi = 1$. As before, let $b_1, \ldots, b_6$ be a basis of m over k, and write

$$\xi = \sum_{i,j} a_{ij} b_i \sigma^j.$$

Suppose that $v \in V_f$ does not split in ℓ, and let $\xi_v \in U_{F'_v}$ be the image of ξ_v under the inclusion

$$\mathcal{D} \hookrightarrow \mathcal{D} \otimes_\ell k_v(s) \cong M_{3\times 3}(k_v(s)).$$

Proposition. If $\tilde{v}(\det(B)) = 0$ and if $\tilde{v}(\det(\Theta)) = 0$, then

$$\xi_v(\mathfrak{o}_{\tilde{v}}^3) = \mathfrak{o}_{\tilde{v}}^3 \quad \text{iff} \quad a_{ij} \in k \cap \mathfrak{o}_v \text{ for all } i, j.$$

Let us define the principal arithmetic subgroup Λ so that

- $P_v = SL(3, \mathfrak{o}_v)$ whenever $v \in \mathcal{T} \setminus \mathcal{T}_0$ splits in ℓ,

- $P_v = \{g \in SL(3, \mathfrak{o}_{\tilde{v}}) \ : \ g^* F'_v g = F'_v\}$ whenever v does not split in ℓ,

- $P_{v_0} = G(k_{v_0})$ if $\mathcal{T}_0 = \{v_0\}$.

Let $\xi \in G(k) \subset \mathcal{D}$. So $\xi \in \Lambda$ iff

- $\xi_v(\mathfrak{o}_v^3) = \mathfrak{o}_v^3$ whenever $v \in \mathcal{T} \setminus \mathcal{T}_0$ splits in ℓ,

- $\xi(\mathfrak{o}_{\tilde{v}}^3) = \mathfrak{o}_{\tilde{v}}^3$ whenever v does not split in ℓ, Example. $(a = 7, p = 2)$ case. We choose $b_1, \ldots, b_6 = 1, \zeta, \ldots, \zeta^5$ and $\theta_0, \theta_1, \theta_2 = s, s(\zeta - 1), (\zeta - 1)^2$ as before. Then $\tilde{v}(\det(B)) = 0$ and $\tilde{v}(\det(\Theta)) = 0$ for all $v \in V_f \setminus \mathcal{T}_0$ except $v = u_7$. Writing

$$\xi = \sum_{i,j} a_{ij} b_i \sigma^j,$$

$\xi_v \in P_v$ for all $v \neq u_2, u_7 \Leftrightarrow a_{ij} \in \mathbb{Q} \cap \mathbb{Z}_p$ for all primes $p \neq 2, 7$
$\Leftrightarrow a_{ij} \in \mathbb{Z}[1/2, 1/7]$ for each i, j.

When $v = u_2$, the condition $\xi_v \in P_v$ always holds.

In the case $v = u_7$, when is $\xi_v \in P_v$?

Taking $v = u_7$, ξ_v equals $J_v \Psi(\xi) J_v^{-1}$, where $J_v = \Theta C_v$. This is a matrix with entries in $k_v(s) = \mathbb{Q}_7(s)$, and we can write

$$\xi_v = \begin{pmatrix} x_{11} + y_{11}s & x_{12} + y_{12}s & x_{13} + y_{13}s \\ x_{21} + y_{21}s & x_{22} + y_{22}s & x_{23} + y_{23}s \\ x_{31} + y_{31}s & x_{32} + y_{32}s & x_{33} + y_{33}s \end{pmatrix}$$

where $x_{ij}, y_{ij} \in \mathbb{Q}_7$ for each i, j. Each of these is a linear combination of the coefficients a_{ij}'s of ξ. So we can write

$$\boldsymbol{x} = M\boldsymbol{a},$$

where $\boldsymbol{a}$ and $\boldsymbol{x}$ are column vectors of length 18, made from the coefficients a_{ij} and from the numbers x_{ij} and y_{ij}, and where M is an 18×18 matrix with entries in $\mathbb{Q}_7$. In this case, the entries of M are explicit polynomials in the $c \in \mathbb{Q}_7$ used in solving $\mathsf{N}_{\mathbb{Q}_7(s,Z)/\mathbb{Q}_7(s)}(\eta) = D$.

$$\xi_v(\mathfrak{o}_{\tilde{v}}^3) = \mathfrak{o}_{\tilde{v}}^3 \quad \Leftrightarrow \quad \xi_v(\mathfrak{o}_{\tilde{v}}^3) \subset \mathfrak{o}_{\tilde{v}}^3 \quad \Leftrightarrow \quad \xi_v \text{ has entries in } \mathfrak{o}_{\tilde{v}}.$$

In this case, $\mathfrak{o}_{\tilde{v}} = \{x + ys : x, y \in \mathbb{Z}_7\} \subset \mathbb{Q}_7(s)$, and so

$$\xi_v(\mathfrak{o}_{\tilde{v}}^3) = \mathfrak{o}_{\tilde{v}}^3 \quad \Leftrightarrow \quad x_{ij}, y_{ij} \in \mathbb{Z}_7 \text{ for all } i, j \quad \Leftrightarrow \quad \boldsymbol{x} = M\boldsymbol{a} \text{ has entries in } \mathbb{Z}_7.$$

If $L \in GL(18, \mathbb{Z}_7)$, then

$$M\boldsymbol{a} \text{ has entries in } \mathbb{Z}_7 \quad \Leftrightarrow \quad LM\boldsymbol{a} \text{ has entries in } \mathbb{Z}_7.$$

We can choose $L \in GL(18, \mathbb{Z}_7)$ so that $LM = \mathcal{E}$ is in "reduced row echelon form". Then

$$\xi_v(\mathfrak{o}_{\tilde{v}}^3) = \mathfrak{o}_{\tilde{v}}^3 \quad \Leftrightarrow \quad \mathcal{E}\boldsymbol{a} \text{ has entries in } \mathbb{Z}_7.$$

We only need a 7-adic approximation M_7 (mod 49 is enough) to M to get $\mathcal{E}$. The following Magma commands give us $\mathcal{E}$.

M_7:=`Matrix(IntegerRing(49),18,18,[...]);`

$\mathcal{E}$:=`EchelonForm`(M_7);

We used the order

$$a_{10}, \ldots, a_{60}, a_{11}, \ldots, a_{61}, a_{12}, \ldots, a_{62}$$

for the coefficients of ξ.

We get:

$$\mathcal{E} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 3 & 6 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 6 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 3 & 5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 6 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 5 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 6 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7
\end{pmatrix}.$$

To summarize:

For each $v \in V_f \setminus \mathcal{T}_0$ which splits in ℓ, let x_v be the vertex $[\mathfrak{o}_v^3]$ of X_v.

For each $v \in V_f$ which does not split in ℓ, let x_v be the type 1 vertex $\mathfrak{o}_{\tilde{v}}^3$ of X_v.

Let $\Lambda = \{\xi \in G(k) : \xi_v.x_v = x_v \text{ for all } v \in V_f \setminus \mathcal{T}_0\}$.

This is a principal arithmetic subgroup in which each P_v is maximal, and no x_v's are of type 2. So the set we call $\mathcal{T}_1$ is $\emptyset$ here.

The elements ξ of Λ are the

$$\xi = \sum_{i=1}^{6} \sum_{j=0}^{2} a_{i,j} \zeta^{i-1} \sigma^j \in \mathcal{D}$$

such that $\iota(\xi)\xi = 1$, $\mathrm{Nrd}(\xi) = 1$, $a_{i,j} \in \mathbb{Z}[1/2, 1/7]$ for all i, j, and such that $\mathcal{E}\boldsymbol{a}$ has entries in $\mathbb{Z}_7$.

We want to find elements not only of Λ but of its normalizer Γ in $SU(2,1)$. Recall equation $(*)$:

$$3^{\alpha-1} d_{k,\ell} = [\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T}} e'(P_v)$$

In the case $(a = 7, p = 2)$, with $\mathcal{T}_1 = \emptyset$, $\mathcal{T}$ equals $\mathcal{T}_0 = \{2\}$, and the term $e'(P_v)$ for the 2-adic v is $(2-1)^2(2+1) = 3$. Also, $d_{k,\ell} = 21$ and $\alpha = 2$. We get $[\bar{\Gamma} : \Pi] = 21$.

We can identify $\bar{\Gamma}$ with

$$\begin{aligned}\{\xi = \sum a_{i,j} \zeta^{i-1} \sigma^j \in \mathcal{D} \ :\ & \iota(\xi)\xi = 1, \\ & a_{i,j} \in \mathbb{Z}[1/2, 1/7] \text{ for all } i, j, \\ & \mathcal{E}\boldsymbol{a} \text{ has entries in } \mathbb{Z}_7\}/\mathcal{Z},\end{aligned}$$

where $\mathcal{Z} = \{t1 : t \in \ell \ \&\ \bar{t}t = 1\}$.