50 Fake Planes: Finding enough elements of $\bar{\Gamma}$

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- We have a concrete division algebra $\mathcal{D}$.
- We are interested in a certain arithmetic subgroup $\bar{\Gamma} \subseteq P(\mathcal{D}^\times)$.
- We have conditions on $g \in \mathcal{D}^\times$ which say which elements are in $\bar{\Gamma}$.
- Somehow we find a few elements of $\bar{\Gamma}$. Call that set A.

For the calculations discussed here, we want to think of the elements of A as matrices in $PU(2,1)$. This means using the map $\mathbb{PU}(k) \to \mathbb{PU}(k_v) \cong PU(2,1)$ for a certain real place v. Concretely, our elements of A come as matrices, and all we need to do is (i) consider their entries as complex numbers, (ii) if $k = \mathbf{Q}[\sqrt{b}]$, choose the appropriate sign for $\sqrt{b}$, and (iii) conjugate by a matrix which converts the preserved form of signature $(2,1)$ to the standard form of signature $(2,1)$.

Let $0 \in B^2(\mathbf{C})$ be the origin. Let $d(\cdot,\cdot)$ be the invariant or hyperbolic distance on $B^2(\mathbf{C})$. We measure the "size" of $g \in \Gamma$ by $d(0, g(0))$. For purposes of size comparison, this is the same as using the Hilbert–Schmidt norm for matrices in $PU(2,1)$.

Two days back, for one case, Cartwright explained a method for finding the elements of $\bar{\Gamma}$ in order of their size. However, in most cases, we have no reason to believe that the elements of A are the smallest elements in $\bar{\Gamma}$.

Starting with A, and using inverses and products, we proceed to generate more elements of Γ.

- We maintain a list of the elements we have found.
- This list is initialized using A.
- We keep the list sorted by size.
- We fix an arbitrary limit N, say $N = 10\,000$ for the length of the list.
- When the list is full, and we have a new element to insert, we drop the last, that is biggest, element of the list.
- When all possible new elements have size that would put them beyond the end of the list, the algorithm terminates.

Discreteness of $\bar{\Gamma}$ guarantees that the algorithm terminates. In truth, my program generates new elements in batches, and updates the master list only after a batch is complete. Cartwright's program may work differently.

Let S' be the set of elements on the final list. Choose r_1 so that

$$r_1 < \max\{d(0, g(0)\,;\; g \in S'\}$$

and let

$$S = \{g \in S'\,;\; d(0, g(0)) \leq r_1\}$$

Then S satisfies

- $d(0, g(0)) \leq r_1$ for $g \in S$.
- $S = S^{-1}$.
- If $g, h \in S$ and $d(0, gh(0)) \leq r_1$, then $gh \in S$.

From this point on, we work with S and forget about A. Let $\Gamma = \langle S \rangle \subset \bar{\Gamma}$. We hope $\Gamma = \bar{\Gamma}$, but for this lecture, we'll just think about Γ. This is *not* the same group that was called Γ in earlier lectures and in [Prasad, Yeung, 2007].

We hope that

$$S = \{g \in \Gamma\,;\; d(0, g(0)) \leq r_1\}$$

When is this true? How can we prove it?

Consider

$$\mathcal{F}_S = \{z \in B(\mathbf{C}^2)\,;\; d(z, 0) \leq d(z, g(0)) \text{ for every } g \in S\}$$

If one used all the elements of $g \in \Gamma$ instead of just $g \in S$, this would be a Dirichlet fundamental domain for Γ.

Let

$$r_0 = \max\{d(0, z)\,;\; z \in \mathcal{F}_S\}$$

the **radius** of $\mathcal{F}_S$.

As will be explained later, it is possible to calculate r_0 numerically. Elements $g \in S$ for which $d(0, g(0)) > 2r_0$ have no effect on $\mathcal{F}_S$.

If $r_0 = +\infty$, then something has gone wrong. Either

- Γ is not cocompact. Thus $\Gamma \neq \bar{\Gamma}$ and $[\bar{\Gamma} : \Gamma] = \infty$; or
- r_1 is too small, most likely because N was chosen too small; or
- because N was chosen too small, S does not contain all of $\{g \in \Gamma\,;\ d(0, g(0)) \leq r_1\}$.

The first possibility can easily arise, almost always because the original set A isn't a generating set for $\bar{\Gamma}$ or for a finite index subgroup of $\bar{\Gamma}$. The last two possibilities can be dealt with in principle by increasing N, and this always worked in practice for the fake plane project.

Question: can one use methods anything like these in the case of non-uniform lattices?

From **[Cartwright, Steger, 2017]** *Finding Generators and Relations for Groups Acting on the Hyperbolic Ball*, arXiv:1701.02452.

Theorem: Suppose $S \subseteq PU(2,1)$ is a finite set satisfying

- $d(0, g(0)) \leq r_1$ for $g \in S$.
- $S = S^{-1}$.
- If $g, h \in S$ and $d(0, gh(0)) \leq r_1$, then $gh \in S$.

Let $\Gamma = \langle S \rangle$, let $\mathcal{F}_S$ and r_0 be as above, and suppose:

- $r_1 > 2r_0$.

Then

$$S = \{g \in \Gamma\,;\ d(0, g(0)) \leq r_1\}$$

Moreover, using S as the generators and all true identities of the form $g_1 g_2 g_3 = 1$ for $g_1, g_2, g_3 \in S$ as relations, we obtain a presentation of Γ.

This theorem is a close cousin of (a particular case of) Macbeath's theorem. The key difference is that Macbeath uses a set like:

$$S' = \{g \in \Gamma\,;\ g(X) \cap X \neq \emptyset\}$$

whereas our hypotheses on S can be checked on S itself, without knowing a priori what the rest of Γ looks like. If it was possible to apply Macbeath's theorem in our case, we would do so using $X = \{z\,;\ d(0,z) \leq r_0\}$.

The only properties of $B(\mathbf{C}^2)$ used in the proof are

- $B(\mathbf{C}^2)$ is simply connected (as in Macbeath's theorem), and
- $B(\mathbf{C}^2)$ is a geodesic metric space.

From here on, assume the hypotheses of the theorem.

Lemma 1: Γ is generated by $S_0 = \{g \in S\,;\ d(0, g(0)) \leq 2r_0\}$.

Note: the remainder of the talk followed the proof of the theorem as found in [Cartwright, Steger, 2017]. It had many pictures, and was given on the chalkboards.