Enumerating the Fake Projective Planes

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A fake projective plane is a smooth compact complex surface, *not* biholomorphic to the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$, with Betti numbers 1, 0, 1, 0, 1.

Mumford, 1979:

- gave first example,
- showed number of fpp's is finite.

Ishida and Kato, 1998, gave two examples.

Keum, 2006, gave an example.

Gopal Prasad and Sai-Kee Yeung, 2007, showed

- all fpp's fall into 41 "classes".
- classes defined using unitary groups in either
  - division algebras, or
  - matrix algebras.
- 28 classes of division algebra type. All are non-empty.
- 13 classes of matrix algebra type. They conjectured these empty.

Classes involve: fields k and ℓ, with $[\ell : k] = 2$, and extra data.

Either $k = \mathbb{Q}$ or $\dim_{\mathbb{Q}}(k) = 2$.

Tim Steger and I (2010):

(a) found all fpp's in each class.

(b) showed matrix algebra classes are empty.

Altogether, there are 100 fpp's (up to biholomorphism).

There are only 50 fpp's up to homeomorphism. We give presentations for each of the 50 fundamental groups.

Set

$$F_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$U(2,1) := \{g \in M_{3\times 3}(\mathbb{C}) : g^* F_0 g = F_0\},$$

$$PU(2,1) = U(2,1)/Z, \quad \text{where } Z = \{tI : |t| = 1\}.$$

$PU(2,1)$ acts on $B(\mathbb{C}^2) = \{(z_1, z_2) \ : \ |z_1|^2 + |z_2|^2 < 1\}$.

Theorem (Klingler, Yeung). The fundamental group Π of an fpp is a torsion-free cocompact arithmetic subgroup of $PU(2,1)$.

So an fpp is a ball quotient $B(\mathbb{C}^2)/\Pi$ for such a Π.

Explaining “arithmetic”.

central simple algebra: a finite dimensional algebra $\mathcal{A}$ over a field ℓ such that

- Centre of $\mathcal{A}$ is $\{t1 : t \in \ell\}$,

- no non-trivial proper two sided ideals.

Examples:

- $M_{n\times n}(\ell)$,

- division algebras.

Proposition. $\mathcal{A}$ central simple algebra $\Rightarrow$

$$\mathcal{A} \cong M_{n\times n}(\mathcal{D})$$

for some division algebra $\mathcal{D}$ over ℓ.

Corollary. $\mathcal{A}$ central simple algebra and $\dim_\ell \mathcal{A} = 9 \Rightarrow$

$$\mathcal{A} \cong M_{3\times 3}(\ell) \quad \textbf{or} \quad \mathcal{A} \text{ is a division algebra.}$$

When ℓ is a totally complex quadratic extension $k(s)$ of a totally real field k, an **involution ι of the second kind** on $\mathcal{A}$ is a map $\iota : \mathcal{A} \to \mathcal{A}$ such that

- $\iota(\iota(\xi)) = \xi$,
- $\iota(\xi\eta) = \iota(\eta)\iota(\xi)$
- $\iota(\xi + \eta) = \iota(\eta) + \iota(\xi)$, and
- $\iota(t\xi) = \bar{t}\,\iota(\xi)$,

for all $\xi, \eta \in \mathcal{A}$ and $t \in \ell$. Here $\bar{t} = a - bs$ if $t = a + bs \in \ell$.

Example: $\mathcal{A} = M_{3\times 3}(\ell)$ and $\iota(x) = x^*$.

Example: $\mathcal{A} = M_{3\times 3}(\ell)$, and

$$\iota(x) = F^{-1} x^* F,$$

where $F \in GL(3, \ell)$ and $F^* = F$.

Fact: Any involution of the second kind on $M_{3\times 3}(\ell)$ has this form.

For this ι:

$$\iota(x)x = 1 \quad \Leftrightarrow \quad x^* F x = F.$$

If $\mathcal{A}$ is a central simple algebra, there is a map $\mathrm{Nrd} : \mathcal{A} \to \ell$ which generalizes the determinant map $\det : M_{n\times n}(\ell) \to \ell$.

Proposition. For any field L containing ℓ:

(a) $\mathcal{A} \otimes_\ell L$ is central simple algebra over L,

(b) we can choose L and isomorphism $f : \mathcal{A} \otimes_\ell L \cong M_{n\times n}(L)$.

(c) for L, f as in (b), define

$$\mathrm{Nrd}(x) = \det f(x) \quad \text{for } x \in \mathcal{A},$$

This does not depend on the particular L and f we choose.

Now when we say that the fundamental group Π of a fake projective plane is arithmetic, we mean that

there are fields k and ℓ, with k totally real and ℓ a totally complex quadratic extension of k, and there is a central simple algebra $\mathcal{A}$ of dimension 9 over ℓ, and there is an involution ι of the second kind on $\mathcal{A}$, so that in the algebraic group G defined over k so that

$$G(k) = \{\xi \in \mathcal{A} : \iota(\xi)\xi = 1 \text{ and } \mathrm{Nrd}(\xi) = 1\},$$

there is principal arithmetic subgroup Λ of $G(k)$ which is commensurable with Π.

The term "principal arithmetic subgroup" will be explained later. It involves the groups $G(k_v)$ for the places v of k.

Explaining "commensurable" :

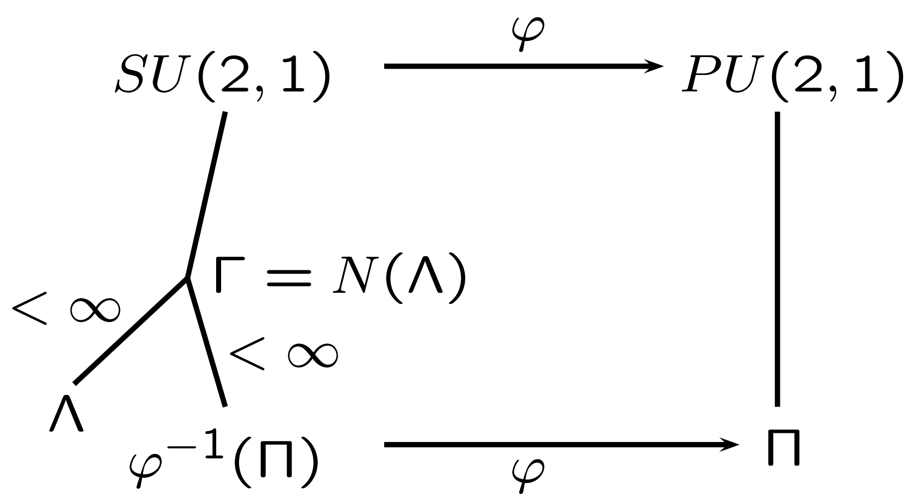
$\varphi : SU(2,1) \to PU(2,1)$: the canonical map $g \mapsto gZ$.

$$
\begin{array}{ccc}
SU(2,1) & \xrightarrow{\ \varphi\ } & PU(2,1) \\
\Big| & & \Big| \\
\varphi^{-1}(\Pi) & \xrightarrow[\ \varphi\ ]{} & \Pi
\end{array}
$$

We have a principal arithmetic subgroup $\Lambda \subset G(k)$, and for one archimedean place v of k we have an embedding

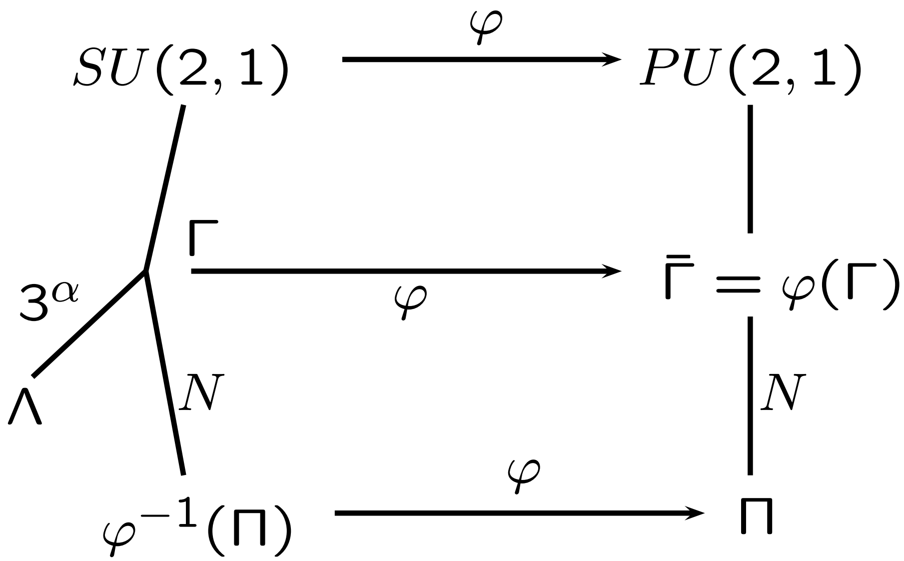
$$\Lambda \subset G(k) \hookrightarrow G(k_v) \cong G(\mathbb{R}) \cong SU(2,1).$$

Let Γ be the normalizer in $SU(2,1)$ of Λ.



For any other archimedean place v of k, we require

$$G(k_v) \cong SU(3).$$



Prasad and Yeung showed that $[\Gamma : \Lambda]$ is a power of 3.

The Euler-Poincaré characteristic of an fpp is

$$1 - 0 + 1 - 0 + 1 = 3.$$

Hirzebruch Proportionality Theorem:

$$\chi(B(\mathbb{C}^2)/\Pi) = 3\mathrm{vol}(\mathcal{F}_\Pi)$$

where vol is appropriately normalized hyperbolic volume on $B(\mathbb{C}^2)$, and $\mathcal{F}_\Pi$ is a fundamental domain for the action of Π on $B(\mathbb{C}^2)$. So

$$1 = \mathrm{vol}(\mathcal{F}_\Pi) = m(PU(2,1)/\Pi).$$

For discrete subgroups Γ of $G = PU(2,1)$, Haar measure μ_G on G induces a G-invariant measure $m_{G/\Gamma}$ on G/Γ, so that

$$\int_G f \, d\mu_G = \int_{G/\Gamma} f^\Gamma \, dm_{G/\Gamma},$$

where

$$f^\Gamma(g\Gamma) = \sum_{\gamma \in \Gamma} f(g\gamma).$$

If $\Gamma_1 \subset \Gamma_2$, then

$$m(G/\Gamma_1) = [\Gamma_2 : \Gamma_1] \, m(G/\Gamma_2).$$

Relation of $m_{G/\Gamma}$ on G/Γ to hyperbolic volume vol on $B(\mathbb{C}^2)$:

$$m(G/\Gamma) = \mathrm{vol}(\mathcal{F}_\Gamma)$$

where $\mathcal{F}_\Gamma \subset B(\mathbb{C}^2)$ is a fundamental domain for the action of Γ.

Prasad & Yeung mostly work with $SU(2,1)$, not $PU(2,1)$. Invariant measures are set up so that

$$m(SU(2,1)/\varphi^{-1}(\Pi)) = \frac{1}{3}m(PU(2,1)/\Pi),$$

($\varphi : SU(2,1) \to PU(2,1)$ canonical map).

So if $\Pi \subset PU(2,1)$ is the fundamental group of an fpp, then

$$m(SU(2,1)/\varphi^{-1}(\Pi)) = \frac{1}{3}.$$

Prasad has formulas for the numbers $m(SU(2,1)/\Lambda)$, where Λ is a principal arithmetic subgroup of the group $G(k)$.

Use the two inclusions $\Lambda \subset \Gamma$ and $\varphi^{-1}(\Pi) \subset \Gamma$:

$$m(SU(2,1)/\Lambda) = [\Gamma : \Lambda]\, m(SU(2,1)/\Gamma) = 3^{\alpha} m(SU(2,1)/\Gamma)$$

and

$$\frac{1}{3} = m(SU(2,1)/\varphi^{-1}(\Pi)) = [\Gamma : \varphi^{-1}(\Pi)]\, m(SU(2,1)/\Gamma).$$

Use $[\Gamma : \varphi^{-1}(\Pi)] = [\bar{\Gamma} : \Pi]$ to get

$$3^{\alpha-1} = [\bar{\Gamma} : \Pi]\, m(SU(2,1)/\Lambda).$$

$V_f :=$ set of non-archimedean places of the field k. Then $V_f \leftrightarrow$

- set of non-trivial non-archimedean valuations on k, or

- set of prime ideals $\mathfrak{p}$ in the ring $\mathfrak{o}_k$ of algebraic integers in k.

For $v \in V_f$, $k_v :=$ corresponding completion of k.

Let $(P_v)_{v \in V_f}$ be a "coherent" family of "parahoric subgroups" $P_v \subset G(k_v)$. A **principal arithmetic subgroup** of $G(k)$ has the form

$$\Lambda = \prod_{v \in V_f} P_v = \{g \in G(k) : g_v \in P_v \text{ for each } v \in V_f\}.$$

Here g_v is image in $G(k_v)$ of $g \in G(k)$.

Prasad's formula:

$$m(SU(2,1)/\Lambda) = \mu_{k,\ell} \prod_{v \in \mathcal{T}} e'(P_v),$$

where $\mu_{k,\ell}$ is a **rational** number depending only on k and ℓ, where the $e'(P_v)$'s are certain explicit integers, and where $\mathcal{T} \subset V_f$ is finite.

So if Π is the fundamental group of an fpp, then

$$3^{\alpha-1} = [\bar{\Gamma} : \Pi]\, \mu_{k,\ell} \prod_{v \in \mathcal{T}} e'(P_v).$$

Corollary. The numerator of $\mu_{k,\ell}$ is a power of 3.

We next explain the bounds Prasad and Yeung found for α.

If v splits in ℓ, then either

$$(a): \quad G(k_v) \cong SL(3, k_v), \quad \text{OR} \quad (b): \quad G(k_v) \quad \text{is compact.}$$

(b) only occurs if G comes from a division algebra $\mathcal{D}$, and $\mathcal{D} \otimes_\ell k_v$ is still a division algebra. This occurs for a finite **nonzero** number of $v \in V_f$.

$\boldsymbol{\mathcal{T}_0}$:= set of v's for which (b) holds.

If $v \in \mathcal{T}_0$, then $v \in \mathcal{T}$, $P_v = G(k_v)$ and $e'(P_v) = (q_v - 1)^2(q_v + 1)$.

Here q_v := size of residual field of k_v.

If v splits in ℓ and $G(k_v) \cong SL(3, k_v)$, then $G(k_v)$ acts on a building X_v which is "of type $\tilde{A}_2$" — it is a simplicial complex made up of vertices, edges and triangles.

A parahoric subgroup is the stabilizer in $G(k_v)$ of a simplex.

If P_v is the stabilizer of a vertex, then $v \notin \mathcal{T}$.

- If P_v is the stabilizer of an edge, then $e'(P_v) = q_v^2 + q_v + 1$,

- If P_v is the stabilizer of a triangle, then $e'(P_v) = (q_v^2 + q_v + 1)(q_v + 1)$,

and in both these cases, v is in $\mathcal{T}$.

If $v \in V_f$ does not split in $\ell = k(s)$, then

$$G(k_v) \cong \{g \in SL(3, k_v(s)) : \det(g) = 1 \text{ and } g^* F_v g = F_v\}$$

for an Hermitian $F_v \in GL(3, k_v(s))$. Now $G(k_v)$ acts on a building X_v which is a tree — a simplicial complex made up of vertices and edges. The vertices have two "types", edges having one vertex of each type.

If v ramifies in ℓ, tree is homogeneous, each vertex has $q_v + 1$ neighbours.

If v does not ramify in ℓ, then

Each type 1 vertex has $q_v^3 + 1$ neighbours, each of type 2.

Each type 2 vertex has $q_v + 1$ neighbours, each of type 1.

The group P_v is the stabilizer of a vertex or of an edge.

A non-split v belongs to $\mathcal{T}$ when

- P_v is the stabilizer of a type 2 vertex and v does not ramify. Then $e'(P_v) = (q_v^3 + 1)/(q_v + 1) = q_v^2 - q_v + 1$.

- P_v is the stabilizer of an edge. Then

  (i) $e'(P_v) = q_v^3 + 1$ if v does not ramify, and

  (ii) $e'(P_v) = q_v + 1$ if v ramifies in ℓ.

If v ramifies in ℓ and P_v is stabilizer of a vertex, $v \notin \mathcal{T}$. **N.B.:** stabilizers of type 1 vertices are **not conjugate** to stabilizers of type 2 vertices.

Let

$$\beta = \sharp\{v \in V_f : v \text{ splits in } \ell \text{ and } P_v \text{ stabilizes a triangle}\}.$$

Prasad and Yeung (equation (0) in their §2.3) give the bound

$$3^\alpha \leq h_{\ell,3} 3^{1+\#\mathcal{T}_0+\beta},$$

where $h_{\ell,3}$ is the order of the subgroup of the class group of ℓ consisting of elements of order dividing 3.

$v \in \mathcal{T} \Rightarrow e'(P_v) \geq 3$, so

$$\prod_{v \in \mathcal{T}} e'(P_v) \geq 3^{\#\mathcal{T}_0+\beta}$$

and

$$\mu_{k,\ell}[\bar{\Gamma} : \Pi] 3^{\#\mathcal{T}_0+\beta} \;\leq\; \mu_{k,\ell}[\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T}} e'(P_v) \;=\; 3^{\alpha-1} \leq\; h_{\ell,3} 3^{\#\mathcal{T}_0+\beta}.$$

So

$$\mu_{k,\ell}[\bar{\Gamma} : \Pi] \leq h_{\ell,3}.$$

In particular,

$$\mu_{k,\ell} \leq h_{\ell,3}.$$

The rational number $\mu_{k,\ell}$ equals

$$\frac{D_\ell^{5/2}\zeta_k(2)L_{\ell|k}(3)}{(16\pi^5)^n D_k},$$

involving the absolute values D_k and D_ℓ of the discriminants of k and ℓ, and $n = \dim_{\mathbb{Q}}(k)$. Prasad and Yeung use $\mu_{k,\ell} \leq h_{\ell,3}$ to obtain strong bounds on n, D_k and D_ℓ.

This allowed them to show that either

(a) $k = \mathbb{Q}$ and $\ell = \mathbb{Q}(\sqrt{-a})$ for some $a \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31\}$, or

(b) (k, ℓ) is one of a list of 40 pairs $\mathcal{C}_1, \ldots, \mathcal{C}_{40}$, for all of which $2 \leq \dim_{\mathbb{Q}}(k) \leq 4$.

The numerator of $\mu_{k,\ell}$ is power of 3 $\Rightarrow$ several of the above excluded. In particular, 25 of the 40 $\mathcal{C}_i$'s were eliminated.

In the cases not eliminated, the numerator of $\mu_{k,\ell}$ is equal to 1. Writing $\mu_{k,\ell} = 1/d_{k,\ell}$,

$$3^{\alpha-1} d_{k,\ell} = [\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T}} e'(P_v). \qquad (*)$$

When $k = \mathbb{Q}$, G must come from a division algebra $\mathcal{D}$ (see §4.1 of Prasad & Yeung), and so the non-empty set $\mathcal{T}_0$ is in $\mathcal{T}$.

Using this, they could eliminate more of the above possibilities.

For example, if $k = \mathbb{Q}$ and $\ell = \mathbb{Q}(\sqrt{-a})$ for $a = 3$, then $\mu_{k,\ell} = 1/216 = 1/(2^3 \times 3^3)$. So

$$3^{\alpha+2} \times 2^3 = [\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T}} e'(P_v)$$

is divisible by $e'(P_v) = (q_v - 1)^2(q_v + 1)$ for a $v \in \mathcal{T}_0$. If $q_v = 2n + 1$ is odd, then $e'(P_v) = 8n^2(n + 1)$ is divisible by 16. So q_v must be even. So v is the 2-adic valuation on k. This is inert in ℓ because -3 has no square root in $\mathbb{Q}_2$. But the v's in $\mathcal{T}_0$ have to split in ℓ.

So the case $k = \mathbb{Q}$, $\ell = \mathbb{Q}(\sqrt{-3})$ is eliminated.

Lemma. In each of the cases coming from a division algebra, the set $\mathcal{T}_0$ has just one element, and we can identify this element explicitly.

Proof. This is done case by case.

1. If $d_{k,\ell}$ is not divisible by 16 and if $v \in \mathcal{T}_0$, then q_v has to be even, and so v is a 2-adic valuation. If there is only one 2-adic valuation on k, we are done. Only in the case $\mathcal{C}_{21}$ (for which $d_{k,\ell} = 12$) are there two 2-adic valuations, but neither of them splits.

2. No $d_{k,\ell}$ is divisible by 16^2, and so there is at most one $v \in \mathcal{T}_0$ with q_v odd. For example, in the $\mathcal{C}_3$ case, $d_{k,\ell} = 32$. There is just one 2-adic valuation on k in this case, but it ramifies in ℓ, and so is not in $\mathcal{T}_0$.

Identifying the v in $\mathcal{T}_0$: E.g. in $\mathcal{C}_3$ case, we already know $\mathcal{T}_0$ consists of a single p-adic valuation v for p odd.

Fact: $q \geq 3$ odd & $(q-1)^2(q+1)$ divides $3^{\alpha-1} \times 32 \Rightarrow q = 3,\ 5$ or 7.

In $\mathcal{C}_3$ case, the primes 3 and 7 are inert in k, so $q_v = 9$ and 49 for the 3-adic and 7-adic valuations, respectively.

Conclusion: v has to be the 5-adic valuation on k.

Prasad and Yeung could eliminate several cases (of those coming from a division algebra) because there is no $v \in V_f$ which splits in ℓ such that $(q_v-1)^2(q_v+1)$ divides $3^{\alpha-1}d_{k,\ell}$.

Lemma. No $v \in V_f \setminus \mathcal{T}_0$ which splits in ℓ can be in $\mathcal{T}$. In particular, the number β of $v \in V_f$ which split in ℓ and for which P_v stabilizes a triangle is zero.

Proof. A prime p divides LHS $3^{\alpha-1} d_{k,\ell}$ of $(*)$ $\Rightarrow p =$ 2, 3, 5 or 7.

If $q_v^2 + q_v + 1$ divides some $3^{\alpha-1} d_{k,\ell}$, then $q_v \in \{2, 4\}$.

If $v \in \mathcal{T} \setminus \mathcal{T}_0$ splits in ℓ, then $q_v^2 + q_v + 1$ divides $e'(P_v)$, and so RHS of $(*)$. So $q_v = 2$ or 4. So v is 2-adic and $q_v^2 + q_v + 1 = 7$ or 21. Only in the cases $(k, \ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{-7}))$, $\mathcal{C}_{20}$ and $\mathcal{C}_{31}$ does 7 divide $d_{k,\ell}$. In each of these cases, there is just one 2-adic valuation on k, and it is in $\mathcal{T}_0$.

Corollary. The number α appearing in $(*)$ satisfies

$$\alpha = \begin{cases} 1 & \text{in the matrix algebra case,} \\ 2 & \text{in the division algebra case.} \end{cases}$$

Partial Proof. We have the bound $3^{\alpha} \leq h_{\ell,3} 3^{1+\sharp\mathcal{T}_0+\beta}$, and we now know that $\beta = 0$, and that $\sharp\mathcal{T}_0 = 1$ in the division algebra case. By definition, $\sharp\mathcal{T}_0 = 0$ in the matrix algebra cases. Now $h_\ell = 1$ in all our cases except $\mathcal{C}_{26}$ (when $h_\ell = 2$ and so $h_{\ell,3} = 1$) and $k = \mathbb{Q}$, $\ell = \mathbb{Q}(\sqrt{-23})$ (when $h_\ell = 3 = h_{\ell,3}$). This shows that $\alpha \leq 1 + \sharp\mathcal{T}_0$ holds except when $k = \mathbb{Q}$ and $\ell = \mathbb{Q}(\sqrt{-23})$. The proof that $\alpha = 1 + \#\mathcal{T}_0 = 2$ also holds in that special case, and that $\alpha \geq 1 + \#\mathcal{T}_0$ in all our cases, is given in §5.4 of Prasad-Yeung's paper.

The other method used by Prasad and Yeung to eliminate cases was based on the following:

Lemma. Suppose that Π is a torsion-free subgroup of finite index in a group $\bar{\Gamma}$. Let K be a finite subroup of $\bar{\Gamma}$. Then $|K|$ divides $[\bar{\Gamma} : \Pi]$.

Proof. There is an action $g\Pi \mapsto kg\Pi$ of K on the set $\bar{\Gamma}/\Pi$ of cosets. No $k \in K \setminus \{1\}$ can fix any $g\Pi$. For $kg\Pi = g\Pi$ implies that $g^{-1}kg \in \Pi$, contradicting the torsion-free hypothesis. So if $\bar{\Gamma}/\Pi$ is the union of s K-orbits, then $[\bar{\Gamma} : \Pi] = s|K|$.

For example, in the case $\mathcal{C}_{31}$, $d_{k,\ell}$ equals 147, and so the left hand side of $(*)$ equals $3^{\alpha-1} \times 147 = 3^{\alpha} \times 7^2$. So the primes dividing $[\bar{\Gamma} : \Pi]$ can only by 3 and 7. Prasad and Yeung produce an element g of $\bar{\Gamma}$ of order 2. Applying the lemma to $K = \langle g \rangle$, we get a contradiction.

Using the above methods, Prasad and Yeung were able to reduce the possibilities for (k, ℓ) to the following:

Division algebra cases:

a) $k = \mathbb{Q}$, $\ell = \mathbb{Q}(\sqrt{-a}\,)$ for $a \in \{1, 2, 7, 15, 23\}$.

b) $\mathcal{C}_2$, $\mathcal{C}_{10}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{20}$.

Matrix algebra cases:

(c) $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_8$, $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$.

They conjectured that there are no fake projective planes arising from the matrix algebra cases. Tim and I confirmed this conjecture.

Division algebra cases. $\mathcal{T}_0 = \{v_0\}$ for the p-adic $v_0 \in V_f$, for p below:

name	k	ℓ	p	$d_{k,\ell}$
$(a = 1, p = 5)$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-1}))$	5	96
$(a = 2, p = 3)$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-2})$	3	16
$(a = 7, p = 2)$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-7})$	2	21
$(a = 15, p = 2)$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-15})$	2	3
$(a = 23, p = 2)$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-23})$	2	1
$\mathcal{C}_2$	$\mathbb{Q}(\sqrt{5})$	$k(\sqrt{-3})$	2	135
$\mathcal{C}_{10}$	$\mathbb{Q}(\sqrt{2})$	$k(\sqrt{-5+2\sqrt{2}})$	2	3
$\mathcal{C}_{18}$	$\mathbb{Q}(\sqrt{6})$	$k(\sqrt{-3})$	3	48
$\mathcal{C}_{20}$	$\mathbb{Q}(\sqrt{7})$	$k(\sqrt{-1})$	2	21

Matrix algebra cases: $\mathcal{T}_0 = \emptyset$.

name	k	ℓ	$d_{k,\ell}$
$\mathcal{C}_1$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	600
$\mathcal{C}_3$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{5}, i)$	32
$\mathcal{C}_8$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{2}, i)$	128
$\mathcal{C}_{11}$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{3}, i)$	864
$\mathcal{C}_{18}$	$\mathbb{Q}(\sqrt{6})$	$\mathbb{Q}(\sqrt{6}, \zeta_3)$	48
$\mathcal{C}_{21}$	$\mathbb{Q}(\sqrt{33})$	$\mathbb{Q}(\sqrt{33}, \zeta_3)$	12

Here ζ_n is a primitive n-th root of 1.

In the division algebra cases, we can divide both sides of the equation $(*)$ by $e'(P_{v_0}) = (q_{v_0} - 1)^2(q_{v_0} + 1)$ for the $v_0 \in \mathcal{T}_0$, obtaining

$$3d_{k,\ell}/e'(P_{v_0}) = [\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T} \setminus \mathcal{T}_0} e'(P_v). \qquad (\dagger)$$

We've seen $v \in \mathcal{T} \setminus \mathcal{T}_0 \Rightarrow$, v doesn't split in ℓ. So v inert in ℓ or v ramifies in ℓ. In inert case:

- $e'(P_v) = q_v^2 - q_v + 1$ if P_v stabilizes a type 2 vertex, and
- $e'(P_v) = q_v^3 + 1 = (q_v^2 - q_v + 1)(q_v + 1)$ if P_v stabilizes an edge.

Fact: If $q \notin \{1, 2, 3, 5\}$, then $q^2 - q + 1$ is divisible by a prime $p \notin \{2, 3, 5, 7\}$. For $q = 2$, 3 and 5, $q^2 - q + 1$ equals 3, 7 and 21, respectively.

Division algebra cases. Possibilities for inert $v \in \mathcal{T} \setminus \mathcal{T}_0$:

name	q_{v_0}	$3d_{k,\ell}/e'(P_{v_0})$	v
$(a = 1, p = 5)$	5	3	$-$
$(a = 2, p = 3)$	3	3	$-$
$(a = 7, p = 2)$	2	21	$3, 5$
$(a = 15, p = 2)$	2	3	$-$
$(a = 23, p = 2)$	2	1	$-$
$\mathcal{C}_2$	4	9	$-$
$\mathcal{C}_{10}$	2	3	$-$
$\mathcal{C}_{18}$	3	9	2
$\mathcal{C}_{20}$	2	21	$3+, 3-$

We also list the $v \in V_f$ which ramify in ℓ:

name	q_{v_0}	$3d_{k,\ell}/e'(P_{v_0})$	v
$(a = 1, p = 5)$	5	3	2
$(a = 2, p = 3)$	3	3	2
$(a = 7, p = 2)$	2	21	7
$(a = 15, p = 2)$	2	3	3, 5
$(a = 23, p = 2)$	2	1	23
$\mathcal{C}_2$	4	9	3
$\mathcal{C}_{10}$	2	3	$17-$
$\mathcal{C}_{18}$	3	9	$-$
$\mathcal{C}_{20}$	2	21	$-$

Example. In the $\mathcal{C}_{18}$ case, $k = \mathbb{Q}(r)$ with $r^2 = 6$, and $d_{k,\ell} = 48$. In the division algebra case, $\alpha = 2$, and so equation $(*)$ tells us that

$$3^2 \times 2^4 = [\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T}} e'(P_v).$$

The only prime powers q for which $(q-1)^2(q+1)$ divides $3^2 \times 2^4$ are $q = 2$ and $q = 3$. The primes 2 and 3 ramify in k, so there is only one 2-adic valuation on k, and only one 3-adic valuation. The 2-adic valuation is inert in ℓ, and so cannot be in $\mathcal{T}_0$. So $\mathcal{T}_0 = \{v_0\}$ for the one 3-adic valuation v_0 on k. Then $(q_{v_0} - 1)^2(q_{v_0} + 1) = 2^4$, and $(\dagger)$ reads

$$9 = [\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T} \setminus \mathcal{T}_0} e'(P_v).$$

The only possibility for a $v \in \mathcal{T} \setminus \mathcal{T}_0$ such that $e'(P_v)$ divides 9 is the one 2-adic valuation.

For the one 2-adic valuation v, for which $q_v = 2$, P_v could stabilize an edge, and $e'(P_v) = q_v^3 + 1 = 9$. Equation (†) then tells us that

$$\mathcal{T} = \{2, 3\} \quad \text{and } [\bar{\Gamma} : \Pi] = 1.$$

If v is the 2-adic valuation, and P_v merely stabilizes a type 2 vertex, then $e'(P_v) = q_v^2 - q_v + 1 = 3$. Equation (†) now tells us that

$$\mathcal{T} = \{2, 3\} \quad \text{and } [\bar{\Gamma} : \Pi] = 3.$$

If v is the 2-adic valuation, and P_v stabilizes a type 1 vertex, then $v \notin \mathcal{T}$, and Equation (†) tells us that

$$\mathcal{T} = \{3\} \quad \text{and } [\bar{\Gamma} : \Pi] = 9.$$

There are two other cases (k, ℓ) in which P_v can be the stabilizer of an edge. When v ramifies in ℓ and P_v is the stabilizer of an edge, then $e'(P_v) = q_v + 1$. In the cases

$$(a = 1, p = 5) \quad \text{and} \quad (a = 2, p = 3),$$

the 2-adic valuation on $k = \mathbb{Q}$ ramifies in ℓ, and $q_v + 1 = 3$ divides $3d_{k,\ell}/e'(P_{v_0}) = 3$.

We may assume that each P_v is the stabilizer of a vertex.

Recall that Λ had the property that $\varphi^{-1}(\Pi) \subset N_{SU(2,1)}(\Lambda)$. Suppose that $\Lambda = \prod_{v \in V_f} P_v$, and that all the P_v's are maximal except for $v = v_1$. Replace P_{v_1} by a maximal $P'_{v_1} \supset P_{v_1}$. Leaving all the other P_v's unchanged, we get a principal arithmetic group Λ' such that $\Lambda \subset \Lambda'$.

Then $\varphi^{-1}(\Pi) \subset N_{SU(2,1)}(\Lambda')$ (see the end of §2.2 in [PY]).

So we may assume that each P_v is maximal for each $v \in V_f$.

These maximal P_v's are the maximal compact subgroups of $G(k_v)$.

When $v \in V_f \setminus \mathcal{T}_0$ splits in ℓ, there are 3 conjugacy classes of maximal compact subgroups in $G(k_v) \cong SL(3, k_v)$, but only one conjugacy class if you allow conjugation by elements of $\bar{G}(k_v) \cong PGL(3, k_v)$.

If $v \in V_f \setminus \mathcal{T}_0$ does not split in ℓ, there are 2 conjugacy classes of maximal compact subgroups in $G(k_v)$, and still two conjugacy classes even if you allow conjugation by elements of $\bar{G}(k_v)$.

If P_v is the stabilizer of a vertex x_v and if P'_v is the stabilizer of another vertex x'_v in the building X_v, then P_v can only be conjugated into P'_v by an element of $G(k_v)$ if x_v and x'_v have the same type.

In this case, v is in $\mathcal{T}$ only when that type is 2.

We use the following set in describing the classification of the fpps:

Definition. $\mathcal{T}_1 :=$ the set of $v \in V_f \setminus \mathcal{T}_0$ which do not split in ℓ for which P_v is the stabilizer of a type 2 vertex.

If $v \in \mathcal{T}_1$, then either $v \in \mathcal{T} \setminus \mathcal{T}_0$ or v ramifies in ℓ.

Proposition. (Proposition 5.3 on [PY]). If $\Lambda = \prod_{v \in V_f} P_v$ and $\Lambda' = \prod_{v \in V_f} P'_v$ are two principal arithmetic subgroups of $G(k)$, with each P_v and P'_v maximal, and if the set $\mathcal{T}_1$ is the same for both, then Λ' and Λ are conjugate by an element of $\bar{G}(k)$.

For each $v \in V_f$ such that v splits in ℓ, pick a vertex $x_v \in X_v$. For each $v \in V_f$ which does not split, pick a type 1 vertex x_v and a type 2 vertex x'_v. Let

$$\Lambda_{\mathcal{T}_1} = \{g \in G(k) : g \in P_v \text{ for all } v \in V_f\},$$

where

$$P_v = \begin{cases} \text{the stabilizer of } x'_v \text{ if } v \in \mathcal{T}_1, \\ \text{the stabilizer of } x_v \text{ if } v \in \mathcal{T} \setminus \mathcal{T}_1. \end{cases}$$

One has to assume that the choices made give a coherent family of parahoric subgroups.

So up to conjugation by an element of $\bar{G}(k)$, $\Lambda_{\mathcal{T}_1}$ is independent of the choices made of the x_v's and x'_v's.

For example, if (k, ℓ) is $(a = 15, p = 2)$, then $d_{k,\ell} = 3$. Equation $(*)$ reads

$$9 = [\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T}} e'(P_v).$$

The only prime power q such that $(q-1)^2(q+1)$ divides 9 is $q = 2$. So $\mathcal{T}_0 = \{2\}$, and

$$3 = [\bar{\Gamma} : \Pi] \prod_{v \in \mathcal{T}, v \neq 2} e'(P_v).$$

So $\mathcal{T} = \mathcal{T}_0 = \{2\}$. However, the primes 3 and 5 ramify in ℓ, and so there are 4 possibilities for $\mathcal{T}_1$:

$$\mathcal{T}_1 = \emptyset,\ \{3\},\ \{5\},\ \text{or } \{3, 5\}.$$

If Π is the fundamental group of an fpp coming from this pair (k, ℓ), then a conjugate of $\varphi^{-1}(\Pi)$ is commensurable with $\Lambda_{\mathcal{T}_1}$ for one of these four $\mathcal{T}_1$'s. So we have four "classes" of fpps coming from (k, ℓ).