Constructing the algebras $\mathcal{D}$ and the involutions $\iota$.

To a fake projective plane there is associated a pair $(k, \ell)$ of fields coming from a short list. There is also an algebra $\mathcal{D}$, an involution $\iota$ and a group $G$, with

$$G(k) = \{\xi \in \mathcal{D} : \iota(\xi)\xi = 1 \& \text{Nrd}(\xi) = 1\}.$$ 

$\mathcal{D}$, $\iota$ and $G$ must satisfy the properties:
• $G(k_v) \cong SL(3, k_v)$ for all $v \in V_f \setminus \mathcal{T}_0$ which split in $\ell$,

• $G(k_v) \cong \{g \in SL(3, k_v(s)) : g^*F_v g = F_v\}$ if $v \in V_f$ does not split in $\ell$,

• $G(k_v)$ is compact for $v \in \mathcal{T}_0$,

• $G(k_v) \cong SU(2, 1)$ for one archimedean place $v$ on $k$, and

• $G(k_v) \cong SU(3)$ for the other archimedean place $v$ on $k$ (if $k \neq \mathbb{Q}$).

We know that $\mathcal{T}_0 = \emptyset$ if $D = M_{3 \times 3}(\ell)$, and that $\mathcal{T}_0$ is a specific singleton if $D$ is a division algebra.
Corollary 6.6 in Chapter 10 of W. Scharlau “Quadratic and Hermitian Forms”:

The above properties determine $\mathcal{D}$ and $\iota$ up to $k$-isomorphism or anti-isomorphism.

Anti-isomorphism must be allowed here because given $\mathcal{D}$, we can define an “opposite” algebra $\mathcal{D}^{\text{op}}$ whose elements $x^{\text{op}}$ are in 1-1 correspondence ($x^{\text{op}} \leftrightarrow x$) with those of $\mathcal{D}$, and in which for all $x, y \in \mathcal{D}$ and $t \in \ell$,

$$x^{\text{op}} + y^{\text{op}} = (x + y)^{\text{op}}, \quad tx^{\text{op}} = (tx)^{\text{op}}, \quad \text{and} \quad x^{\text{op}}y^{\text{op}} = (yx)^{\text{op}}.$$
Suppose that $\mathbb{F}$ is a field and that $m$ is a Galois extension of $\mathbb{F}$ of degree 3, with $\text{Gal}(m/\mathbb{F}) = \langle \varphi \rangle$.

Fix some nonzero $D \in \mathbb{F}$, and form

$$D = \{ a + b\sigma + c\sigma^2 : a, b, c \in m \},$$

which we can make into an associative algebra of dimension 9 over $\mathbb{F}$ in which

$$\sigma^3 = D \quad \text{and} \quad \sigma a = \varphi(a)\sigma \quad \text{for all } a \in m.$$ 

The centre of $D$ is $\mathbb{F}$. We shall see in a moment that $D$ has no non-trivial two-sided ideals — $D$ is a central simple algebra.
There is an ℓ-algebra homomorphism Ψ : \( D \rightarrow M_{3 \times 3}(m) \) such that

\[
\psi(\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ D & 0 & 0 \end{pmatrix} \quad \text{and} \quad \psi(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & \varphi(a) & 0 \\ 0 & 0 & \varphi^2(a) \end{pmatrix}.
\]

So if \( \xi = a + b\sigma + c\sigma^2 \in D \), then

\[
\psi(\xi) = \begin{pmatrix} a & b & c \\ D\varphi(c) & \varphi(a) & \varphi(b) \\ D\varphi^2(b) & D\varphi^2(c) & \varphi^2(a) \end{pmatrix}.
\]
The reduced norm $\text{Nrd}(\xi)$ of $\xi$ is

$$
det(\Psi(\xi)) = a\varphi(a)\varphi^2(a) + Db\varphi(b)\varphi^2(b) + D^2c\varphi(c)\varphi^2(c)
- D(a\varphi(b)\varphi^2(c) + \varphi(a)\varphi^2(b)c + \varphi^2(a)b\varphi(c)).
$$

Then $\text{Nrd} : D \to \ell$, and $\text{Nrd}(\xi\eta) = \text{Nrd}(\xi)\text{Nrd}(\eta)$ for all $\xi, \eta \in D$.

An element $\xi = a + b\sigma + c\sigma^2$ of $D$ is invertible if and only if $\text{Nrd}(\xi) \neq 0$, in which case $\xi^{-1}$ equals

$$
\frac{1}{\text{Nrd}(\xi)}\left((\varphi(a)\varphi^2(a) - D\varphi(b)\varphi^2(c)) + (Dc\varphi^2(c) - b\varphi^2(a))\sigma + (b\varphi(b) - c\varphi(a))\sigma^2\right).
$$
Proposition. Either

(a) $\mathcal{D} \cong M_{3 \times 3}(\ell)$, or

(b) $\mathcal{D}$ is a division algebra.

Case (a) holds if and only if $D$ is the norm $N_{m/\ell}(\eta)$ of an element $\eta$ of $m$.

Proof. If $D = N_{m/\ell}(\eta)$, let

$$C = \begin{pmatrix} \eta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\varphi(\eta) \end{pmatrix}.$$
Let $\theta_0$, $\theta_1$ and $\theta_2$ be basis for $m$ over $\ell$. Form

$$\Theta = \begin{pmatrix} \theta_0 & \varphi(\theta_0) & \varphi^2(\theta_0) \\ \theta_1 & \varphi(\theta_1) & \varphi^2(\theta_1) \\ \theta_2 & \varphi(\theta_2) & \varphi^2(\theta_2) \end{pmatrix}.$$ 

Then

$$\Theta^{-1} = \begin{pmatrix} \zeta_0 & \zeta_1 & \zeta_2 \\ \varphi(\zeta_0) & \varphi(\zeta_1) & \varphi(\zeta_2) \\ \varphi^2(\zeta_0) & \varphi^2(\zeta_1) & \varphi^2(\zeta_2) \end{pmatrix},$$

where $\text{Trace}(\theta_i \zeta_j) = \delta_{ij}$.

$J := \Theta C$. Then $J\Psi(\xi)J^{-1}$ has entries in $\ell$.

E.g., $(J\Psi(\sigma)J^{-1})_{ij} = \text{Trace}(\theta_i \eta \varphi(\zeta_j))$.

So $\xi \mapsto J\Psi(\xi)J^{-1}$ is a $\ell$-linear algebra homomorphism $\mathcal{D} \to M_{3 \times 3}(\ell)$. It is clearly injective, and so an isomorphism, as dimensions match.
If $D$ is not equal to $N_{m/\ell}(\eta)$ for any $\eta \in m$, then

$$\text{Nrd}(1 + b\sigma) = 1 + DN_{m/\ell}(b) \quad \text{and} \quad \text{Nrd}(1 + c\sigma^2) = 1 + D^2N_{m/\ell}(c)$$
cannot be zero for any $b, c \in m$. So any $1 + b\sigma$ or $1 + c\sigma^2$ is invertible. So any nonzero element of $\mathcal{D}$ is invertible, and $\mathcal{D}$ is a division algebra.

**Corollary.** $\mathcal{D}$ is a central simple algebra over $\ell$. 
Suppose \( \ell = k(s) \), where \( s^2 = -\kappa \in k \).

We want an involution \( \iota \) of the second kind on \( \mathcal{D} \).

Assume \( m \) normal extension of \( k \). The conjugation automorphism extends to \( m \). Then either

\[
\varphi(a) = \varphi(\bar{a}) \quad \text{for all } a \in m \quad \text{OR} \quad \varphi(a) = \varphi^2(\bar{a}) \quad \text{for all } a \in m.
\]

\( \text{Gal}(m/k) \) is abelian in the first case, and non-abelian in the second case.
Lemma. If $\text{Gal}(m/k)$ is non-abelian, and if $D \in \ell$ satisfies $\overline{D} = D \neq 0$, then there is an involution of the second kind $\iota : D \to D$ such that

$$\iota(\sigma) = \sigma \quad \text{and} \quad \iota(a) = \bar{a} \quad \text{for all } a \in m.$$ 

Explicitly,

$$\iota(a + b\sigma + c\sigma^2) = \bar{a} + \varphi(\bar{b})\sigma + \varphi^2(\bar{c})\sigma^2.$$ 

Note that

$$\Psi(\iota(\xi)) = F^{-1}\Psi(\xi)^*F \quad \text{for} \quad F = \begin{pmatrix} D & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$
In each of the five $k = \mathbb{Q}$ cases $(a = 1, p = 5),..., (a = 23, p = 2)$, we can choose a field $m$ as above, with $\text{Gal}(m/k)$ non-abelian, and define $D$ using $D = p$. In each of these cases, there is a $\beta \in \ell$ so that $\bar{\beta}\beta = 2p$. Then

$$F = \frac{1}{2} \Delta^* F_0 \Delta$$  for  $\Delta = \begin{pmatrix} \beta & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.

If $\iota(\xi)\xi = 1$ then

$$(\Delta \psi(\xi)\Delta^{-1})^* F_0 (\Delta \psi(\xi)\Delta^{-1}) = F_0.$$
So if
\[ G(k) = \{ \xi \in D : \iota(\xi)\xi = 1 \text{ and } \text{Nrd}(\xi) = 1 \}, \]
then \( \xi \mapsto \Delta\psi(\xi)\Delta^{-1} \) defines an injective homomorphism \( G(k) \to SU(2, 1) \).

In fact, \( G(k_v) \cong G(\mathbb{R}) \cong SU(2, 1) \) for the one archimedean place \( v \) on \( k = \mathbb{Q} \) — see below.

We can take \( \beta = 3 + i \) in the case \((a = 1, p = 5)\), \( \beta = 2 + s \) for \((a = 2, p = 3)\) and \( \beta = 2 \) for the other three cases.
In the 5 cases \((k, \ell)\) in which \(k = \mathbb{Q}\) and \(\ell = \mathbb{Q}(s)\), we can define \(m = \mathbb{Q}(s, Z)\), where \(Z\) satisfies \(P(Z) = 0\) for a cubic monic \(P(X) \in \mathbb{Z}[X]\).

<table>
<thead>
<tr>
<th>(s^2)</th>
<th>(p)</th>
<th>(P(X))</th>
<th>(\varphi(Z))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>5</td>
<td>(X^3 - 3X^2 - 2)</td>
<td>((s + 3 - (4s + 1)Z + sZ^2)/2)</td>
</tr>
<tr>
<td>(-2)</td>
<td>3</td>
<td>(X^3 + X^2 + 2X - 2)</td>
<td>((2(s - 1) + (3s - 2)Z + sZ^2)/4)</td>
</tr>
<tr>
<td>(-7)</td>
<td>2</td>
<td>(X^3 + 3X^2 + 3)</td>
<td>(-(3(s + 7) + (9s + 7)Z + 2sZ^2)/14)</td>
</tr>
<tr>
<td>(-15)</td>
<td>2</td>
<td>(X^3 - 3X - 3)</td>
<td>((4s + (3s - 5)Z - 2sZ^2)/10)</td>
</tr>
<tr>
<td>(-23)</td>
<td>2</td>
<td>(X^3 - X - 1)</td>
<td>((4s + (9s - 23)Z - 6sZ^2)/46)</td>
</tr>
</tbody>
</table>

In each case \(\text{Gal}(m/\ell) = \langle \varphi \rangle\), and \(\text{Gal}(m/\mathbb{Q})\) is non-abelian.

In the case \((a = 7, p = 2)\), we shall use a different cyclic simple algebra, coming from a field \(m\) so that \(\text{Gal}(m/k)\) is abelian.
Lemma. If \( \text{Gal}(m/k) \) is abelian, and if \( D \in \ell \) satisfies \( \bar{D}D = 1 \), then there is an involution \( \iota_0 : D \to D \) of the second kind such that
\[
\iota_0(\sigma) = \sigma^{-1} \quad \text{and} \quad \iota_0(a) = \bar{a} \quad \text{for all} \quad a \in m.
\]
Explicitly,
\[
\iota_0(a + b\sigma + c\sigma^2) = \bar{a} + \bar{D}\varphi(\bar{c})\sigma + \bar{D}\varphi^2(\bar{b})\sigma^2.
\]
It is easy to check that
\[
\Psi(\iota_0(\xi)) = \Psi(\xi)^*.
\]
For reasons explained on the next slide, we shall use the involution
\[
\iota(\xi) = T^{-1}\iota_0(\xi)T,
\]
where \( T \in m \) and \( \bar{T} = T \neq 0 \). Then
\[
\Psi(\iota(\xi)) = F^{-1}\Psi(\xi)^*F \quad \text{for} \quad F = \begin{pmatrix} T & 0 & 0 \\ 0 & \varphi(T) & 0 \\ 0 & 0 & \varphi^2(T) \end{pmatrix}.
\]
Embedding $m$ in $C$, the images of $T$, $\varphi(T)$ and $\varphi^2(T)$ are real because $\overline{\varphi(T)} = \varphi(\overline{T}) = \varphi(T)$.

If $T > 0$, $\varphi(T) > 0$ and $\varphi^2(T) < 0$, then

$$F = \Delta^* F_0 \Delta \quad \text{for} \quad \Delta = \begin{pmatrix} |T|^{1/2} & 0 & 0 \\ 0 & |\varphi(T)|^{1/2} & 0 \\ 0 & 0 & |\varphi^2(T)|^{1/2} \end{pmatrix}.$$ 

So $\xi \mapsto \Delta \Psi(\xi) \Delta^{-1}$ is an injective homomorphism $G(k) \rightarrow SU(2, 1)$.

In the cases $(a = 7, p = 2)$, $C_2$, $C_{10}$, $C_{18}$ and $C_{20}$, we define

$$D = \{ a + b\sigma + c\sigma^2 : a, b, c \in m, \text{ where } \sigma^3 = D \text{ and } \sigma x \sigma^{-1} = \varphi(x) \}$$

for the following $m$’s and $D$’s:
<table>
<thead>
<tr>
<th>name</th>
<th>$m$</th>
<th>$v_0$</th>
<th>$D$</th>
<th>$\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a = 7, p = 2)$</td>
<td>$\mathbb{Q}(\zeta_7)$</td>
<td>2</td>
<td>$(3 + s)/4$</td>
<td>$\zeta_7 \mapsto \zeta_7^2$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$k(\zeta_9)$</td>
<td>2</td>
<td>$(1 + \sqrt{-15})/4$</td>
<td>$\zeta_9 \mapsto \zeta_9^4$</td>
</tr>
<tr>
<td>$C_{10}$</td>
<td>$\ell(W)$</td>
<td>2</td>
<td>$rU/2$</td>
<td>$W \mapsto 2 - W - W^2$</td>
</tr>
<tr>
<td>$C_{18}$</td>
<td>$k(\zeta_9)$</td>
<td>3</td>
<td>$(r + 1 + 2\omega)/3$</td>
<td>$\zeta_9 \mapsto \zeta_9^4$</td>
</tr>
<tr>
<td>$C_{20}$</td>
<td>$k(\zeta_7)$</td>
<td>2</td>
<td>$(3 + \sqrt{-7})/4$</td>
<td>$\zeta_7 \mapsto \zeta_7^2$</td>
</tr>
</tbody>
</table>

In case $C_2$, $k = \mathbb{Q}(r)$, where $r^2 = 5$ and $\ell = k(\omega)$, where $\omega = \zeta_3$.

In case $C_{10}$, $k = \mathbb{Q}(r)$, where $r^2 = 2$, $\ell = k(U)$, where $U^2 = (r + 1)U - 2$, and $W^3 - 3W + 1 = 0$.

In case $C_{18}$, $k = \mathbb{Q}(r)$, where $r^2 = 6$, and $\ell = k(\omega)$.

In case $C_{20}$, $k = \mathbb{Q}(r)$ where $r^2 = 7$, and $\ell = k(i)$. 
Having chosen $m$ as above, with $\text{Gal}(m/k)$ abelian, the subfield $\{ a \in m : \bar{a} = a \}$ has the form $k(W)$, where $W$ satisfies an equation $Q(W) = 0$ for some monic cubic $Q(X) \in \mathbb{Z}[X]$. We choose $T \in k(W)$ as follows:

<table>
<thead>
<tr>
<th>name</th>
<th>$W$</th>
<th>$\varphi(W)$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a = 7, p = 2)$</td>
<td>$\zeta_7 + \zeta_7^{-1}$</td>
<td>$W^2 - 2$</td>
<td>$W$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$\zeta_9 + \zeta_9^{-1}$</td>
<td>$2 - W - W^2$</td>
<td>$-2r + (r - 1)W + 2W^2$</td>
</tr>
<tr>
<td>$C_{10}$</td>
<td>$W$</td>
<td>$2 - W - W^2$</td>
<td>$-r + (1 - r)W + W^2$</td>
</tr>
<tr>
<td>$C_{18}$</td>
<td>$\zeta_9 + \zeta_9^{-1}$</td>
<td>$2 - W - W^2$</td>
<td>$3 - 3r + rW + rW^2$</td>
</tr>
<tr>
<td>$C_{20}$</td>
<td>$\zeta_7 + \zeta_7^{-1}$</td>
<td>$W^2 - 2$</td>
<td>$2 + W - (4 + 3W + W^2)/r$</td>
</tr>
</tbody>
</table>

In the first and last cases, $Q(X) = X^3 + X^2 - 2X - 1$. In the other three cases, $Q(X) = X^3 - 3X + 1$. 
The above choice is made so that, in the four cases $k = \mathbb{Q}(r)$, where $r^2 = N$ ($N = 5, 2, 6$ or $7$), and fixing a solution $W_R \in \mathbb{R}$ of $Q(X) = 0$,

- embedding $k(W)$ in $\mathbb{R}$ by mapping $r$ to $+\sqrt{N}$ and $W$ to $W_R$, the images of $T$, $\varphi(T)$ and $\varphi(T)$ DO NOT all have the same sign, and

- embedding $k(W)$ in $\mathbb{R}$ by mapping $r$ to $-\sqrt{N}$ and $W$ to $W_R$, the images of $T$, $\varphi(T)$ and $\varphi(T)$ DO all have the same sign.

This implies that $G(k_\nu) \cong SU(2, 1)$ for the archimedean valuation $\nu$ corresponding to the first embedding, and $G(k_\nu) \cong SU(3)$ for the archimedean valuation $\nu$ corresponding to the second embedding.
Example. The case \((a = 7, p = 2)\).

Let \(k = \mathbb{Q}\) and \(\ell = \mathbb{Q}(s)\), where \(s^2 = -7\). Let \(m\) be the cyclotomic field \(\mathbb{Q}(\zeta)\), where \(\zeta\) is a primitive 7-th root of 1. Let

\[
s = 1 + 2\zeta + 2\zeta^2 + 2\zeta^4.
\]

Then \(s^2 = -7\). So \(\ell \subset m\).

Now \(\text{Gal}(m/\mathbb{Q})\) is cyclic, generated by \(\chi : \zeta \mapsto \zeta^3\), and \(\phi = \chi^2 : \zeta \mapsto \zeta^2\) generates \(\text{Gal}(m/\ell)\). Form the cyclic algebra \(D\) with this \(m\) and \(\phi\), and with

\[
D = \frac{3 + s}{4}.
\]

Notice that \(\overline{DD} = 1\). Let’s check that \(D\) is not the norm \(N_{m/\ell}(\eta)\) of any element \(\eta\) of \(m\).
The prime $2$ splits in $\ell$, as $2 = \rho \bar{\rho}$ for $\rho = (1 - s)/2 \in \sigma_\ell$.

$D = (3 + s)/4$ equals $-\rho/\bar{\rho}$, so $w(D) = +1$ and $\bar{w}(D) = -1$, where $w \leftrightarrow \rho_\ell$ and $\bar{w} \leftrightarrow \bar{\rho}_\ell$.

Alternatively, $\mathbb{Q}_2$ contains a square root $s_2 = 1 + 0 \times 2 + 1 \times 2^2 + 0 \times 2^3 + \cdots$ of $-7$, and $v_\epsilon(a + bs) = u_2(a + \epsilon bs_2)$, for $\epsilon = \pm 1$, define two distinct extensions to $\ell$ of the $2$-adic valuation $u_2$ on $\mathbb{Q}$.

Then $v_+(D) = 1$ and $v_-(D) = -1$. So $w = v_+$ and $\bar{w} = v_-$.

Magma verifies that $\rho_m$ is prime. So $w$ has a unique extension $\tilde{w}$ to $m$. Then $\tilde{w}(\eta) = \tilde{w}(\varphi(\eta)) = \tilde{w}(\varphi^2(\eta)) \in \mathbb{Z}$ for $\eta \in m$. If $D = N_{m/\ell}(\eta)$, then

$$1 = w(D) = \tilde{w}(D) = \tilde{w}(\eta \varphi(\eta) \varphi^2(\eta)) = 3\tilde{w}(\eta),$$

a contradiction.
We choose $T = \zeta + \zeta^{-1}$, and use the involution

$$
\iota(\xi) = T^{-1} \iota_0(\xi) T, \text{ where } \iota_0(\sigma) = \sigma^{-1}, \text{ and } \iota_0(a) = \bar{a} \text{ for } a \in m.
$$

Then

$$
\Psi(\iota(\xi)) = F^{-1} \Psi(\xi)^* F \text{ for } F = \begin{pmatrix}
T & 0 & 0 \\
0 & \varphi(T) & 0 \\
0 & 0 & \varphi^2(T)
\end{pmatrix}.
$$

Embed $m$ into $\mathbb{C}$, mapping $\zeta$ to $e^{2\pi i / 7}$. Then $T > 0$, $\varphi(T) = T^2 - 2 < 0$ and $\varphi^2(T) = 1 - T - T^2 < 0$. Let

$$
\Delta = \begin{pmatrix}
0 & 0 & |\varphi^2(T)|^{1/2} \\
0 & |\varphi(T)|^{1/2} & 0 \\
|T|^{1/2} & 0 & 0
\end{pmatrix}.
$$

Then $\Delta^* F_0 \Delta = -F$, so $g^* F g = F$ iff $\tilde{g} = \Delta g \Delta^{-1}$ satisfies $\tilde{g}^* F_0 \tilde{g} = F_0$. 
So far: for each of the 9 pairs \((k, \ell)\), we have defined

- a cyclic algebra \(D\),
- an involution \(\iota\) on \(D\).

In each case, \(D\) is a division algebra. The proof: as in the case \((a = 7, p = 2)\), using \(w(D) \neq 0\) for the two extensions \(w\) of the \(v_0 \in T_0\).

Tricky case: \(C_{18}\). Here \(v_0\) is the one 3-adic valuation on \(k = \mathbb{Q}(r), r^2 = 6\). This splits in \(\ell\), and the two extensions \(w\) of \(v_0\) *ramify* in \(m\).
We need to find $G(k_v)$ for the various places $v$ of $k$.

We start from one of our nine $(D, \iota)$.

If $K$ is a field containing $k$, then by definition,

$$G(K) = \{\xi \in D \otimes_k K : \iota_K(\xi)\xi = 1 \text{ and } \text{Nrd}_K(\xi) = 1\}$$

(see §1.2 in [PY]). Because $\iota : D \to D$ is $k$-linear, it induces a unique $K$-linear map $\iota_K : D \otimes_k K \to D \otimes_k K$. It is an anti-automorphism.

We can define $\text{Nrd}_K$ using $\Psi : D \to M_{3 \times 3}(m)$. This induces $\Psi_K : D \otimes_k K \to M_{3 \times 3}(m \otimes_k K)$, and we set $\text{Nrd}_K(\xi) = \det(\Psi_K(\xi))$. Then $\text{Nrd}_K : D \otimes_k K \to \ell \otimes_k K$, and it is multiplicative.
We also need the adjoint group $\tilde{G}$. We can define this by setting

$$\tilde{G}(K) = \{ \alpha \in \text{Aut}_\ell(D \otimes_k K) : \iota_K \circ \alpha = \alpha \circ \iota_K \}.$$ 

for any field $K$ containing $k$. Here $\alpha \in \text{Aut}_\ell(D \otimes_k K)$ means that $\alpha$ is an automorphism of the $K$-algebra $D \otimes_k K$ which is also $\ell$-linear.

Fact:

$$\tilde{G}(k) \cong \{ \xi \in D : \iota(\xi) \xi = 1 \}/\{ t1 : t \in \ell & \bar{t}t = 1 \}.$$ 

This is because any automorphism of $D$ is of the form $\alpha : \eta \mapsto \xi \eta \xi^{-1}$ for some invertible $\xi \in D$ (Skolem-Noether Theorem). Then $\iota \circ \alpha = \alpha \circ \iota$ means that $\iota(\xi) \xi = c1$ for some $c \in k$. Let $\xi' = c\xi/\text{Nrd}(\xi)$. Then $\alpha(\eta) = \xi' \eta \xi'^{-1}$ for all $\eta$, and $\iota(\xi') \xi' = 1$. 

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The form of $G(k_v)$ depends on whether or not $v$ splits in $\ell = k(s)$. Recall that $s^2 = -\kappa$ for some $\kappa \in k$, and $v$ splits in $\ell \iff \ell$ embeds in $k_v \iff -\kappa$ has a square root in $k_v$.

For any field $K$ containing $k$,

(i) If $\ell \hookrightarrow K$, then $\ell \otimes_k K \cong K \oplus K$.

(ii) If $\ell \not\hookrightarrow K$, then $\ell \otimes_k K \cong K(s)$, a field.

In (i), the isomorphism is $1 \otimes x + s \otimes y \mapsto (x + ys_K, x - ys_K)$, where $s_K$ is the image of $s$ under an embedding $\ell \to K$.

In (ii), the isomorphism is $1 \otimes x + s \otimes y \mapsto x + ys$. 

Let $\mathcal{D}$ denote one of our 9 division algebras $\mathcal{D}$.

**Proposition.** If $v \in V_f$ splits in $\ell$, then either

(1) $G(k_v) \cong SL(3, k_v)$, $\bar{G}(k_v) \cong PGL(3, k_v)$, and $\mathcal{D} \otimes_\ell k_v \cong M_{3 \times 3}(k_v)$, or

(2) $G(k_v)$ and $\bar{G}(k_v)$ are compact, and $\mathcal{D} \otimes_\ell k_v$ is a division algebra,

and (2) only happens for the one $v \in T_0$.

We heavily use the explicit form of these isomorphisms, so give some details of the proof.
As $\ell \subset k_v$, we can form $D \otimes_\ell k_v$. The map $\xi \mapsto (\xi, \iota(\xi)^{\text{op}})$ is $k$-linear $D \to D \oplus D^{\text{op}}$, and so induces

$$h : D \otimes_k k_v \to (D \otimes_\ell k_v) \oplus (D \otimes_\ell k_v)^{\text{op}},$$

and we get the commutative diagram

$$\begin{array}{ccc}
D \otimes_k k_v & \xrightarrow{h} & (D \otimes_\ell k_v) \oplus (D \otimes_\ell k_v)^{\text{op}} \\
\iota_k & & \downarrow \quad \quad \quad \downarrow (x, y^{\text{op}}) \mapsto (y, x^{\text{op}}) \\
D \otimes_k k_v & \xrightarrow{h} & (D \otimes_\ell k_v) \oplus (D \otimes_\ell k_v)^{\text{op}}
\end{array}$$

and the map $h$ is an isomorphism of $k_v$-algebras.
We also get a commutative diagram

\[
\begin{array}{ccc}
D \otimes_k k_v & \xrightarrow{h} & (D \otimes \ell k_v) \oplus (D \otimes \ell k_v)^{op} \\
Nrd_{k_v} \downarrow & & \downarrow (x, y^{op}) \mapsto (\text{Nrd}(x), \text{Nrd}(y)) \\
\ell \otimes_k k_v & \xrightarrow{f} & k_v \oplus k_v \\
\end{array}
\]

where \( f(1 \otimes_k x + s \otimes_k y) = (x + s_{vy}, x - s_{vy}) \), where \( s_v \in \ell \) is the image of \( s \).

So if \( \xi \in D \otimes_k k_v \) and \( h(\xi) = (x, y^{op}) \), then

\[ \iota_{k_v}(\xi) \xi = 1 \iff yx = 1, \quad \text{and} \]
\[ \text{Nrd}_{k_v}(\xi) = 1 \iff \text{Nrd}(x) = \text{Nrd}(y) = 1. \]
Corollary. If \( v \) splits in \( \ell \), then

\[
G(k_v) \cong \{ x \in \mathcal{D} \otimes_k k_v : \text{Nrd}(x) = 1 \},
\]

\[
\{ \xi \in \mathcal{D} \otimes_k k_v : \iota_{k_v}(\xi)\xi = 1 \} \cong (\mathcal{D} \otimes_\ell k_v)^\times.
\]

and

\[
\bar{G}(k_v) \cong (\mathcal{D} \otimes_\ell k_v)^\times / k_v^\times.
\]

Note that \( \mathcal{D} \otimes_\ell k_v \) is a central simple algebra of dimension 9 over \( k_v \), and so is isomorphic to \( M_{3 \times 3}(k_v) \) or is a division algebra.
Recall embedding $\Psi : \mathcal{D} \to M_{3 \times 3}(m)$.

**Case a:** the embedding $\ell \hookrightarrow k_v$ extends to an embedding $m \hookrightarrow k_v$.

Then

$$
\mathcal{D} \xrightarrow{\Psi} M_{3 \times 3}(m) \hookrightarrow M_{3 \times 3}(k_v)
$$

induces isomorphisms

$$
\mathcal{D} \otimes_\ell k_v \cong M_{3 \times 3}(k_v), \quad G(k_v) \cong SL(3, k_v), \quad \text{and} \quad \bar{G}(k_v) \cong PGL(3, k_v).
$$

Moreover

$$
\{ \xi \in \mathcal{D} \otimes_k k_v : \iota_{k_v}(\xi)\xi = 1 \} \cong GL(3, k_v).
$$

So we get an embedding $\xi \mapsto \xi_v$ of $\{ \xi \in \mathcal{D} : \iota(\xi)\xi = 1 \}$ in $GL(3, k_v)$ which maps $G(k)$ into $SL(3, k_v)$. Here $\xi_v$ is the image of $\Psi(\xi)$ in $M_{3 \times 3}(k_v)$.
Case b: When $m = k(s, Z)$ does not embed in $k_v$, then $m \otimes \ell k_v \cong k_v(Z)$ is a field which is a cubic Galois extension of $k_v$, and $D \otimes \ell k_v$ is the cyclic simple algebra $\{a + b\sigma + c\sigma^2 : a, b, c \in k_v(Z)\}$.

When $D = N_{k_v(Z)/k_v}(\eta_v)$ of some $\eta_v \in k_v(Z)$, $D \otimes \ell k_v \cong M_{3 \times 3}(k_v)$, the isomorphism induced by

$$D \xrightarrow{\psi} M_{3 \times 3}(m) \hookrightarrow M_{3 \times 3}(k_v(s)) \xrightarrow{J_v \cdot J_v^{-1}} M_{3 \times 3}(k_v(s)),$$

where $J_v = \Theta C_v$ for

$$C_v = \begin{pmatrix} \eta_v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\varphi(\eta_v) \end{pmatrix} \quad \text{and} \quad \Theta = \begin{pmatrix} \theta_0 & \varphi(\theta_0) & \varphi^2(\theta_0) \\ \theta_1 & \varphi(\theta_1) & \varphi^2(\theta_1) \\ \theta_2 & \varphi(\theta_2) & \varphi^2(\theta_2) \end{pmatrix},$$

and $\theta_0, \theta_1, \theta_2$ is a basis of $m$ over $\ell$. 
We again have isomorphisms

$$\mathcal{D} \otimes_k k_v \cong M_{3 \times 3}(k_v), \quad G(k_v) \cong SL(3, k_v), \quad \bar{G}(k_v) \cong PGL(3, k_v),$$

and

$$\{\xi \in \mathcal{D} \otimes_k k_v : \iota_{k_v}(\xi)\xi = 1\} \cong GL(3, k_v).$$

Again we have an embedding $\xi \mapsto \xi_v$ of $\{\xi \in \mathcal{D} : \iota(\xi)\xi = 1\}$ in $GL(3, k_v)$ mapping $G(k)$ into $SL(3, k_v)$. Now $\xi_v$ is the image of $J_v \psi(\xi) J_v^{-1}$ in $M_{3 \times 3}(k_v)$. 
To see that $D$ is a norm when $v \neq v_0$ splits in $\ell$, and $m \not\hookrightarrow k_v$, we use the following theorem from local class field theory (see Weil “Basic Number Theory”, pp. 225–226):

**Theorem.** If $K$ is a non-archimedean local field, and if $L$ is a cyclic extension of $K$ of degree $n$, then the image of $L^\times$ under the norm map $N_{L/K} : L^\times \to K^\times$ has index $n$ in $K^\times$. When $L$ is an unramified cyclic extension, then that image equals $\{x \in K^\times : v(x) \equiv 0 \pmod{n}\}$.

If $v \neq v_0$ splits in $\ell$ and $m \not\hookrightarrow k_v$, then by choice of $D$ we have $w(D) = 0$ for both extensions $w$ of $v$ to $\ell$. Moreover, neither $w$ ramifies in $m$, as we see checking case by case. So the extension $k_v(Z)$ of $k_v = \ell_w$ is unramified, and so $N_{k_v(Z)/k_v}(\eta_v) = D$ for some $\eta_v \in k_v(Z)$, by the theorem.
If \( v = v_0 \), then \( m \not
rightarrow k_v \), by choice of \( m \), and \( w(D) \neq 0 \) by choice of \( D \), for both extensions \( w \) of \( v_0 \) to \( \ell \). Assuming that \( w \) does not ramify in \( m \), the theorem shows that \( D \) is not a norm.

There is only one case when \( w \) ramifies: \( C_{18} \). Here \( k = \mathbb{Q}(r) \), where \( r^2 = 6 \), and \( v_0 \) is the unique 3-adic valuation on \( k \) (3 ramifies in \( k \)). This splits in \( \ell \) since \( 3 = (\omega - 1)(\bar{\omega} - 1) \). The extensions \( w \) and \( \bar{w} \) corresponding to \( p = (\omega - 1)\sigma_\ell \) and \( \bar{p} \) both ramify in \( m = k(\zeta_9) \) because \( N_{m/\ell}(\zeta_9 - 1) = \omega - 1 \).

In the \( C_{18} \) case, we carefully identify the index 3 subgroup of \( k_{v_0}^\times \) which is the image of the norm map, and show that \( D \) is not in that image.
If $D$ is not a norm, then $D \otimes \ell k_v$ is a division algebra over the local field $k_v$, and so

$$G(k_v) \cong \{ \xi \in D \otimes \ell k_v : \text{Nrd}(\xi) = 1 \}$$

is compact. To see this, let $\xi_1, \ldots, \xi_9$ be a basis for $D \otimes \ell k_v$, and let $\mathfrak{o}_v = \{ t \in k_v : |t|_v \leq 1 \}$. Here $|t|_v = q_v^{-v(t)}$. Then $\mathfrak{o}_v$ is compact, and hence so is

$$S = \{ \sum_\nu t_\nu \xi_\nu : t_\nu \in \mathfrak{o}_v \text{ for all } \nu, \text{ and } |t_\nu|_v = 1 \text{ for at least one } \nu \}.$$

Now $\text{Nrd}(\xi) \neq 0$ for all $\xi \neq 0$, since $D \otimes \ell k_v$ is a division algebra. So there is a number $m > 0$ so that $|\text{Nrd}(\xi)|_v \geq m$ for all $\xi \in S$. Now suppose that $\xi = \sum_\nu t_\nu \xi_\nu$ satisfies $\text{Nrd}(\xi) = 1$. Let $T = \max\{|t_\nu|_v : \nu = 1, \ldots, 9\}$. Then $c\xi \in S$ for some $c \in k_v$ satisfying $|c|_v = 1/T$. Hence $|c|^3|_v = |\text{Nrd}(c\xi)|_v \geq m$. Hence $T = 1/|c|_v \leq 1/m^{1/3}$. So $\{ \xi \in D \otimes \ell k_v : \text{Nrd}(\xi) = 1 \}$ is closed and bounded, and so compact.
$G(k_v)$ and $\bar{G}(k_v)$ when $v$ does NOT split in $\ell$.

For a basis $\xi_1, \ldots, \xi_9$ of $D$ over $\ell$, setting

$$h\left(\sum_\nu \xi_\nu \otimes_k x_\nu + \sum_\nu (s_\xi_\nu) \otimes_k y_\nu\right) = \sum_\nu \xi_\nu \otimes_\ell (x_\nu + sy_\nu)$$

we get an isomorphism, and setting

$$\tilde{\iota}\left(\sum_\nu \xi_\nu \otimes_k x_\nu + \sum_\nu (s_\xi_\nu) \otimes_k y_\nu\right) = \sum_\nu \iota_\nu(\xi_\nu) \otimes_\ell (x_\nu - sy_\nu)$$

we get an involution of the second kind, and the commutative diagram

\[ \begin{array}{ccc} D \otimes_k k_v & \xrightarrow{h} & D \otimes_\ell k_v(s) \\ \iota_{k_v} \downarrow & & \downarrow \tilde{\iota} \\ D \otimes_k k_v & \xrightarrow{h} & D \otimes_\ell k_v(s) \end{array} \]
and also the commutative diagram

\[
\begin{array}{ccc}
\mathcal{D} \otimes_k k_v & \xrightarrow{h} & \mathcal{D} \otimes_\ell k_v(s) \\
\downarrow \text{Nrd}_{k_v} & & \downarrow \text{Nrd} \\
\ell \otimes_k k_v & \xrightarrow{f} & k_v(s)
\end{array}
\]

where now \( f \) is the isomorphism \( 1 \otimes x + s \otimes y \mapsto x + sy \).
Corollary. If \( v \) does not split in \( \ell \), then
\[
G(k_v) \cong \{ \xi \in D \otimes_\ell k_v(s) : \bar{i}(\xi)\xi = 1 \& Nrd(\xi) = 1 \}.
\]
and
\[
\bar{G}(k_v) \cong \{ \xi \in D \otimes_\ell k_v(s) : \bar{i}(\xi)\xi = 1 \}/\{ t1 : t \in k_v(s) \& \bar{t}t = 1 \}.
\]
\( D \otimes_\ell k_v(s) \cong M_{3 \times 3}(k_v(s)) \) or \( D \otimes_\ell k_v(s) \) is a division algebra.

It is never a division algebra, as noted in \( \S2.2 \) of [PY]. We can also see this using theorem about norms, and explicit calculations when \( v \) ramifies in \( m \).
The isomorphism $D \otimes k_v(s) \cong M_{3 \times 3}(k_v(s))$ induces isomorphisms

$$G(k_v) \cong \{\xi \in SL(3, k_v(s)) : g^*F'_v g = F'_v\},$$

$$\bar{G}(k_v) \cong \{g \in GL(3, k_v(s)) : g^*F'_v g = F'_v\}/{\{t1 : t \in k_v(s) \& \bar{t}t = 1\}}$$

and

$$\{\xi \in D \otimes k \quad : \quad \iota_{k_v}(\xi)\xi = 1\} \cong \{g \in GL(3, k_v(s)) : g^*F'_v g = F'_v\}$$

for a suitable Hermitian $F'_v \in GL(3, k_v(s))$. In particular, we have an embedding $\xi \mapsto \xi_v$ of

$$\{\xi \in D \quad : \quad \iota(\xi)\xi = 1\} \quad \text{into} \quad \{g \in GL(3, k_v(s)) : g^*F'_v g = F'_v\}.$$
The matrix $F'_v$ depends on whether or not $m$ embeds in $k_v(s)$.

**Case a:** the embedding $\ell = k(s) \leftrightarrow k_v(s)$ extends to an embedding $m \leftrightarrow k_v(s)$. Then

$$D \xrightarrow{\Psi} M_{3 \times 3}(m) \leftrightarrow M_{3 \times 3}(k_v(s))$$

induces the above isomorphisms, with $F'_v$ the image of $F \in M_{3 \times 3}(m)$ in $M_{3 \times 3}(k_v(s))$.

In particular, if $v$ is an archimedean place of $k$, then $\ell$ does not embed in $k_v \cong \mathbb{R}$, but $m = k(s, W)$ embeds in $k_v(s) \cong \mathbb{C}$. After a further conjugation by the matrix we called $\Delta$ above, we may assume that $F'_v = F_0$, if $k = \mathbb{Q}$, or that $F'_v = F_0$ for one $v$ and $F'_v = I$ for the other $v$, when $k = \mathbb{Q}(r)$. 
Case b: the field $m$ does not embed in $k_v(s)$. Then writing $m = k(s, Z)$, $m \otimes \ell k_v(s)$ is the field $k_v(s, Z)$, and $D \otimes \ell k_v(s)$ is isomorphic to $M_{3 \times 3}(k_v(s))$.

As $\Psi(\xi)$ is unitary with respect to a matrix $F$, $J_v \Psi(\xi) J_v^{-1}$ is unitary with respect to

$$F'_v = J_v^{-1} F_v J_v^{-1} = \Theta^{-1} C_v^{-1} F_v C^{-1} \Theta^{-1},$$

where $F_v$ is the image of $F \in M_{3 \times 3}(m)$ in $M_{3 \times 3}(k_v(s, Z))$. 