50 Fake Planes: Two floating-point calculations for $\mathcal{F}$

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Recall that $d(\cdot,\cdot)$ stands for invariant distance on $B(\mathbf{C}^2)$, $S \subset \bar{\Gamma} \subset PU(2,1)$ is a finite set, and

$$\mathcal{F} = \mathcal{F}_S = \{z \in B(\mathbf{C}^2)\,;\ d(0,z) \le d(g(0),z)) \text{ for } g \in S\}$$

When everything has gone well, $\mathcal{F}$ will be a fundamental domain for $\Gamma = \langle S \rangle$.

We need to calculate two numbers associated to $\mathcal{F}$:

$$r_0 = \operatorname{radius}(\mathcal{F}) = \max\{d(0,z)\,;\ z \in \mathcal{F}\}$$
$$\operatorname{vol}(\mathcal{F}) = \operatorname{covol}(\Gamma)$$

The value of r_0 is needed for several things. As discussed yesterday, it is used to to verify that $\mathcal{F}_S$ is really a fundamental domain for Γ. Suppose this works out.

Obviously $\Gamma = \langle S \rangle \subseteq \bar{\Gamma}$, and if $\operatorname{vol}(\mathcal{F}) < \infty$, then $[\bar{\Gamma} : \Gamma] < \infty$. If we can verify that $\operatorname{covol}(\Gamma) = \operatorname{vol}(\mathcal{F}) = \operatorname{covol}(\bar{\Gamma})$, then $\Gamma = \bar{\Gamma}$, which is what we really want. For each $\bar{\Gamma}$, the value of $\operatorname{covol}(\bar{\Gamma})$ is given by [Prasad, Yeung, 2017].

For $z \in B(\mathbf{C}^2)$ we can write $z = tu$ with $t \geq 0$ and $|u| = (|u_1|^2 + |u_2|^2)^{1/2} = 1$. This expresses z in polar coordinates (t, u). For fixed u, the ray $(tu)_{t \geq 0}$ is a geodesic. One has

$$d(0, tu) = \operatorname{arctanh}(t) = \frac{1}{2} \log \frac{1+t}{1-t}$$

(With this scaling one sees that $d(0, z) \approx |z|$ for $|z| \ll 1$.)

As noted briefly yesterday, $\mathcal{F}$ is **star-like**:

$$\mathcal{F} \cap \{tu\,;\ t \geq 0\} = \{tu\,;\ 0 \leq t \leq t(u)\} \qquad \text{where } 0 < t(u) \leq 1.$$

If $t(u) = 1$ for even a single value of u, then $\mathcal{F}$ is not cocompact, and further calculation is useless. Otherwise the function $t(u)$ is used for both calculations. Clearly

$$r_0 = \operatorname{radius}(\mathcal{F}) = \operatorname{arctanh} t_0 = \operatorname{arctanh}\Big(\max_{|u|=1} t(u)\Big)$$

How does one calculate $t(u)$? For $z, w \in B(\mathbf{C}^2)$ we use the formula

$$\cosh^2(d(w,z)) = \frac{|1 - w^* z|^2}{(1-|w|^2)(1-|z|^2)} = \frac{|1 - (\bar{w}_1 z_1 + \bar{w}_2 z_2)|^2}{(1-|w|^2)(1-|z|^2)}$$

Thus

$$d(0,z) \geq d(w,z) \iff \frac{1}{1-|z|^2} \geq \frac{|1-w^*z|^2}{(1-|w|^2)(1-|z|^2)}$$
$$\iff 1 - |w|^2 \geq |1 - w^* z|^2 \iff |w^* z|^2 - 2\,\mathrm{Re}(w^* z) + |w|^2 \leq 0$$

We first note that for fixed w, this defines a convex set of z's in the Euclidean sense. Indeed, if $w = \left(\begin{smallmatrix} s \\ 0 \end{smallmatrix} \right)$ the last condition translates to

$$s^2 |z_1|^2 - 2\,\mathrm{Re}(s z_1) + s^2 \leq 0$$

picking out a disk for z_1 and putting no condition on z_2.

Now fix $g \in S$, let $w = g(0)$, fix $u \in \mathbf{C}^2$ with $|u| = 1$, and let $z = tu$. Then

$$d(0, tu) \leq d(g(0), tu) \iff t^2|w^*u|^2 - 2t\,\mathrm{Re}(w^*u) + |w|^2 \geq 0 \iff t \leq t_g(u)$$

where one solves the quadratic equation to calculate $t_g(u)$.

The formula for $t(u)$ is then

$$t(u) = \min_{g \in S} t_g(u)$$

Convexity Lemma: Suppose that $|u| = |v| = |u'| = 1$ and that v lies on the geodesic arc from u to u' in $\partial B(\mathbf{C}^2)$. If $t_g(u), t_g(u') \leq M$, then $t_g(v) \leq M$.

Proof. Let $w = g(0)$. Then $d(0, z) \geq d(w, z)$ holds for $z = Mu$ and for $z = Mu'$. Suppose $0 \leq a \leq 1$. By the convexity mentioned above, $d(0, z) \geq d(w, z)$ holds also for $z = M(au + (1-a)u')$. For some such a, $v = (au + (1-a)u')/|au + (1-a)u'|$. Thus

$$d(0, M|au + (1-a)u'|\, v) \geq d(w, M|au + (1-a)u'|\, v)$$

which shows that $t_g(v) \leq M|au + (1-a)u'| \leq M$. $\square$

Algorithm for calculating $t_0 = \max_{|u|=1} t(u)$. Before starting, fix a desired accuracy ϵ, say $\epsilon = 10^{-12}$.

- One maintains a list of simplexes of $\partial B(\mathbf{C}^2)$.
- The list is initialized so that the simplexes cover $\partial B(\mathbf{C}^2)$.
- For each vertex u of this decomposition, one calculates $t(u)$ and uses those values to calculate t_{max}, the largest such value.
- From time to time we subdivide a simplex into 8 smaller simplexes. Whenever we do this, we calculate the values $t(u)$ for the new vertices, and use them to update t_{max}.

- We work with the simplexes on the list one at a time.
- Our attention is on one such simplex. Let u be its central point and find $g \in S$ so that $t(u) = t_g(u)$. If we have $t_g(v) \leq t_{\text{max}} + \epsilon$ for each vertex of this simplex, then $t(v) \leq t_g(v) \leq t_{\text{max}} + \epsilon$ for every point v in the simplex: we can just discard this simplex.
- Otherwise, we subdivide the simplex into 8 simplexes, add them to the end of our list, and drop the original simplex.
- The algorithm terminates when the list is empty.

At termination $t_{\text{max}} \leq t_0 \leq t_{\text{max}} + \epsilon$.

To make this calculation mathematically rigorous, one would have to use interval arithmetic throughout. All I did was add something like 10^{-6} to t_{max} and use that as an upper bound on t_0.

How about $\mathrm{vol}(\mathcal{H})$? Of course, this is to be calculated using the invariant volume element. In polar coordinates (t, u) with $z = tu$, that volume element is:

$$dV(z) = \frac{2}{\pi^2} \frac{t^3}{(1-t^2)^3} \, dt \, d\Theta(u)$$

where Θ is the usual invariant measure on the unit 3-sphere, so $\Theta(\partial B(\mathbf{C}^2)) = 2\pi^2$.

Thus

$$\begin{aligned} \int_{\mathcal{F}} dV(z) &= \frac{2}{\pi^2} \int_{|u|=1} \int_0^{t(u)} \frac{t^3}{(1-t^2)^3} \, dt \, d\Theta(u) \\ &= \frac{1}{2\pi^2} \int_{|u|=1} \frac{t(u)^4}{(1-t(u)^2)^2} \, d\Theta(u) \end{aligned}$$

I used an extremely simple hand-written numerical method to calculate the last integral. This procedure lacks mathematical rigor. But the offense is minimal. Remember that $\mathrm{vol}(\mathcal{F}) = \mathrm{covol}(\Gamma) = [\bar{\Gamma} : \Gamma]\,\mathrm{covol}(\bar{\Gamma})$.

Consider one actual example where $\mathrm{covol}(\bar{\Gamma}) = 1/3$. Thus, $\mathrm{vol}(\mathcal{F})$ is necessarily an integral multiple of $1/3$. The numerical integration gave:

$$0.333327467996977$$

for the volume. I don't think it's worth considering the possibility that this is a really bad approximation to $2/3$ instead of a reasonably good approximation to $1/3$.

Still, if someone wanted to fill in this gap, the Convexity Lemma could be used to get a rigorous upper estimate on the integral.

This concludes my confession.

50 Fake Planes: The invariant volume element

Why is the invariant volume element what it is? Why is it normalized as it is?

Suppose $U(2,1)$ preserves the sesquilinear form given by $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Denote the form by $\langle \cdot, \cdot \rangle$.

We identify $B(\mathbf{C}^2)$ with the projectivized version of $\{z \in \mathbf{C}^3\,;\ \langle z, z \rangle > 0\}$ via:

$$\begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} \in \mathbf{C}^3 \leftrightarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in B(\mathbf{C}^2)$$

Consider the matrix $g_r = \begin{pmatrix} \cosh r & 0 & \sinh r \\ 0 & 1 & 0 \\ \sinh r & 0 & \cosh r \end{pmatrix} \in U(2,1)$ and calculate:

$$g_r \cdot \begin{pmatrix} w_1 \\ w_2 \\ 1 \end{pmatrix} = \begin{pmatrix} (\cosh r) w_1 + \sinh r \\ w_2 \\ (\sinh r) w_1 + \cosh r \end{pmatrix}$$

$$g_r \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} ((\cosh r) w_1 + \sinh r)/((\sinh r) w_1 + \cosh r) \\ w_2/((\sinh r) w_1 + \cosh r) \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$g_r(0) = g_r \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \tanh r \\ 0 \end{pmatrix}$$

$$\partial(z_1, z_2)/\partial(w_1, w_2)|_{w=0} = \begin{pmatrix} 1/\cosh^2 r & 0 \\ 0 & 1/\cosh r \end{pmatrix}$$

$$|\det \partial(z_1, z_2)/\partial(w_1, w_2)|_{w=0}|^2 = \operatorname{sech}^6 r$$

The last number gives the ratio between Euclidean 4-volume for z and Euclidean 4-volume for w. If we normalize by setting $dV(w) = dV_{\text{Euclid}}(w)$ at $w = 0$, then invariance of dV forces

$$dV(z) = (\cosh^6 r)\, dV_{\text{Euclid}}(z) \qquad \text{at } z = \begin{pmatrix} \tanh r \\ 0 \end{pmatrix}$$

$$dV(z) = \frac{dV_{\text{Euclid}}(z)}{(1 - t^2)^3} \qquad \text{at } z = \begin{pmatrix} t \\ 0 \end{pmatrix}.$$

Since rotations in $U(2)$ preserve both dV and $dV_{\text{Euclidean}}$,

$$dV(z) = \frac{dV_{\text{Euclid}}(z)}{(1-t^2)^3} \qquad \text{whenever } |z| = t.$$

If we use polar coordinate (t, u) for $z = tu$ with $|u| = 1$, then

$$dV_{\text{Euclidean}}(z) = t^3\, dt\, d\Theta(u) \qquad dV(z) = \frac{t^3}{(1-t^2)^3}\, dt\, d\Theta(u)$$

What about the normalization? The Hirzebruch proportionality principle says that we should calculate the normalization by looking at the compact form of $B(\mathbf{C}^2)$, namely $P^2(\mathbf{C})$.

If $G = U(2,1)$, then $G_{\textsf{comp}} = U(3)$. The form preserved by $G_{\textsf{comp}}$ is the one given by $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. In analogy with the noncompact case, we obtain $P^2(\mathbf{C})$ as the projectived version of $\{z \in \mathbf{C}^3\,;\ \langle z, z\rangle > 0\} = \mathbf{C}^3 \sim \{0\}$. The above calculations can be repeated with little change. In place of the matrix g_r used there we use $\left(\begin{smallmatrix} \cos r & 0 & \sin r \\ 0 & 1 & 0 \\ -\sin r & 0 & \cos r \end{smallmatrix}\right) \in U(3)$. If once again we use polar coordinates (t, u) for $z = tu$ with $|u| = 1$ the result is:

$$dV_{\textsf{comp}}(z) = \frac{t^3}{(1+t^2)^3}\, dt\, d\Theta(u)$$

This formula doesn't work for the line at infinity, but that isn't a problem since we intend to integrate the volume form.

Before integrating, I redefine the form using what turns out to be the right proportionality constant:

$$dV_{\mathsf{comp}}(z) = \frac{2}{\pi^2} \frac{t^3}{(1+t^2)^3}\, dt\, d\Theta(u)$$

Then

$$\begin{aligned}\int_{P^2(\mathbf{C})} dV_{\mathsf{comp}}(z) &= \frac{2}{\pi^2} \int_{|u|=1} \int_0^{+\infty} \frac{t^3}{(1+t^2)^3}\, dt\, d\Theta(u) \\ &= \frac{2}{\pi^2} \int_{|u|=1} \frac{1}{4} \frac{t^4}{(1+t^2)^2} \Bigg|_{t=0}^{t=+\infty} d\Theta(u) \\ &= \frac{1}{2\pi^2} \int_{|u|=1} d\Theta(u) = 1\end{aligned}$$

So for $P^2(\mathbf{C})$, with the volume form normalized as above, Euler characteristic $\chi = 3$ corresponds to volume 1.

The Hirzebruch principle says that if we use the same normalization in the noncompact case, namely:

$$dV_{\mathsf{comp}}(z) = \frac{2}{\pi^2} \frac{t^3}{(1-t^2)^3}\, dt\, d\Theta(u)$$

then also in this case Euler characteristic $\chi = 3$ corresponds to volume 1. Fake planes should have volume 1; their fundamental groups should have covolume 1.

However, we're not done yet, because we've omitted a crucial detail in the statement of Hirzebruch's principle. We have to use the canonical identification between the tangent spaces of $B(\mathbf{C}^2)$ and $P^2(\mathbf{C})$ at their respective origins.

Let

$$
\begin{aligned}
G_{\mathbf{C}} &= GL(3,\mathbf{C}) & \mathfrak{g}_{\mathbf{C}} &= \mathsf{gl}(3,\mathbf{C}) \\
G &= U(2,1) & \mathfrak{g} &= \mathfrak{u}(2,1) \\
G_{\mathsf{comp}} &= U(3) & \mathfrak{g}_{\mathsf{comp}} &= \mathfrak{u}(3) \\
K &= U(2)\times U(1) & \mathfrak{k} &= \mathfrak{u}(2)\oplus\mathfrak{u}(1)
\end{aligned}
$$

It is crucial that $G, G_{\mathsf{comp}} \subset G_{\mathbf{C}}$ and likewise for the Lie algebras.

We have $B(\mathbf{C}^2) \cong G/K$ and $P^2(\mathbf{C}) \cong G_{\text{comp}}/K$. Let

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ \bar{z}_1 & \bar{z}_2 & 0 \end{pmatrix} \right\}$$

$$i\mathfrak{p} = \left\{ \begin{pmatrix} 0 & 0 & iz_1 \\ 0 & 0 & iz_2 \\ i\bar{z}_1 & i\bar{z}_2 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & 0 & w_1 \\ 0 & 0 & w_2 \\ -\bar{w}_1 & -\bar{w}_2 & 0 \end{pmatrix} \right\}$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and this identifies $\mathfrak{p}$ with the tangent space at the origin of $G/K \cong B(\mathbf{C}^2)$. Similarly $\mathfrak{g}_{\text{comp}} = \mathfrak{k} \oplus i\mathfrak{p}$ and this identifies $i\mathfrak{p}$ with the tangent space at the origin of $G_{\text{comp}}/K \cong P^2(\mathbf{C})$.

Specifically, the tangent vector at the origin of $B(\mathbf{C}^2)$ corresponding to $\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ \bar{z}_1 & \bar{z}_2 & 0 \end{pmatrix} \in \mathfrak{p}$ is obtained as

$$\frac{d}{dr} \exp \begin{pmatrix} 0 & 0 & rz_1 \\ 0 & 0 & rz_2 \\ \bar{r}z_1 & \bar{r}z_2 & 0 \end{pmatrix} (0) = \frac{d}{dr} \exp \begin{pmatrix} 1 & 0 & rz_1 \\ 0 & 1 & rz_2 \\ \bar{r}z_1 & \bar{r}z_2 & 1 \end{pmatrix} (0) = \frac{d}{dr} \begin{pmatrix} rz_1 \\ rz_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

A nearly identical calculation shows that the tangent vector at the origin of $P^2(\mathbf{C})$ corresponding to $\begin{pmatrix} 0 & 0 & iz_1 \\ 0 & 0 & iz_2 \\ i\bar{z}_1 & i\bar{z}_2 & 0 \end{pmatrix} \in i\mathfrak{p}$ is simply $\begin{pmatrix} iz_1 \\ iz_2 \end{pmatrix}$.

Conclusion: the canonical identification between the tangent spaces at the origins of $B(\mathbf{C}^2)$ and $P^2(\mathbf{C})$ is, up to a factor of i, the obvious identification coming from the inclusion $B(\mathbf{C}^2) \subset P^2(\mathbf{C})$.

This means that

$$dV(z) = \frac{2}{\pi^2}\frac{1}{(1-|z|^2)^3}dV_{\textsf{Euclidean}}(z) \quad \text{on } B(\mathbf{C}^2)$$

$$\text{and} \quad dV_{\textsf{comp}}(z) = \frac{2}{\pi^2}\frac{1}{(1+|z|^2)^3}dV_{\textsf{Euclidean}}(z) \quad \text{on } P^2(\mathbf{C})$$

match at the respective origins. That is what the definition requires, so our little calculation is complete.

[Prasad, 1989] gives a formula for the normalized volume form which is valid for all semisimple groups. If one traces back and works out the meaning of the notations in that formula, it is not hard to apply in simple cases like this one.