50 Fake Planes: Finding enough elements of $\tilde{\Gamma}$

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• We have a concrete division algebra \( \mathcal{D} \).

• We are interested in a certain arithmetic subgroup \( \bar{\Gamma} \subseteq P(\mathcal{D}^\times) \).

• We have conditions on \( g \in \mathcal{D}^\times \) which say which elements are in \( \bar{\Gamma} \).

• Somehow we find a few elements of \( \bar{\Gamma} \). Call that set \( A \).

For the calculations discussed here, we want to think of the elements of \( A \) as matrices in \( PU(2,1) \). This means using the map \( \mathbb{P}U(k) \to \mathbb{P}U(k_v) \cong PU(2,1) \) for a certain real place \( v \).

Concretely, our elements of \( A \) come as matrices, and all we need to do is (i) consider their entries as complex numbers, (ii) if \( k = \mathbb{Q}[\sqrt{b}] \), choose the appropriate sign for \( \sqrt{b} \), and (iii) conjugate by a matrix which converts the preserved form of signature (2,1) to the standard form of signature (2,1).
Let $0 \in B^2(\mathbb{C})$ be the origin. Let $d(\cdot, \cdot)$ be the invariant or hyperbolic distance on $B^2(\mathbb{C})$. We measure the "size" of $g \in \Gamma$ by $d(0, g(0))$. For purposes of size comparison, this is the same as using the Hilbert–Schmidt norm for matrices in $PU(2, 1)$.

Two days back, for one case, Cartwright explained a method for finding the elements of $\bar{\Gamma}$ in order of their size. However, in most cases, we have no reason to believe that the elements of $\mathcal{A}$ are the smallest elements in $\bar{\Gamma}$.
Starting with $A$, and using inverses and products, we proceed to generate more elements of $\Gamma$.

- We maintain a list of the elements we have found.
- This list is initialized using $A$.
- We keep the list sorted by size.
- We fix an arbitrary limit $N$, say $N = 10000$ for the length of the list.
- When the list is full, and we have a new element to insert, we drop the last, that is biggest, element of the list.
- When all possible new elements have size that would put them beyond the end of the list, the algorithm terminates.

Discreteness of $\bar{\Gamma}$ guarantees that the algorithm terminates. In truth, my program generates new elements in batches, and updates the master list only after a batch is complete. Cartwright’s program may work differently.
Let $S'$ be the set of elements on the final list. Choose $r_1$ so that

$$r_1 < \max\{d(0, g(0) ; g \in S')\}$$

and let

$$S = \{g \in S' ; d(0, g(0)) \leq r_1\}$$

Then $S$ satisfies

- $d(0, g(0)) \leq r_1$ for $g \in S$.
- $S = S^{-1}$.
- If $g, h \in S$ and $d(0, gh(0)) \leq r_1$, then $gh \in S$.

From this point on, we work with $S$ and forget about $A$. Let $\Gamma = \langle S \rangle \subset \bar{\Gamma}$. We hope $\Gamma = \bar{\Gamma}$, but for this lecture, we’ll just think about $\Gamma$. This is not the same group that was called $\Gamma$ in earlier lectures and in [Prasad, Yeung, 2007].
We hope that

$$S = \{ g \in \Gamma ; \ d(0, g(0)) \leq r_1 \}$$

When is this true? How can we prove it?

Consider

$$\mathcal{F}_S = \{ z \in B(\mathbb{C}^2) ; \ d(z, 0) \leq d(z, g(0)) \text{ for every } g \in S \}$$

If one used all the elements of $g \in \Gamma$ instead of just $g \in S$, this would be a Dirichlet fundamental domain for $\Gamma$.

Let

$$r_0 = \max \{ d(0, z) ; \ z \in \mathcal{F}_S \}$$

the radius of $\mathcal{F}_S$.

As will be explained later, it is possible to calculate $r_0$ numerically. Elements $g \in S$ for which $d(0, g(0)) > 2r_0$ have no effect on $\mathcal{F}_S$. 
If $r_0 = +\infty$, then something has gone wrong. Either

- $\Gamma$ is not cocompact. Thus $\Gamma \neq \tilde{\Gamma}$ and $[\tilde{\Gamma} : \Gamma] = \infty$; or

- $r_1$ is too small, most likely because $N$ was chosen too small; or

- because $N$ was chosen too small, $S$ does not contain all of $\{g \in \Gamma; d(0, g(0)) \leq r_1\}$.

The first possibility can easily arise, almost always because the original set $A$ isn’t a generating set for $\tilde{\Gamma}$ or for a finite index subgroup of $\tilde{\Gamma}$. The last two possibilities can be dealt with in principle by increasing $N$, and this always worked in practice for the fake plane project.

**Question:** can one use methods anything like these in the case of non-uniform lattices?

**Theorem:** Suppose $S \subseteq PU(2,1)$ is a finite set satisfying

- $d(0, g(0)) \leq r_1$ for $g \in S$.
- $S = S^{-1}$.
- If $g, h \in S$ and $d(0, gh(0)) \leq r_1$, then $gh \in S$.

Let $\Gamma = \langle S \rangle$, let $\mathcal{F}_S$ and $r_0$ be as above, and suppose:

- $r_1 > 2r_0$.

Then

$$S = \{ g \in \Gamma ; d(0, g(0)) \leq r_1 \}$$

Moreover, using $S$ as the generators and all true identities of the form $g_1g_2g_3 = 1$ for $g_1, g_2, g_3 \in S$ as relations, we obtain a presentation of $\Gamma$. 

This theorem is a close cousin of (a particular case of) Macbeath’s theorem. The key difference is that Macbeath uses a set like:

\[ S' = \{ g \in \Gamma ; g(X) \cap X \neq \emptyset \} \]

whereas our hypotheses on \( S \) can be checked on \( S \) itself, without knowing a priori what the rest of \( \Gamma \) looks like. If it was possible to apply Macbeath’s theorem in our case, we would do so using \( X = \{ z ; d(0, z) \leq r_0 \} \).

The only properties of \( B(C^2) \) used in the proof are

- \( B(C^2) \) is simply connected (as in Macbeath’s theorem), and
- \( B(C^2) \) is a geodesic metric space.

From here on, assume the hypotheses of the theorem.

**Lemma 1:** \( \Gamma \) is generated by \( S_0 = \{ g \in S ; d(0, g(0)) \leq 2r_0 \} \).
Note: the remainder of the talk followed the proof of the theorem as found in [Cartwright, Steger, 2017]. It had many pictures, and was given on the chalkboards.