Exponential Stability of large BV Solutions in a Model of Granular Flow

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Joint work with: F. Ancona (Padova) & C. Christoforou (Cyprus)

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“PDE/Probability Interactions: Particle Systems, Hyperbolic Cons. Laws”

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MATEMATICA
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Università
degli Studi
di Padova

A toy model towards (?) stability for more general systems

- A Model for Granular Flow: Introduction
- A Model for Granular Flow: Mathematical Analysis
- Stability Results
- Stability Granular Flow

A Model for Granular Flow: Last contributors

Hadeler-Kutter [1999, Granular Matter]

‘Hadeler is a first-generation pioneer in mathematical biology’

Special issue in his memory on J. of Mathematical Biology

Amadori-Shen [2009, Communications in PDEs]

A Model for Granular Flow: Last contributors

. . . physicists Bouchaud, Cates, Prakash, Edwards, Boutreux, de Gennes, . . .

A Model for Granular Flow: What we are describing

Wiki: Khimsar Sand Dunes Village, India—Ankur2436

Kelso Dunes Avalanche Deposits, California—A. Wilson, The College of Wooster

A Model for Granular Flow: What we are describing

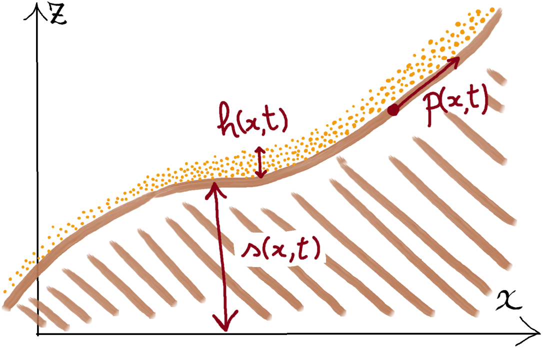
Video: Alessandro Ielpi, Laurentian University (Canada)

`https://www.youtube.com/watch?v=curEvUdhro4`

Dry sand: A grain flow induced from the brink of an eolian bedform in the Carcross Sand Dunes, Yukon Territory (June 2016)

Also: gravel in dunes, snow in avalanches,. . .

A Model for Granular Flow: PDE formulation



$h = h(x,t) > 0$: thickness of the rolling layer (on the top)

$s = s(x,t) > 0$: height of the standing layer (at the bottom)

$p = p(x,t)$: slope of the standing layer (at the bottom)

A Model for Granular Flow: PDE formulation

[Hadeler–Kuttler, 1999]

$h = h(x,t) > 0$: thickness of the rolling layer (on the top)
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$$\begin{cases} h_t & -\mathrm{div}\,(h\nabla s) = (|\nabla s| - 1)h \\ s_t & +(|\nabla s| - 1)h = 0 \end{cases} \qquad t \geq 0,\ x \in \mathbb{R}^2$$

normalized model; **critical slope**: $|\nabla s| = 1$

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normalized model; **critical slope**: $|\nabla s| = 1$

- we study one space dimension
- we differentiate the second equation
- we study $p := s_x$, slope of the standing layer, in place of s

A Model for Granular Flow: PDE formulation

$h = h(x,t) > 0$: thickness of the rolling layer (on the top)
$p = p(x,t) > 0$: slope of the standing layer (at the bottom)

$$\begin{cases} h_t - (hp)_x = (p-1)h, \\ p_t + ((p-1)h)_x = 0, \end{cases} \quad t \geq 0, \ x \in \mathbb{R}$$

and assign data

$$h(x,0) = \overline{h}(x)\,, \qquad p(x,0) = \overline{p}(x) \qquad \text{for} \quad x \ \in \mathbb{R}$$

A Model for Granular Flow: PDE formulation

$\delta_0 > \overline{h} \geq 0$: initial thickness of the rolling layer (on the top)
$\overline{p} > p_0 > 0$: initial slope of the standing layer (at the bottom)

$$\begin{cases} h_t - (hp)_x = (p-1)h, \\ p_t + ((p-1)h)_x = 0, & t \geq 0,\ x \in \mathbb{R} \\ h(x,0) = \overline{h}(x)\,, \quad p(x,0) = \overline{p}(x) \end{cases} \qquad \text{(GF)}$$

'mesoscopic' description ⇝ hyperbolic system of balance laws

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A toy model towards (?) stability for more general systems

System of balance laws:

$$u_t + A(u)u_x = g(u), \qquad u = (h,p)$$

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with eigenvalues

$$\lambda_{1,2}(h,p) = \frac{h - p \mp \sqrt{(p-h)^2 + 4h}}{2} \qquad \lambda_1 \approx -p; \lambda_2 \approx \frac{h}{p}$$

strictly hyperbolic in $\Omega = \{(h,p) : \ h \geq 0, \ p > p_0 > 0\}$

$$1\text{–char. field} = \begin{cases} \text{GNL} & \text{for } p > 1 \\ \text{LD} & \text{for } p = 1 \\ \text{GNL} & \text{for } p < 1 \end{cases}$$

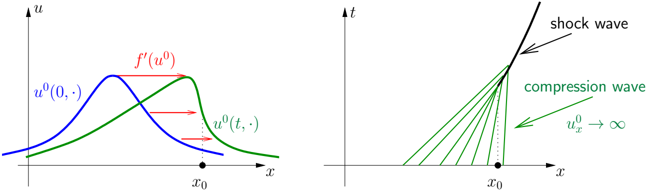
$$2\text{–char. field} = \begin{cases} \text{GNL} & \text{for } h \neq 0 \\ \text{LD} & \text{for } h = 0 \end{cases}$$

A Model for Granular Flow: What difficulties? I

- ☺ **Classical Solutions** for special initial data **[Shen, 2008]**
- ☹ Lack of regularity in general for conservation laws

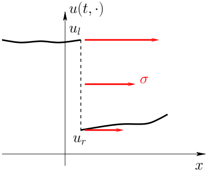
$$u(t,x) \;\; \text{smooth sol} \qquad \Longrightarrow \qquad \partial_t\, u + f'(u)\,\partial_x\, u = 0$$

Gradient Catastrophe also for single, convex equations



A Model for Granular Flow: What difficulties? II

We consider solutions in the sense of distributions



$\lambda_k(u^-) \geq \sigma \geq \lambda_k(u^+)$

$$\int_0^{+\infty}\int_{\mathbb{R}} \left[u\varphi_t + f(u)\varphi_x\right] dxdt = 0\,, \qquad \varphi \in \mathcal{C}^1_c(]0,+\infty[\times\mathbb{R})$$

A Model for Granular Flow: What difficulties? II

We consider solutions in the sense of distributions

- ☺ **well-posedness** theory developed for small BV data for entropy weak solutions (Lax '56, Liu). For CL:

Existence	Kružkov, 1970; Glimm, 1965; Bianchini-Bressan, 2000;
Uniqueness	Bressan & coll. 1992-1998; (. . .)
Stability	Liu–Yang 1999, Bressan–Liu–Yang 1999 for fields LD or GN

- ☹ The problem makes sense with locally large total variation
- ☹ The source is not dissipative
- ☹ The fields have linear degeneracy and genuine nonlinearity

A Model for Granular Flow: What difficulties? III

- ☺ **Global in time existence** of entropy solutions large in BV [Amadori-Shen, 2009]
- ☹ No uniqueness was proved, neither semigroup properties, nor stability

Theorem (Amadori–Shen, CPDE (2009))

For all $M_0, p_0 > 0$ *there exists* $\delta_0 > 0$ *small enough such that if*

$$\mathrm{TotVar}\ \bar{h} + \mathrm{TotVar}\ (\bar{p} - 1) \leq M_0,$$

$$0 \leq \bar{h} \leq \delta_0, \qquad p_0 \leq \bar{p} \leq M_0$$

hold then the Cauchy problem for (GF) has an entropy weak solution $(h(t,x), p(t,x))$ *defined for all* $t \geq 0$*.*

Moreover, there exists $\delta_0^*, p_0^*, M_1 > 0$ such that

$$0 \leq h(t,x) \leq \delta_0^* \qquad p_0^* \leq p(t,x) \leq M_1 \qquad \forall t > 0$$

Basic Functionals for Amadori-Shen, 2009

Total Variation: $V(u) \doteq \sum_{\alpha \text{ jumps of } u} |\rho_\alpha|$

Interaction Potential: $\mathcal{Q}(u) \doteq \mathcal{Q}_{hh} + \mathcal{Q}_{hp} + \mathcal{Q}_{pp}$

$$\mathcal{Q}_{hh} \doteq \sum_{\substack{k_\alpha = k_\beta = 1 \\ x_\alpha < x_\beta}} \omega_{\alpha\beta} |\rho_\alpha \rho_\beta| \, , \, \mathcal{Q}_{hp}(u) \doteq \sum_{\substack{k_\alpha = 2, \, k_\beta = 1 \\ x_\alpha < x_\beta}} |\rho_\alpha \rho_\beta| ,$$

$$\mathcal{Q}_{pp}(u) \doteq \sum_{\alpha \text{ or } \beta \text{ shock, } k_\alpha = k_\beta = 2} |\rho_\alpha \rho_\beta|$$

$$\omega_{\alpha,\beta} = \begin{cases} \delta_0 \min\{|p^\ell_\alpha - 1|, |p^\ell_\beta - 1|\} & \rho_\alpha, \rho_\beta \text{ 1-shocks lying both} \\ & \text{either in } \{p > 1\} \text{ or } \{p < 1\} \\ 0 & \text{otherwise} \end{cases}$$

Note: weighted functional $\mathcal{Q}_{hh}$ ⇝ existence for large BV data

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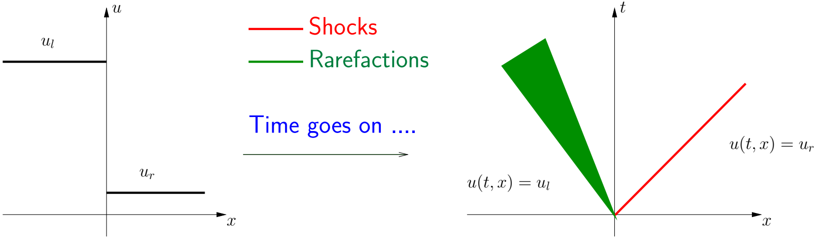
Glimm functional is: $\mathcal{G}(u) \doteq V(u) + c\mathcal{Q}(u)$

A Model for Granular Flow: What helps? Special features

☺ "simple" solutions to Riemann Problems



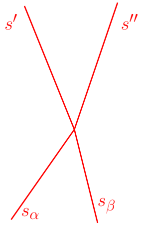
Riemann datum

Packs of i-th waves

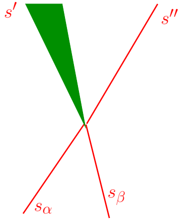
☺ h, the thickness of the rolling layer, is small

Wave interactions

- GNL fields: waves do not change nature after interactions
- Non GNL 1-field in GF: shock waves of the first family can become rarefaction waves (and vice versa) after interactions with waves of the second family, or also contact discontinuities

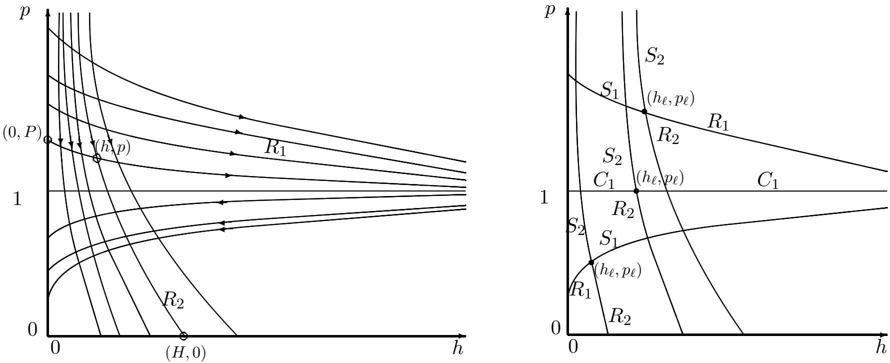


GNL fields



Non GNL fields

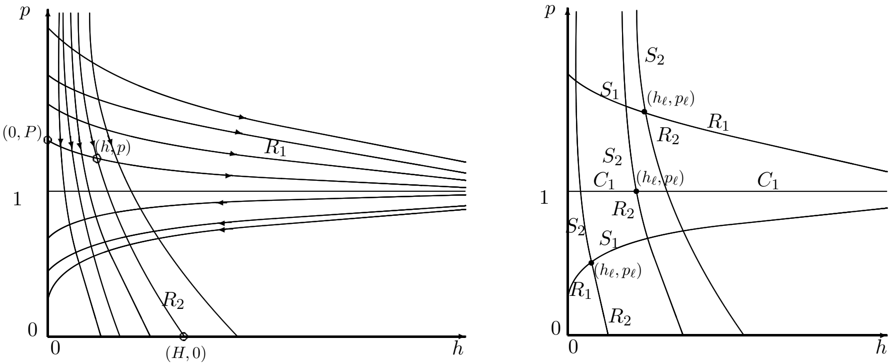
Characteristic and Wave Curves



Left: Rarefaction curves of the two families

Right: Right states connected to the left state (h_ℓ, p_ℓ) by an entropy admissible 1-wave or 2-wave of the homogeneous system

Characteristic and Wave Curves



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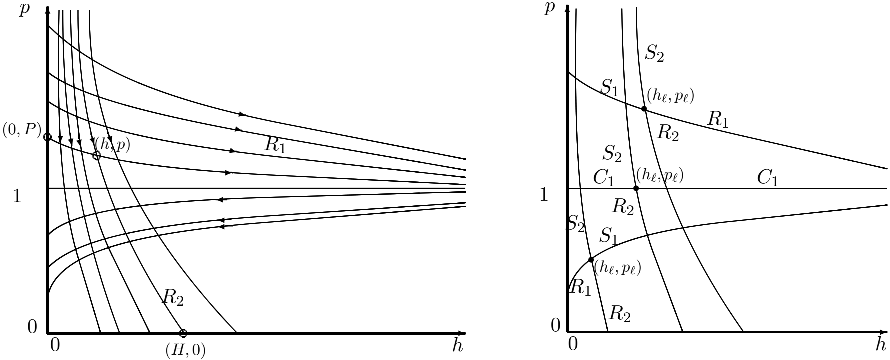

Left: Rarefaction curves of the two families

Right: Right states connected to the left state (h_ℓ, p_ℓ) by an entropy admissible 1-wave or 2-wave of the homogeneous system

A Model for Granular Flow: What difficulties? Summary

- ☹ no smooth solutions in general

⇝ entropy weak solutions

- ☹ possibly large total variation
- ☹ it has linear degeneracy and nonlinearity
- ☹ non dissipative source

⇝ special features of the problem

Existence of global solutions established [Amadori-Shen, 2009]

Goal: Uniqueness & Semigroup & L^1-stability in the initial data

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strictly hyperbolic in $\Omega = \{(h,p) : \ h \geq 0, \ p > p_0 > 0\}$

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$$\text{2–char. field} = \begin{cases} \text{GNL} & D\lambda_2 \cdot \mathbf{r}_2 < 0 & \text{for } h \neq 0 \\ \text{LD} & D\lambda_2 \cdot \mathbf{r}_2 = 0 & \text{for } h = 0 \end{cases}$$

Existing L^1–Stability Results Homotopy Method

Careful a-priori estimates on weighted norm of generalized tangent vectors to the flow generated by the system of conservation laws

- conservation laws GNL or LD, small BV
- non-GNL only 2×2 or Temple conservation laws, small BV
- a single work on GN Temple conservation laws in large BV
- a single work on 2×2 GN balance laws, small BV

[Amadori, Ancona, Bianchini, **Bressan**, Colombo, Corli, Crasta, Goatin, Gosse, Guerra, Marson, Piccoli 1996-2010]

Existing L^1–Stability Results Others

Probabilistic approach
Diagonal strictly hyperbolic systems with large monotonic data

- conservation laws non-GNL, large BV data but monotonic

`[Bolley-Brenier-Loeper 2005, Jourdain-Reygner 2016]`

"Vasseur" approach `[refer to his course, not` L^1`]`

Existing L^1–Stability Results Lyapunov-like

Construction of nonlinear functional, equivalent to $\mathbf{L^1}$ distance, decreasing in time along pairs of solutions

1. Conservation laws GNL or LD
   `[Liu-Yang 1999, Bressan-Liu-Yang 1999]`
2. Conservation laws GNL or LD, special data, in large BV
   [Lewicka-Trivisa 2002, Lewicka 2004, 2005]
3. Balance laws GNL or LD, dissipative source
   [Amadori-Guerra 2002]
4. Balance laws GNL or LD with non-resonant source
   [Amadori-Gosse-Guerra 2002]
5. Balance laws of Temple class non-GNL, in large BV
   [Colombo-Corli 2004]

Existing L^1–Stability Results: Bressan-Liu-Yang 1999

Lyapunov-like functional that controls the growth of the $\mathbf{L}^1$-distance between pairs of approximate solutions

$$\Phi = \Phi(u,v) \qquad u,v \in \mathbf{L}^1 \text{ piecewise constant}$$

$$\frac{1}{C} \cdot \left\|u - v\right\|_{\mathbf{L}^1} \leq \Phi(u,v) \leq C \cdot \left\|u - v\right\|_{\mathbf{L}^1}$$

(C depends on system, on TV of u, v, on $\mathbf{L}^\infty$ norm of u_h, v_h)

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Features: [on ε-front-tracking]

- At interaction times: $t \mapsto \Phi(u_k(t), v_k(t)) \searrow$
- Between interaction times: $\frac{d}{dt}\Phi(u_k(t), v_k(t)) \leq \mathcal{O}(1)\varepsilon$

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(C depends on system, on TV of u, v, on $\mathbf{L}^\infty$ norm of u_h, v_h)

☹ large BV data

☹ fields either linearly degenerate or genuinely nonlinear

☹ no source

Basic Functionals for (GF) in Bressan-Liu-Yang 1999

Total Variation: $V(u) \doteq \sum_{\alpha \text{ jumps of } u} |\rho_\alpha|$ [strength of waves in u]

Interaction Potential: $\mathcal{Q}(u) \doteq \mathcal{Q}_{hh} + \mathcal{Q}_{hp} + \mathcal{Q}_{pp}$ $\begin{bmatrix}\text{controls future interactions}\\ \text{among waves in } u\end{bmatrix}$

Glimm functional: $\mathcal{G}(u) \doteq V(u) + c\mathcal{Q}(u)$ $\begin{bmatrix}\text{controls over time}\\ \text{the variation of } u\end{bmatrix}$

$$\Phi(u(t), v(t)) \doteq \int_{-\infty}^{+\infty} \left[|\eta_1|(t,x)W_1(t,x) + |\eta_2|(t,x)W_2(t,x)\right] dx$$

$$W_i \doteq 1 + \kappa_1 \begin{bmatrix}\text{strength of waves in } u \text{ and } v\\ \text{which approach the } i\text{-wave } \eta_i\end{bmatrix} + \kappa_1\kappa_2(\mathcal{Q}(u) + \mathcal{Q}(v))$$

$\eta_i \doteq$ [distance along the i-th field among states $u(x,t)$ and $v(x,t)$]

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$\eta_i \doteq$ [distance along the i-th field among states $u(x,t)$ and $v(x,t)$]

A toy model towards (?) stability for more general systems

Almost all available results deal with GNL or LD CLs

Goal: Construct Lyapunov-like functional Φ for GF system in BV

- from [Amadori-Shen, 2009]: approximate solutions combining
  - front-tracking algorithm
  - operator splitting scheme with time steps $t_k = k\Delta t$
- For the homogeneous system
  - $\Phi(u_k(t), v_k(t))$ shall decrease at interaction times
  - between interactions, $\frac{d}{dt}\Phi(u_k(t), v_k(t)) \leq \mathcal{O}(1)\varepsilon$
- Estimating at time-steps, Φ exponentially increases in time

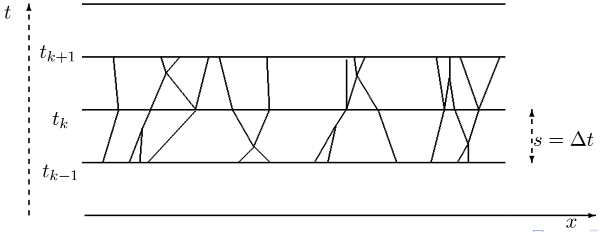
Approximate solutions: (h^s, p^s)

Homogeneous System

$$\begin{cases} h_t - (hp)_x = 0 \\ p_t + ((p-1)h)_x = 0. \end{cases} \qquad [t_{k-1}, t_k)$$

Next, at time t_k the function (h^s, p^s) is updated as follows

$$\begin{cases} h^s(t_k) = h^s(t_k-) + \Delta t[p^s(t_k-) - 1]h^s(t_k-) \\ p^s(t_k) = p^s(t_k-). \end{cases}$$



Functions needed for existence

Total Variation: $V(u) \doteq \sum_{\alpha \text{ jumps of } u} |\rho_\alpha|$

$\left[\begin{array}{c}\text{measures}\\ \text{strength of waves in } u\end{array}\right]$

Interaction Potential: $\mathcal{Q}(u) \doteq \mathcal{Q}_{hh} + \mathcal{Q}_{hp} + \mathcal{Q}_{pp}$

$\left[\begin{array}{c}\text{controls interactions}\\ \text{possibly occurring in the}\\ \text{future among waves in } u\end{array}\right]$

Glimm functional: $\mathcal{G}(u) \doteq V(u) + c\mathcal{Q}(u)$

$\left[\begin{array}{c}\text{controls over time}\\ \text{the variation of } u\end{array}\right]$

Stability Functional

New!

u, v approximate solutions; $\mathbf{S}_i(\cdot;\cdot)$ i-shock curve

η_1 and η_2 scalar functions defined implicitly by

$$v(t,x) = \mathbf{S}_2(\eta_2(t,x);\cdot) \circ \mathbf{S}_1(\eta_1(t,x); u(t,x))$$

Define

$$t \quad \mapsto \quad \Phi(u,v) \doteq \sum_{i=1}^{2} \int_{-\infty}^{\infty} \left[W_1(x) \left|\eta_1(x)\right| + W_2(x) \left|\eta_2(x)\right| \right] \; dx$$

where the weights W_i have the following form:

$$W_1(t,x) \doteq 1 + \kappa_{1\mathcal{A}} \cdot \mathcal{A}_1(t,x) + \kappa_{1\mathcal{G}} \cdot [\mathcal{G}(u(t)) + \mathcal{G}(v(t))]$$
$$W_2(t,x) \doteq 1 + \kappa_{2\mathcal{A}} \cdot \mathcal{A}_2(t,x) + \kappa_{2\mathcal{G}} \cdot [\mathcal{G}(u(t)) + \mathcal{G}(v(t))]$$

Stability Functional

New!

u, v approximate solutions; $\mathbf{S}_i(\cdot;\cdot)$ i-shock curve

η_1 and η_2 scalar functions defined implicitly by

$$v(t,x) = \mathbf{S}_2(\eta_2(t,x);\cdot) \circ \mathbf{S}_1(\eta_1(t,x); u(t,x))$$

Define

$$t \quad \mapsto \quad \Phi\big(u,v\big) \doteq \sum_{i=1}^{2} \int_{-\infty}^{\infty} \Big[W_1(x)\big|\eta_1(x)\big| + W_2(x)\big|\eta_2(x)\big|\Big] \; dx$$

Φ is equivalent to the L^1 norm

$$C_0\|u(t)-v(t)\|_{L^1} \le \Phi(u(t),v(t)) \le \bar{C}_0\|u(t)-v(t)\|_{L^1}$$

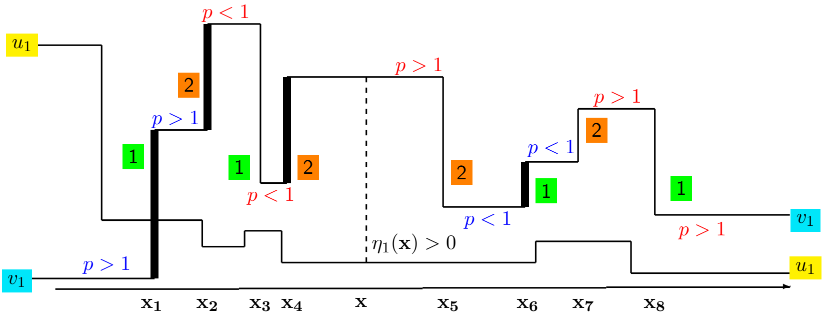
Weights in Φ: $W_i(x) \doteq 1 + \kappa_{i\mathcal{A}}\mathcal{A}_i(x) + \kappa_{i\mathcal{G}}[\mathcal{G}(u) + \mathcal{G}(v)]$

$$\mathcal{A}_1(x) \doteq \sum_\alpha |\rho_\alpha| \cdot |p^\ell_\alpha - 1| \quad \text{summing over} \left\{ \text{1-waves in } u \text{ and in } v \text{ which approach the 1-wave } \eta_1(x) \right\} +$$

$$+ \sum_\alpha |\rho_\alpha| \quad \text{summing over} \left\{ \text{2-waves in } u \text{ and in } v \text{ which approach the 1-wave } \eta_1(x) \right\},$$

$$\mathcal{A}_2(x) \doteq \sum_\alpha |\rho_\alpha| \quad \text{summing over} \left\{ \text{1-waves and 2-waves in } u \text{ and in } v \text{ which approach the 2-wave } \eta_2(x) \right\},$$

Approaching Waves in $\mathcal{A}_1$: in v towards $\eta_1(\mathbf{x}) > 0$



Regions $p < 1$, $p > 1$ are connected by 2−waves crossing $\{p = 1\}$

1−waves: $\gamma \to \lambda_1(\gamma; \cdot)$ strictly increasing on $\{p > 1\}$ $\quad \mathbf{x}_\alpha < \mathbf{x}$

1−waves: $\gamma \to \lambda_1(\gamma; \cdot)$ strictly decreasing on $\{p < 1\}$ $\quad \mathbf{x}_\alpha > \mathbf{x}$

Three categories of times:

A: at interaction times: $t \mapsto \Phi(u(t), v(t)) \searrow$

B: at times between interactions: $\frac{d}{dt}\Phi(u(t), v(t)) \leq \mathcal{O}(1) \cdot \varepsilon$

$$\Phi\big(u(t,\cdot), v(t,\cdot)\big) \leq \Phi\big(u(s,\cdot), v(s,\cdot)\big) + \mathcal{O}(1) \cdot \varepsilon\,(t-s)\ ,$$

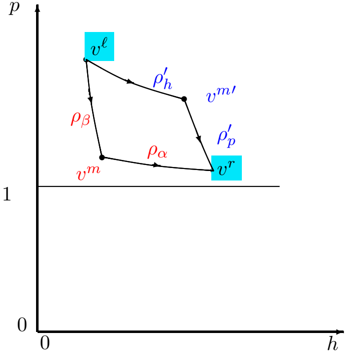
$\forall\ t_k < s < t < t_{k+1}$.

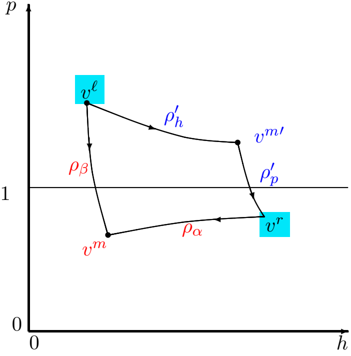
C: at time steps t_k, we prove that

$$\Phi(u(t_k+), v(t_k+)) - \Phi(u(t_k-), v(t_k-)) \leq \mathcal{O}(1)\,\Delta t\,\Phi(u(t_k-), v(t_k-))$$

[A:] at interaction times

$$\mathcal{A}_1(\tau+;x) - \mathcal{A}_1(\tau-;x) = |p^\ell_\beta - 1||\rho'_h| \leq \mathcal{O}(1)|\rho_\beta||\rho_\alpha|$$



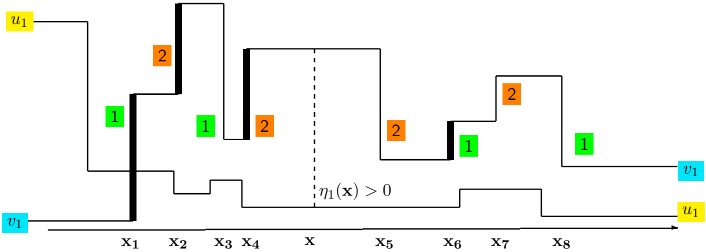


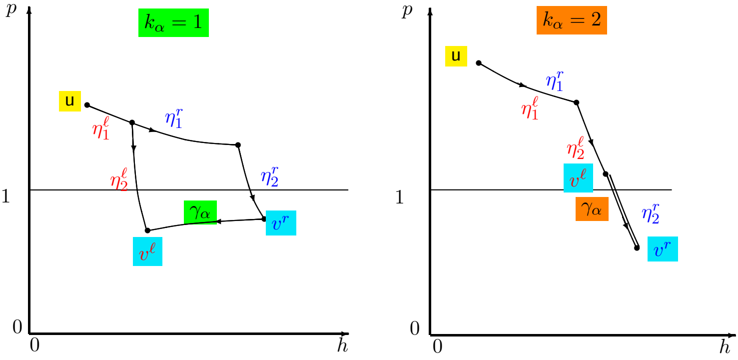
Examples of $2-1$ interactions

[B:] at times between interactions

$$\frac{d}{dt}\Phi(u(t),v(t)) = \sum_{\alpha \text{ jumps of } u \text{ and } v} \left(E_{\alpha,1} + E_{\alpha,2} \right) \leq \mathcal{O}(1)\cdot\varepsilon$$

$$E_{\alpha,i} \doteq W_i^{\alpha,r}|\eta_i^{\alpha,r}|(\lambda_i^{\alpha,r} - \dot{x}_\alpha) - W_i^{\alpha,\ell}|\eta_i^{\alpha,\ell}|(\lambda_i^{\alpha,\ell} - \dot{x}_\alpha) \qquad \text{errors}$$





Left: The jump at x_α is a 1-shock: $v^r = S_1(\gamma_\alpha; v^\ell)$
Right: The jump at x_α is a 2-shock: $v^r = S_2(\gamma_\alpha; v^\ell)$

[B:] at times between interactions

Generalized Interaction Estimates:

(i) The jump at x_α is a **1-shock**: $v^r = S_1(\gamma_\alpha; v^\ell)$

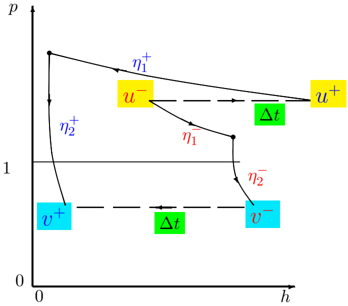
$$|\eta_1^r - \eta_1^\ell - \gamma_\alpha| + |\eta_2^r - \eta_2^\ell| \le C\left[|p_\alpha - 1|^2|\eta_1^\ell + \gamma_\alpha||\eta_1^\ell\gamma_\alpha| + h_{\mathsf{max}}|\eta_2^\ell\gamma_\alpha|\right]$$

(ii) The jump at x_α is along **2-shocks**: $v^r = S_2(\gamma_\alpha; v^\ell)$

$$|\eta_1^r - \eta_1^\ell| + |\eta_2^r - \eta_2^\ell - \gamma_\alpha| \le C|h_\alpha + \eta_1^\ell|^2|\eta_2^\ell\gamma_\alpha||\eta_2^\ell + \gamma_\alpha|$$

[C]: at time steps t_k

$$\Phi(u,v)(t_k^+) - \Phi(u,v)(t_k^-) \le \mathcal{O}(1)\,\Delta t\,\Phi(u,v)(t_k^-)$$



The shock curves connecting the states u^-, v^- before a time step of size Δt, and the states u^+, v^+ after such time step

Semigroup $\mathcal{S}$ for Homogeneous System

Theorem 1 (Ancona–C.–Christoforou, Preprint 2018)

$\forall\, M_0 \quad \exists\, \delta_0,\, \delta_p > 0,\ \exists\, \delta_0^*,\, \delta_p^*,\, M_0^*, L > 0, \quad \exists!$*map* $(t, \overline{u}) \mapsto \mathcal{S}_t \overline{u}$

$$\mathcal{S} : [0, +\infty) \times \left\{ \begin{array}{l} \mathrm{TotVar}\left(\frac{\overline{h}}{\overline{p}-1}\right) \leq M_0 \\ 0 \leq \overline{h} \leq \delta_0 \\ |\overline{p} - 1| \leq \delta_p \end{array} \right\} \to \left\{ \begin{array}{l} \mathrm{TotVar}\left(\frac{h}{p-1}\right) \leq M_0^* \\ 0 \leq h \leq \delta_0^* \\ |p - 1| \leq \delta_p^* \end{array} \right\}$$

which enjoys the following properties:

(i) $\mathcal{S}_0 \overline{u} = \overline{u}, \quad \mathcal{S}_{t+s} \overline{u} = \mathcal{S}_t \big(\mathcal{S}_s \overline{u} \big)$ *"semigroup"*

(ii) $\left\| \mathcal{S}_t \overline{u} - \mathcal{S}_s \overline{v} \right\|_{\mathbf{L}^1} \leq L \cdot \left(|s - t| + \| \overline{u} - \overline{v} \|_{\mathbf{L}^1} \right)$

(iii) $(h, p) \doteq \mathcal{S}_t \overline{u}(x)$ *entropy solution of conservation laws (GF)*

Semigroup $\mathcal{P}$ for Non–Homogeneous System

Theorem 2 (Ancona–C.–Christoforou, Preprint 2018)

$\forall\, M_0 \quad \exists\, \delta_0,\, \delta_p > 0,\ \exists\, \delta_0^*,\, \delta_p^*,\, M_0^*, L', C, \quad \exists!$*map* $(t, \overline{u}) \mapsto \mathcal{P}_t \overline{u}$

$$\mathcal{P} : [0, +\infty) \times \left\{ \begin{array}{l} \mathrm{TotVar}\left(\frac{\overline{h}}{\overline{p}-1}\right) \leq M_0 \\ 0 \leq \overline{h} \leq \delta_0 \\ |\overline{p} - 1| \leq \delta_p \end{array} \right\} \to \left\{ \begin{array}{l} \mathrm{TotVar}\left(\frac{h}{p-1}\right) \leq M_0^* \\ 0 \leq h \leq \delta_0^* \\ |p - 1| \leq \delta_p^* \end{array} \right\}$$

which enjoys the following properties:

(i) $\mathcal{P}_0 \overline{u} = \overline{u}, \quad \mathcal{P}_{t+s} u = \mathcal{P}_t \big(\mathcal{P}_s \overline{u}\big)$ *"semigroup"*

(ii) $\big\| \mathcal{P}_t \overline{u} - \mathcal{P}_s \overline{v} \big\|_{\mathbf{L}^1} \leq L' \cdot \big(e^{C t} \| \overline{u} - \overline{v} \|_{\mathbf{L}^1} + (t - s) \big)$

(iii) $(h, p) \doteq \mathcal{P}_t \overline{u}(x)$ *entropy weak solution of balance laws (GF)*

What we want to improve?

- what happens with boundary conditions?
- the Lipschitz constant shall really blow up in time?
- of course, there are other interesting models. . .
  . . . could we do it ‘more in general’?

THANK YOU