Constructing the algebras $\mathcal{D}$ and the involutions ι.

To a fake projective plane there is associated a pair (k, ℓ) of fields coming from a short list. There is also an algebra $\mathcal{D}$, an involution ι and a group G, with

$$G(k) = \{\xi \in \mathcal{D} : \iota(\xi)\xi = 1 \ \& \ \mathrm{Nrd}(\xi) = 1\}.$$

$\mathcal{D}$, ι and G must satisfy the properties:

- $G(k_v) \cong SL(3, k_v)$ for all $v \in V_f \setminus \mathcal{T}_0$ which split in ℓ,

- $G(k_v) \cong \{g \in SL(3, k_v(s)) : g^* F_v g = F_v\}$ if $v \in V_f$ does not split in ℓ,

- $G(k_v)$ is compact for $v \in \mathcal{T}_0$,

- $G(k_v) \cong SU(2,1)$ for one archimedean place v on k, and

- $G(k_v) \cong SU(3)$ for the other archimedean place v on k (if $k \neq \mathbb{Q}$).

We know that $\mathcal{T}_0 = \emptyset$ if $\mathcal{D} = M_{3\times 3}(\ell)$, and that $\mathcal{T}_0$ is a specific singleton if $\mathcal{D}$ is a division algebra.

Corollary 6.6 in Chapter 10 of W. Scharlau "Quadratic and Hermitian Forms":

The above properties determine $\mathcal{D}$ and ι up to k-isomorphism or anti-isomorphism.

Anti-isomorphism must be allowed here because given $\mathcal{D}$, we can define an "opposite" algebra $\mathcal{D}^{\text{op}}$ whose elements x^{op} are in 1-1 correspondence ($x^{\text{op}} \leftrightarrow x$) with those of $\mathcal{D}$, and in which for all $x, y \in \mathcal{D}$ and $t \in \ell$,

$$x^{\text{op}} + y^{\text{op}} = (x+y)^{\text{op}}, \ tx^{\text{op}} = (tx)^{\text{op}}, \text{ and } x^{\text{op}}y^{\text{op}} = (yx)^{\text{op}}.$$

Suppose that ℓ is a field and that m is a Galois extension of ℓ of degree 3, with $\mathrm{Gal}(m/\ell) = \langle \varphi \rangle$.

Fix some nonzero $D \in \ell$, and form

$$\mathcal{D} = \{a + b\sigma + c\sigma^2 : a, b, c \in m\},$$

which we can make into an associative algebra of dimension 9 over ℓ in which

$$\sigma^3 = D \quad \text{and} \quad \sigma a = \varphi(a)\sigma \quad \text{for all } a \in m.$$

The centre of $\mathcal{D}$ is ℓ. We shall see in a moment that $\mathcal{D}$ has no non-trivial two-sided ideals — $\mathcal{D}$ is a central simple algebra.

There is an ℓ-algebra homomorphism $\Psi : \mathcal{D} \to M_{3\times 3}(m)$ such that

$$\Psi(\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ D & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Psi(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & \varphi(a) & 0 \\ 0 & 0 & \varphi^2(a) \end{pmatrix}.$$

So if $\xi = a + b\sigma + c\sigma^2 \in \mathcal{D}$, then

$$\Psi(\xi) = \begin{pmatrix} a & b & c \\ D\varphi(c) & \varphi(a) & \varphi(b) \\ D\varphi^2(b) & D\varphi^2(c) & \varphi^2(a) \end{pmatrix}.$$

The reduced norm $\mathrm{Nrd}(\xi)$ of ξ is

$$\begin{aligned}\det(\Psi(\xi)) = a\varphi(a)\varphi^2(a) + Db\varphi(b)\varphi^2(b) + D^2c\varphi(c)\varphi^2(c) \\ - D(a\varphi(b)\varphi^2(c) + \varphi(a)\varphi^2(b)c + \varphi^2(a)b\varphi(c)).\end{aligned}$$

Then $\mathrm{Nrd} : \mathcal{D} \to \ell$, and $\mathrm{Nrd}(\xi\eta) = \mathrm{Nrd}(\xi)\mathrm{Nrd}(\eta)$ for all $\xi, \eta \in \mathcal{D}$.

An element $\xi = a + b\sigma + c\sigma^2$ of $\mathcal{D}$ is invertible if and only if $\mathrm{Nrd}(\xi) \neq 0$, in which case ξ^{-1} equals

$$\frac{1}{\mathrm{Nrd}(\xi)}\Big(\Big(\varphi(a)\varphi^2(a) - D\varphi(b)\varphi^2(c)\Big) + \Big(Dc\varphi^2(c) - b\varphi^2(a)\Big)\sigma + \Big(b\varphi(b) - c\varphi(a)\Big)\sigma^2\Big).$$

Proposition. Either

(a) $\mathcal{D} \cong M_{3\times 3}(\ell)$, or

(b) $\mathcal{D}$ is a division algebra.

Case (a) holds if and only if D is the norm $N_{m/\ell}(\eta)$ of an element η of m.

Proof. If $D = N_{m/\ell}(\eta)$, let

$$C = \begin{pmatrix} \eta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\varphi(\eta) \end{pmatrix}.$$

Let θ_0, θ_1 and θ_3 be basis for m over ℓ. Form

$$\Theta = \begin{pmatrix} \theta_0 & \varphi(\theta_0) & \varphi^2(\theta_0) \\ \theta_1 & \varphi(\theta_1) & \varphi^2(\theta_1) \\ \theta_2 & \varphi(\theta_2) & \varphi^2(\theta_2) \end{pmatrix}.$$

Then

$$\Theta^{-1} = \begin{pmatrix} \zeta_0 & \zeta_1 & \zeta_2 \\ \varphi(\zeta_0) & \varphi(\zeta_1) & \varphi(\zeta_2) \\ \varphi^2(\zeta_0) & \varphi^2(\zeta_1) & \varphi^2(\zeta_2) \end{pmatrix},$$

where $\text{Trace}(\theta_i\zeta_j) = \delta_{ij}$.

$J := \Theta C$. Then $J\Psi(\xi)J^{-1}$ has entries in ℓ.

E.g., $(J\Psi(\sigma)J^{-1})_{ij} = \text{Trace}(\theta_i\eta\varphi(\zeta_j))$.

So $\xi \mapsto J\Psi(\xi)J^{-1}$ is a ℓ-linear algebra homomomomorphism $\mathcal{D} \to M_{3\times 3}(\ell)$. It is clearly injective, and so an isomorphism, as dimensions match.

If D is not equal to $N_{m/\ell}(\eta)$ for any $\eta \in m$, then

$$\mathrm{Nrd}(1 + b\sigma) = 1 + DN_{m/\ell}(b) \quad \text{and} \quad \mathrm{Nrd}(1 + c\sigma^2) = 1 + D^2 N_{m/\ell}(c)$$

cannot be zero for any $b, c \in m$. So any $1 + b\sigma$ or $1 + c\sigma^2$ is invertible. So any nonzero element of $\mathcal{D}$ is invertible, and $\mathcal{D}$ is a division algebra.

Corollary. $\mathcal{D}$ is a central simple algebra over ℓ.

Suppose $\ell = k(s)$, where $s^2 = -\kappa \in k$.

We want an involution ι of the second kind on $\mathcal{D}$.

Assume m normal extension of k. The conjugation automorphism extends to m. Then either

$$\overline{\varphi(a)} = \varphi(\bar{a}) \quad \text{for all } a \in m \quad \text{OR} \quad \overline{\varphi(a)} = \varphi^2(\bar{a}) \quad \text{for all } a \in m.$$

Gal(m/k) is abelian in the first case, and non-abelian in the second case.

Lemma. If Gal(m/k) is non-abelian, and if $D \in \ell$ satisfies $\overline{D} = D \neq 0$, then there is an involution of the second kind $\iota : \mathcal{D} \to \mathcal{D}$ such that

$$\iota(\sigma) = \sigma \quad \text{and} \quad \iota(a) = \bar{a} \quad \text{for all } a \in m.$$

Explicitly,

$$\iota(a + b\sigma + c\sigma^2) = \bar{a} + \varphi(\bar{b})\sigma + \varphi^2(\bar{c})\sigma^2.$$

Note that

$$\Psi(\iota(\xi)) = F^{-1}\Psi(\xi)^* F \quad \text{for} \quad F = \begin{pmatrix} D & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In each of the five $k = \mathbb{Q}$ cases $(a = 1, p = 5)$,...,$(a = 23, p = 2)$, we can choose a field m as above, with Gal(m/k) non-abelian, and define $\mathcal{D}$ using $D = p$. In each of these cases, there is a $\beta \in \ell$ so that $\bar{\beta}\beta = 2p$. Then

$$F = \frac{1}{2}\Delta^* F_0 \Delta \quad \text{for} \quad \Delta = \begin{pmatrix} \beta & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

If $\iota(\xi)\xi = 1$ then

$$(\Delta\Psi(\xi)\Delta^{-1})^* F_0 (\Delta\Psi(\xi)\Delta^{-1}) \;=\; F_0.$$

So if

$$G(k) = \{\xi \in \mathcal{D} : \iota(\xi)\xi = 1 \quad \text{and} \quad \text{Nrd}(\xi) = 1\},$$

then $\xi \mapsto \Delta\Psi(\xi)\Delta^{-1}$ defines an injective homomorphism $G(k) \to SU(2,1)$.

In fact, $G(k_v) \cong G(\mathbb{R}) \cong SU(2,1)$ for the one archimedean place v on $k = \mathbb{Q}$ — see below.

We can take $\beta = 3 + i$ in the case $(a = 1, p = 5)$, $\beta = 2 + s$ for $(a = 2, p = 3)$ and $\beta = 2$ for the other three cases.

In the 5 cases (k, ℓ) in which $k = \mathbb{Q}$ and $\ell = \mathbb{Q}(s)$, we can define $m = \mathbb{Q}(s, Z)$, where Z satisfies $P(Z) = 0$ for a cubic monic $P(X) \in \mathbb{Z}[X]$.

s^2	p	$P(X)$	$\varphi(Z)$
-1	5	$X^3 - 3X^2 - 2$	$(s + 3 - (4s + 1)Z + sZ^2)/2$
-2	3	$X^3 + X^2 + 2X - 2$	$(2(s - 1) + (3s - 2)Z + sZ^2)/4$
-7	2	$X^3 + 3X^2 + 3$	$-(3(s + 7) + (9s + 7)Z + 2sZ^2)/14$
-15	2	$X^3 - 3X - 3$	$(4s + (3s - 5)Z - 2sZ^2)/10$
-23	2	$X^3 - X - 1$	$(4s + (9s - 23)Z - 6sZ^2)/46$

In each case $\text{Gal}(m/\ell) = \langle\varphi\rangle$, and $\text{Gal}(m/\mathbb{Q})$ is non-abelian.

In the case $(a = 7, p = 2)$, we shall use a different cyclic simple algebra, coming from a field m so that $\text{Gal}(m/k)$ is abelian.

Lemma. If $\mathrm{Gal}(m/k)$ is abelian, and if $D \in \ell$ satisfies $\bar{D}D = 1$, then there is an involution $\iota_0 : \mathcal{D} \to \mathcal{D}$ of the second kind such that

$$\iota_0(\sigma) = \sigma^{-1} \quad \text{and} \quad \iota_0(a) = \bar{a} \quad \text{for all } a \in m.$$

Explicitly,

$$\iota_0(a + b\sigma + c\sigma^2) = \bar{a} + \bar{D}\varphi(\bar{c})\sigma + \bar{D}\varphi^2(\bar{b})\sigma^2.$$

It is easy to check that

$$\Psi(\iota_0(\xi)) = \Psi(\xi)^*.$$

For reasons explained on the next slide, we shall use the involution

$$\iota(\xi) = T^{-1}\iota_0(\xi)T,$$

where $T \in m$ and $\bar{T} = T \neq 0$. Then

$$\Psi(\iota(\xi)) = F^{-1}\Psi(\xi)^*F \quad \text{for} \quad F = \begin{pmatrix} T & 0 & 0 \\ 0 & \varphi(T) & 0 \\ 0 & 0 & \varphi^2(T) \end{pmatrix}.$$

Embedding m in $\mathbb{C}$, the images of T, $\varphi(T)$ and $\varphi^2(T)$ are real because $\overline{\varphi(T)} = \varphi(\bar{T}) = \varphi(T)$.

If $T > 0$, $\varphi(T) > 0$ and $\varphi^2(T) < 0$, then

$$F = \Delta^* F_0 \Delta \quad \text{for} \quad \Delta = \begin{pmatrix} |T|^{1/2} & 0 & 0 \\ 0 & |\varphi(T)|^{1/2} & 0 \\ 0 & 0 & |\varphi^2(T)|^{1/2} \end{pmatrix}.$$

So $\xi \mapsto \Delta\Psi(\xi)\Delta^{-1}$ is an injective homomorphism $G(k) \to SU(2,1)$.

In the cases $(a = 7, p = 2)$, $\mathcal{C}_2$, $\mathcal{C}_{10}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{20}$, we define

$$\mathcal{D} = \{a + b\sigma + c\sigma^2 \ : \ a, b, c \in m, \ \text{where } \sigma^3 = D \text{ and } \sigma x \sigma^{-1} = \varphi(x)\}$$

for the following m's and D's:

name	m	v_0	D	φ
$(a=7, p=2)$	$\mathbb{Q}(\zeta_7)$	2	$(3+s)/4$	$\zeta_7 \mapsto \zeta_7^2$
$\mathcal{C}_2$	$k(\zeta_9)$	2	$(1+\sqrt{-15})/4$	$\zeta_9 \mapsto \zeta_9^4$
$\mathcal{C}_{10}$	$\ell(W)$	2	$rU/2$	$W \mapsto 2 - W - W^2$
$\mathcal{C}_{18}$	$k(\zeta_9)$	3	$(r+1+2\omega)/3$	$\zeta_9 \mapsto \zeta_9^4$
$\mathcal{C}_{20}$	$k(\zeta_7)$	2	$(3+\sqrt{-7})/4$	$\zeta_7 \mapsto \zeta_7^2$

In case $\mathcal{C}_2$, $k = \mathbb{Q}(r)$, where $r^2 = 5$ and $\ell = k(\omega)$, where $\omega = \zeta_3$,

In case $\mathcal{C}_{10}$, $k = \mathbb{Q}(r)$, where $r^2 = 2$, $\ell = k(U)$, where $U^2 = (r+1)U - 2$, and $W^3 - 3W + 1 = 0$.

In case $\mathcal{C}_{18}$, $k = \mathbb{Q}(r)$, where $r^2 = 6$, and $\ell = k(\omega)$,

In case $\mathcal{C}_{20}$, $k = \mathbb{Q}(r)$ where $r^2 = 7$, and $\ell = k(i)$.

Having chosen m as above, with Gal(m/k) abelian, the subfield $\{a \in m : \bar{a} = a\}$ has the form $k(W)$, where W satisfies an equation $Q(W) = 0$ for some monic cubic $Q(X) \in \mathbb{Z}[X]$. We choose $T \in k(W)$ as follows:

name	W	$\varphi(W)$	T
$(a = 7, p = 2)$	$\zeta_7 + \zeta_7^{-1}$	$W^2 - 2$	W
$\mathcal{C}_2$	$\zeta_9 + \zeta_9^{-1}$	$2 - W - W^2$	$-2r + (r-1)W + 2W^2$
$\mathcal{C}_{10}$	W	$2 - W - W^2$	$-r + (1-r)W + W^2$
$\mathcal{C}_{18}$	$\zeta_9 + \zeta_9^{-1}$	$2 - W - W^2$	$3 - 3r + rW + rW^2$
$\mathcal{C}_{20}$	$\zeta_7 + \zeta_7^{-1}$	$W^2 - 2$	$2 + W - (4 + 3W + W^2)/r$

In the first and last cases, $Q(X) = X^3 + X^2 - 2X - 1$. In the other three cases, $Q(X) = X^3 - 3X + 1$.

The above choice is made so that, in the four cases $k = \mathbb{Q}(r)$, where $r^2 = N$ ($N = 5$, 2, 6 or 7), and fixing a solution $W_{\mathbb{R}} \in \mathbb{R}$ of $Q(X) = 0$,

- embedding $k(W)$ in $\mathbb{R}$ by mapping r to $+\sqrt{N}$ and W to $W_{\mathbb{R}}$, the images of T, $\varphi(T)$ and $\varphi(T)$ DO NOT all have the same sign, and

- embedding $k(W)$ in $\mathbb{R}$ by mapping r to $-\sqrt{N}$ and W to $W_{\mathbb{R}}$, the images of T, $\varphi(T)$ and $\varphi(T)$ DO all have the same sign.

This implies that $G(k_v) \cong SU(2,1)$ for the archimedean valuation v corresponding to the first embedding, and $G(k_v) \cong SU(3)$ for the archimedean valuation v corresponding to the second embedding.

Example. The case $(a = 7, p = 2)$.

Let $k = \mathbb{Q}$ and $\ell = \mathbb{Q}(s)$, where $s^2 = -7$. Let m be the cyclotomic field $\mathbb{Q}(\zeta)$, where ζ is a primitive 7-th root of 1. Let

$$s = 1 + 2\zeta + 2\zeta^2 + 2\zeta^4.$$

Then $s^2 = -7$. So $\ell \subset m$.

Now $\mathrm{Gal}(m/\mathbb{Q})$ is cyclic, generated by $\chi : \zeta \mapsto \zeta^3$, and $\varphi = \chi^2 : \zeta \mapsto \zeta^2$ generates $\mathrm{Gal}(m/\ell)$. Form the cyclic algebra $\mathcal{D}$ with this m and φ, and with

$$D = \frac{3+s}{4}.$$

Notice that $\bar{D}D = 1$. Let's check that D is not the norm $N_{m/\ell}(\eta)$ of any element η of m.

The prime 2 splits in ℓ, as $2 = \rho\bar{\rho}$ for $\rho = (1 - s)/2 \in \mathfrak{o}_\ell$.

$D = (3 + s)/4$ equals $-\rho/\bar{\rho}$, so $w(D) = +1$ and $\bar{w}(D) = -1$, where $w \leftrightarrow \rho\mathfrak{o}_\ell$ and $\bar{w} \leftrightarrow \bar{\rho}\mathfrak{o}_\ell$.

Alternatively, $\mathbb{Q}_2$ contains a square root $s_2 = 1+0\times 2+1\times 2^2+0\times 2^3+\cdots$ of -7, and $v_\epsilon(a + bs) = u_2(a + \epsilon b s_2)$, for $\epsilon = \pm 1$, define two distinct extensions to ℓ of the 2-adic valuation u_2 on $\mathbb{Q}$.

Then $v_+(D) = 1$ and $v_-(D) = -1$. So $w = v_+$ and $\bar{w} = v_-$.

Magma verifies that $\rho\mathfrak{o}_m$ is prime. So w has a unique extension $\tilde{w}$ to m. Then $\tilde{w}(\eta) = \tilde{w}(\varphi(\eta)) = \tilde{w}(\varphi^2(\eta)) \in \mathbb{Z}$ for $\eta \in m$. If $D = N_{m/\ell}(\eta)$, then

$$1 = w(D) = \tilde{w}(D) = \tilde{w}(\eta\varphi(\eta)\varphi^2(\eta)) = 3\tilde{w}(\eta),$$

a contradiction.

We choose $T = \zeta + \zeta^{-1}$, and use the involution

$$\iota(\xi) = T^{-1}\iota_0(\xi)T, \text{ where } \iota_0(\sigma) = \sigma^{-1}, \text{ and } \iota_0(a) = \bar{a} \text{ for } a \in m.$$

Then

$$\Psi(\iota(\xi)) = F^{-1}\Psi(\xi)^* F \quad \text{for} \quad F = \begin{pmatrix} T & 0 & 0 \\ 0 & \varphi(T) & 0 \\ 0 & 0 & \varphi^2(T) \end{pmatrix}.$$

Embed m into $\mathbb{C}$, mapping ζ to $e^{2\pi i/7}$. Then $T > 0$, $\varphi(T) = T^2 - 2 < 0$ and $\varphi^2(T) = 1 - T - T^2 < 0$. Let

$$\Delta = \begin{pmatrix} 0 & 0 & |\varphi^2(T)|^{1/2} \\ 0 & |\varphi(T)|^{1/2} & 0 \\ |T|^{1/2} & 0 & 0 \end{pmatrix}.$$

Then $\Delta^* F_0 \Delta = -F$, so $g^* F g = F$ iff $\tilde{g} = \Delta g \Delta^{-1}$ satisfies $\tilde{g}^* F_0 \tilde{g} = F_0$.

So far: for each of the 9 pairs (k, ℓ), we have defined

- a cyclic algebra $\mathcal{D}$,
- an involution ι on $\mathcal{D}$.

In each case, $\mathcal{D}$ is a division algebra. The proof: as in the case $(a = 7, p = 2)$, using $w(D) \neq 0$ for the two extensions w of the $v_0 \in \mathcal{T}_0$.

Tricky case: $\mathcal{C}_{18}$. Here v_0 is the one 3-adic valuation on $k = \mathbb{Q}(r)$, $r^2 = 6$. This splits in ℓ, and the two extensions w of v_0 *ramify* in m.

We need to find $G(k_v)$ for the various places v of k.

We start from one of our nine $(\mathcal{D}, \iota)$.

If K is a field containing k, then by definition,

$$G(K) = \{\xi \in \mathcal{D} \otimes_k K \ : \ \iota_K(\xi)\xi = 1 \text{ and } \mathrm{Nrd}_K(\xi) = 1\}$$

(see §1.2 in [PY]). Because $\iota : \mathcal{D} \to \mathcal{D}$ is k-linear, it induces a unique K-linear map $\iota_K : \mathcal{D} \otimes_k K \to \mathcal{D} \otimes_k K$. It is an anti-automorphism.

We can define Nrd_K using $\Psi : \mathcal{D} \to M_{3\times 3}(m)$. This induces $\Psi_K : \mathcal{D} \otimes_k K \to M_{3\times 3}(m \otimes_k K)$, and we set $\mathrm{Nrd}_K(\xi) = \det(\Psi_K(\xi))$. Then $\mathrm{Nrd}_K : \mathcal{D} \otimes_k K \to \ell \otimes_k K$, and it is multiplicative.

We also need the adjoint group $\bar{G}$. We can define this by setting

$$\bar{G}(K) = \{\alpha \in \mathsf{Aut}_\ell(\mathcal{D} \otimes_k K) \ : \ \iota_K \circ \alpha = \alpha \circ \iota_K\}.$$

for any field K containing k. Here $\alpha \in \mathsf{Aut}_\ell(\mathcal{D} \otimes_k K)$ means that α is an automorphism of the K-algebra $\mathcal{D} \otimes_k K$ which is also ℓ-linear.

Fact:

$$\bar{G}(k) \cong \{\xi \in \mathcal{D} : \iota(\xi)\xi = 1\}/\{t1 \ : \ t \in \ell \ \& \ \bar{t}t = 1\}.$$

This is because any automorphism of $\mathcal{D}$ is of the form $\alpha : \eta \mapsto \xi\eta\xi^{-1}$ for some invertible $\xi \in \mathcal{D}$ (Skolem-Noether Theorem). Then $\iota \circ \alpha = \alpha \circ \iota$ means that $\iota(\xi)\xi = c1$ for some $c \in k$. Let $\xi' = c\xi / \mathsf{Nrd}(\xi)$. Then $\alpha(\eta) = \xi'\eta\xi'^{-1}$ for all η, and $\iota(\xi')\xi' = 1$.

The form of $G(k_v)$ depends on whether or not v splits in $\ell = k(s)$. Recall that $s^2 = -\kappa$ for some $\kappa \in k$, and
v splits in $\ell \Leftrightarrow \ell$ embeds in $k_v \Leftrightarrow -\kappa$ has a square root in k_v.

For any field K containing k,

(i) If $\ell \hookrightarrow K$, then $\ell \otimes_k K \cong K \oplus K$.

(ii) If $\ell \not\hookrightarrow K$, then $\ell \otimes_k K \cong K(s)$, a field.

In (i), the isomorphism is $1 \otimes x + s \otimes y \mapsto (x + ys_K, x - ys_K)$, where s_K is the image of s under an embedding $\ell \to K$.

In (ii), the isomorphism is $1 \otimes x + s \otimes y \mapsto x + ys$.

Let $\mathcal{D}$ denote one of our 9 division algebras $\mathcal{D}$.

Proposition. If $v \in V_f$ splits in ℓ, then either

(1) $G(k_v) \cong SL(3, k_v)$, $\bar{G}(k_v) \cong PGL(3, k_v)$, and $\mathcal{D} \otimes_\ell k_v \cong M_{3\times 3}(k_v)$, or

(2) $G(k_v)$ and $\bar{G}(k_v)$ are compact, and $\mathcal{D} \otimes_\ell k_v$ is a division algebra,

and (2) only happens for the one $v \in \mathcal{T}_0$.

We heavily use the explicit form of these isomorphisms, so give some details of the proof.

As $\ell \subset k_v$, we can form $\mathcal{D} \otimes_\ell k_v$. The map $\xi \mapsto (\xi, \iota(\xi)^{\mathsf{op}})$ is k-linear $\mathcal{D} \to \mathcal{D} \oplus \mathcal{D}^{\mathsf{op}}$, and so induces

$$h : \mathcal{D} \otimes_k k_v \to (\mathcal{D} \otimes_\ell k_v) \oplus (\mathcal{D} \otimes_\ell k_v)^{\mathsf{op}},$$

and we get the commutative diagram

$$\begin{array}{ccc}
\mathcal{D} \otimes_k k_v & \xrightarrow{\quad h \quad} & (\mathcal{D} \otimes_\ell k_v) \oplus (\mathcal{D} \otimes_\ell k_v)^{\mathsf{op}} \\
\Big\downarrow \iota_{k_v} & & \Big\downarrow (x, y^{\mathsf{op}}) \mapsto (y, x^{\mathsf{op}}) \\
\mathcal{D} \otimes_k k_v & \xrightarrow{\quad h \quad} & (\mathcal{D} \otimes_\ell k_v) \oplus (\mathcal{D} \otimes_\ell k_v)^{\mathsf{op}}
\end{array}$$

and the map h is an isomorphism of k_v-algebras.

We also get a commutative diagram

$$\begin{array}{ccc} \mathcal{D} \otimes_k k_v & \xrightarrow{\ h\ } & (\mathcal{D} \otimes_\ell k_v) \oplus (\mathcal{D} \otimes_\ell k_v)^{\mathsf{op}} \\ {\scriptstyle \mathsf{Nrd}_{k_v}} \downarrow & & \downarrow {\scriptstyle (x, y^{\mathsf{op}}) \mapsto (\mathsf{Nrd}(x), \mathsf{Nrd}(y))} \\ \ell \otimes_k k_v & \xrightarrow{\ f\ } & k_v \oplus k_v \end{array}$$

where $f(1 \otimes_k x + s \otimes_k y) = (x + s_v y, x - s_v y)$, where $s_v \in \ell$ is the image of s.

So if $\xi \in \mathcal{D} \otimes_k k_v$ and $h(\xi) = (x, y^{\mathsf{op}})$, then

$\iota_{k_v}(\xi)\xi = 1 \Leftrightarrow yx = 1$, and
$\mathsf{Nrd}_{k_v}(\xi) = 1 \Leftrightarrow \mathsf{Nrd}(x) = \mathsf{Nrd}(y) = 1$.

Corollary. If v splits in ℓ, then

$$G(k_v) \cong \{x \in \mathcal{D} \otimes_\ell k_v : \mathsf{Nrd}(x) = 1\},$$

$$\{\xi \in \mathcal{D} \otimes_k k_v \ : \ \iota_{k_v}(\xi)\xi = 1\} \cong (\mathcal{D} \otimes_\ell k_v)^\times.$$

and

$$\bar{G}(k_v) \cong (\mathcal{D} \otimes_\ell k_v)^\times / k_v^\times.$$

Note that $\mathcal{D} \otimes_\ell k_v$ is a central simple algebra of dimension 9 over k_v, and so is isomorphic to $M_{3\times 3}(k_v)$ or is a division algebra.

Recall embedding $\Psi : \mathcal{D} \to M_{3\times 3}(m)$.

Case a: the embedding $\ell \hookrightarrow k_v$ extends to an embedding $m \hookrightarrow k_v$.

Then

$$\mathcal{D} \xrightarrow{\Psi} M_{3\times 3}(m) \hookrightarrow M_{3\times 3}(k_v)$$

induces isomorphisms

$$\mathcal{D} \otimes_\ell k_v \cong M_{3\times 3}(k_v), \quad G(k_v) \cong SL(3, k_v), \quad \text{and} \quad \bar{G}(k_v) \cong PGL(3, k_v).$$

Moreover

$$\{\xi \in \mathcal{D} \otimes_k k_v \ : \ \iota_{k_v}(\xi)\xi = 1\} \cong GL(3, k_v).$$

So we get an embedding $\xi \mapsto \xi_v$ of $\{\xi \in \mathcal{D} : \iota(\xi)\xi = 1\}$ in $GL(3, k_v)$ which maps $G(k)$ into $SL(3, k_v)$. Here ξ_v is the image of $\Psi(\xi)$ in $M_{3\times 3}(k_v)$.

Case b: When $m = k(s, Z)$ does not embed in k_v, then $m \otimes_\ell k_v \cong k_v(Z)$ is a field which is a cubic Galois extension of k_v, and $\mathcal{D} \otimes_\ell k_v$ is the cyclic simple algebra $\{a + b\sigma + c\sigma^2 : a, b, c \in k_v(Z)\}$.

When $D = N_{k_v(Z)/k_v}(\eta_v)$ of some $\eta_v \in k_v(Z)$, $\mathcal{D} \otimes_\ell k_v \cong M_{3\times 3}(k_v)$, the isomorphism induced by

$$\mathcal{D} \xrightarrow{\Psi} M_{3\times 3}(m) \hookrightarrow M_{3\times 3}(k_v(s)) \xrightarrow{J_v \,\cdot\, J_v^{-1}} M_{3\times 3}(k_v(s)),$$

where $J_v = \Theta C_v$ for

$$C_v = \begin{pmatrix} \eta_v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\varphi(\eta_v) \end{pmatrix} \quad \text{and} \quad \Theta = \begin{pmatrix} \theta_0 & \varphi(\theta_0) & \varphi^2(\theta_0) \\ \theta_1 & \varphi(\theta_1) & \varphi^2(\theta_1) \\ \theta_2 & \varphi(\theta_2) & \varphi^2(\theta_2) \end{pmatrix},$$

and $\theta_0, \theta_1, \theta_2$ is a basis of m over ℓ.

We again have isomorphisms

$$\mathcal{D} \otimes_\ell k_v \cong M_{3\times 3}(k_v), \quad G(k_v) \cong SL(3, k_v), \quad \bar{G}(k_v) \cong PGL(3, k_v),$$

and

$$\{\xi \in \mathcal{D} \otimes_k k_v \;:\; \iota_{k_v}(\xi)\xi = 1\} \cong GL(3, k_v).$$

Again we have an embedding $\xi \mapsto \xi_v$ of $\{\xi \in \mathcal{D} : \iota(\xi)\xi = 1\}$ in $GL(3, k_v)$ mapping $G(k)$ into $SL(3, k_v)$. Now ξ_v is the image of $J_v \Psi(\xi) J_v^{-1}$ in $M_{3\times 3}(k_v)$.

To see that D *is* a norm when $v \neq v_0$ splits in ℓ, and $m \not\leftrightarrow k_v$, we use the following theorem from local class field theory (see Weil "Basic Number Theory", pp. 225–226):

Theorem. If K is a non-archimedean local field, and if L is a cyclic extension of K of degree n, then the image of $L^\times$ under the norm map $N_{L/K} : L^\times \to K^\times$ has index n in $K^\times$. When L is an unramified cyclic extension, then that image equals $\{x \in K^\times : v(x) \equiv 0 \pmod{n}\}$.

If $v \neq v_0$ splits in ℓ and $m \not\leftrightarrow k_v$, then by choice of D we have $w(D) = 0$ for both extensions w of v to ℓ. Moreover, neither w ramifies in m, as we see checking case by case. So the extension $k_v(Z)$ of $k_v = \ell_w$ is unramified, and so $N_{k_v(Z)/k_v}(\eta_v) = D$ for some $\eta_v \in k_v(Z)$, by the theorem.

If $v = v_0$, then $m \not\hookrightarrow k_v$, by choice of m, and $w(D) \neq 0$ by choice of D, for both extensions w of v_0 to ℓ. Assuming that w does not ramify in m, the theorem shows that D is not a norm.

There is only one case when w ramifies: $\mathcal{C}_{18}$. Here $k = \mathbb{Q}(r)$, where $r^2 = 6$, and v_0 is the unique 3-adic valuation on k (3 ramifies in k). This splits in ℓ since $3 = (\omega - 1)(\bar{\omega} - 1)$. The extensions w and $\bar{w}$ corresponding to $\mathfrak{p} = (\omega - 1)\mathfrak{o}_\ell$ and $\bar{\mathfrak{p}}$ both ramify in $m = k(\zeta_9)$ because $N_{m/\ell}(\zeta_9 - 1) = \omega - 1$.

In the $\mathcal{C}_{18}$ case, we carefully identify the index 3 subgroup of $k_{v_0}^\times$ which is the image of the norm map, and show that D is not in that image.

If D is not a norm, then $\mathcal{D} \otimes_\ell k_v$ is a division algebra over the local field k_v, and so

$$G(k_v) \cong \{\xi \in \mathcal{D} \otimes_\ell k_v : \mathrm{Nrd}(\xi) = 1\}$$

is compact. To see this, let $\xi_1, \ldots, \xi_9$ be a basis for $\mathcal{D} \otimes_\ell k_v$, and let $\mathfrak{o}_v = \{t \in k_v : |t|_v \leq 1\}$. Here $|t|_v = q_v^{-v(t)}$. Then $\mathfrak{o}_v$ is compact, and hence so is

$$S = \{\sum_\nu t_\nu \xi_\nu : t_\nu \in \mathfrak{o}_v \text{ for all } \nu, \text{ and } |t_\nu|_v = 1 \text{ for at least one } \nu\}.$$

Now $\mathrm{Nrd}(\xi) \neq 0$ for all $\xi \neq 0$, since $\mathcal{D} \otimes_\ell k_v$ is a division algebra. So there is a number $m > 0$ so that $|\mathrm{Nrd}(\xi)|_v \geq m$ for all $\xi \in S$. Now suppose that $\xi = \sum_\nu t_\nu \xi_\nu$ satisfies $\mathrm{Nrd}(\xi) = 1$. Let $T = \max\{|t_\nu|_v : \nu = 1, \ldots, 9\}$. Then $c\xi \in S$ for some $c \in k_v$ satisfying $|c|_v = 1/T$. Hence $|c|_v^3 = |\mathrm{Nrd}(c\xi)|_v \geq m$. Hence $T = 1/|c|_v \leq 1/m^{1/3}$. So $\{\xi \in \mathcal{D} \otimes_\ell k_v : \mathrm{Nrd}(\xi) = 1\}$ is closed and bounded, and so compact.

$G(k_v)$ and $\bar{G}(k_v)$ when v does NOT split in ℓ.

For a basis $\xi_1, \ldots, \xi_9$ of $\mathcal{D}$ over ℓ, setting

$$h\Big(\sum_\nu \xi_\nu \otimes_k x_\nu + \sum_\nu (s\xi_\nu) \otimes_k y_\nu\Big) = \sum_\nu \xi_\nu \otimes_\ell (x_\nu + s y_\nu)$$

we get an isomorphism, and setting

$$\tilde{\iota}\Big(\sum_\nu \xi_\nu \otimes_k x_\nu + \sum_\nu (s\xi_\nu) \otimes_k y_\nu\Big) = \sum_\nu \iota(\xi_\nu) \otimes_\ell (x_\nu - s y_\nu)$$

we get an involution of the second kind, and the commutative diagram

$$\begin{array}{ccc}
\mathcal{D} \otimes_k k_v & \xrightarrow{\ h\ } & \mathcal{D} \otimes_\ell k_v(s) \\
\Big\downarrow{\scriptstyle \iota_{k_v}} & & \Big\downarrow{\scriptstyle \tilde{\iota}} \\
\mathcal{D} \otimes_k k_v & \xrightarrow{\ h\ } & \mathcal{D} \otimes_\ell k_v(s)
\end{array}$$

and also the commutative diagram

$$\begin{array}{ccc}
\mathcal{D} \otimes_k k_v & \xrightarrow{\quad h \quad} & \mathcal{D} \otimes_\ell k_v(s) \\
\Big\downarrow {\scriptstyle \mathsf{Nrd}_{k_v}} & & \Big\downarrow {\scriptstyle \mathsf{Nrd}} \\
\ell \otimes_k k_v & \xrightarrow{\quad f \quad} & k_v(s)
\end{array}$$

where now f is the isomorphism $1 \otimes x + s \otimes y \mapsto x + sy$.

Corollary. If v does not split in ℓ, then

$$G(k_v) \cong \{\xi \in \mathcal{D} \otimes_\ell k_v(s) \;:\; \tilde{\iota}(\xi)\xi = 1 \;\&\; \mathrm{Nrd}(\xi) = 1\}.$$

and

$$\bar{G}(k_v) \cong \{\xi \in \mathcal{D} \otimes_\ell k_v(s) \;:\; \tilde{\iota}(\xi)\xi = 1\}/\{t1 \;:\; t \in k_v(s) \;\&\; \bar{t}t = 1\}.$$

$\mathcal{D} \otimes_\ell k_v(s) \cong M_{3\times 3}(k_v(s))$ or $\mathcal{D} \otimes_\ell k_v(s)$ is a division algebra.

It is never a division algebra, as noted in §2.2 of [PY]. We can also see this using theorem about norms, and explicit calculations when v ramifies in m.

The isomorphism $\mathcal{D} \otimes_\ell k_v(s) \cong M_{3\times 3}(k_v(s))$ induces isomorphisms

$$G(k_v) \cong \{\xi \in SL(3, k_v(s)) \ : \ g^* F'_v g = F'_v\},$$

$$\bar{G}(k_v) \cong \{g \in GL(3, k_v(s)) \ : \ g^* F'_v g = F'_v\}/\{t1 \ : \ t \in k_v(s) \ \& \ \bar{t}t = 1\}$$

and

$$\{\xi \in \mathcal{D} \otimes_k k_v \ : \ \iota_{k_v}(\xi)\xi = 1\} \cong \{g \in GL(3, k_v(s)) \ : \ g^* F'_v g = F'_v\}$$

for a suitable Hermitian $F'_v \in GL(3, k_v(s))$. In particular, we have an embedding $\xi \mapsto \xi_v$ of

$$\{\xi \in \mathcal{D} \ : \ \iota(\xi)\xi = 1\} \quad \text{into} \quad \{g \in GL(3, k_v(s)) \ : \ g^* F'_v g = F'_v\}.$$

The matrix F_v' depends on whether or not m embeds in $k_v(s)$.

Case a: the embedding $\ell = k(s) \hookrightarrow k_v(s)$ extends to an embedding $m \hookrightarrow k_v(s)$. Then

$$\mathcal{D} \xrightarrow{\Psi} M_{3\times 3}(m) \hookrightarrow M_{3\times 3}(k_v(s))$$

induces the above isomorphisms, with F_v' the image of $F \in M_{3\times 3}(m)$ in $M_{3\times 3}(k_v(s))$.

In particular, if v is an archimedean place of k, then ℓ does not embed in $k_v \cong \mathbb{R}$, but $m = k(s, W)$ embeds in $k_v(s) \cong \mathbb{C}$. After a further conjugation by the matrix we called Δ above, we may assume that $F_v' = F_0$, if $k = \mathbb{Q}$, or that $F_v' = F_0$ for one v and $F_v' = I$ for the other v, when $k = \mathbb{Q}(r)$.

Case b: the field m does not embed in $k_v(s)$. Then writing $m = k(s, Z)$, $m \otimes_\ell k_v(s)$ is the field $k_v(s, Z)$, and $\mathcal{D} \otimes_\ell k_v(s)$ is isomorphic to $M_{3\times 3}(k_v(s))$.

As $\Psi(\xi)$ is unitary with respect to a matrix F, $J_v \Psi(\xi) J_v^{-1}$ is unitary with respect to

$$F'_v = J_v^{*-1} F_v J_v^{-1} = \Theta^{*-1} C_v^{*-1} F_v C_v^{-1} \Theta^{-1},$$

where F_v is the image of $F \in M_{3\times 3}(m)$ in $M_{3\times 3}(k_v(s, Z))$.