An introduction to Ambitwistors and strings

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CIRM Luminy: Twistors meet loops

With David Skinner. arxiv:1311.2564, and T Adamo, E Casali, Y Geyer, A Lipstein, R Monteiro, K Roehrig, & P Tourkine, . . .

Based also on Cachazo, He, Yuan arxiv:1306.2962,. . .

Incorporates Witten's twistor-string 2003.

Some of my collaborations with George

Conformally invariant powers of the Laplacian, I: Existence

…, R Jenne, LJ **Mason**, GAJ **Sparling** - Journal of the London …, 1992 - academic.oup.com

A geometric derivation is given of a family of scalar conformally invariant differential operators on conformal densities with leading part a power of the Laplacian. The derivation produces an operator for each positive integral power in odd dimensions, but only for a finite …

☆ 99 Cited by 458 Related articles All 4 versions »

Nonlinear Schrödinger and Korteweg-de Vries are reductions of self-dual Yang-Mills

LJ **Mason**, GAJ **Sparling** - Physics Letters A, 1989 - Elsevier

The nonlinear Schrödinger (NS) and KdV equations are shown to be reductions of the self-dual Yang-Mills (SDYM) equations. A correspondence between solutions of the NS and KdV equations and certain holomorphic vector bundles on a complex line bundle over the …

☆ 99 Cited by 222 Related articles All 6 versions »

Twistor correspondences for the soliton hierarchies

LJ **Mason**, GAJ **Sparling** - Journal of Geometry and Physics, 1992 - Elsevier

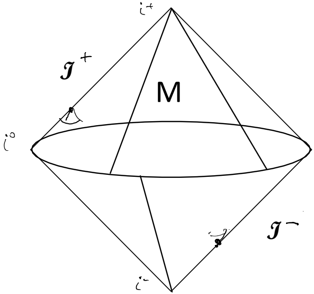
In this article we propose a new overview on the theory of integrable systems based on symmetry reduction of the anti-self-dual Yang—Mills equations and its twistor correspondence. First, the non-linear Schrödinger (NS) equations and the Korteweg de …

☆ 99 Cited by 72 Related articles All 3 versions

Amplitudes provide canonical gauge invariant observables.

Applications:

- Classical scattering problems.
- Gravitational wave signatures.
- Cosmological perturbations.
- Gauge theory and LHC.
- Quantum gravity (de Witt . . .).

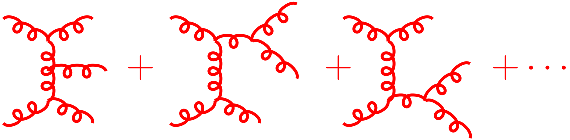


Need for ideas beyond space-time field theory

Feynman diagrams quickly become extremely complicated.
Even for the easier case of Yang-Mills/QCD we have:

Consider the five-gluon tree-level amplitude of QCD. Enters in calculation of multi-jet production at hadron colliders.

Described by following Feynman diagrams:



If you follow the textbooks you discover a disgusting mess.

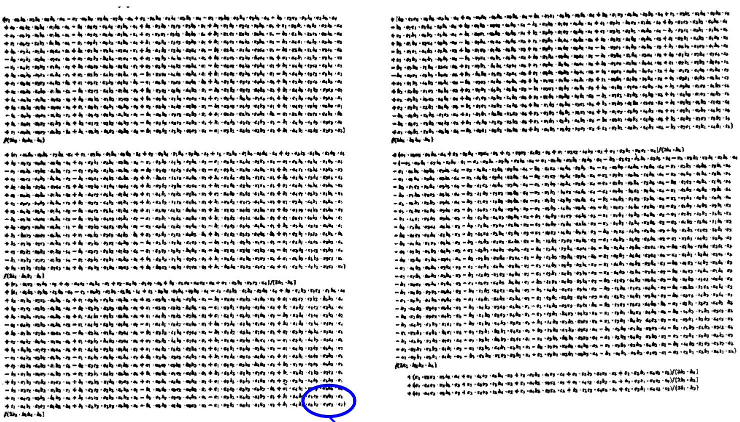
Result of a brute force calculation:

$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$

+ 19 more pages.

- Computationally intractable.
- Theoretically inscrutable.

Result of a brute force calculation:



$k_1 \cdot k_4 \, \varepsilon_2 \cdot k_1 \, \varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5$

$$+ \text{ 19 pages } = \frac{\langle i\, j \rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\langle 3\, 4\rangle\langle 4\, 5\rangle\langle 5\, 1\rangle}\, \delta^4\left(\sum_i k_i\right)$$

[Parke, Taylor 1984].

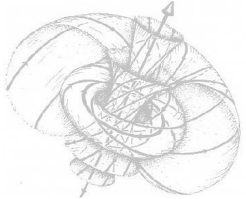
Hidden structure in general tree amplitudes

Parke-Taylor extended above formula to *n* particles MHV (1984)

$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \dots \langle n-1\; n\rangle\langle n\, 1\rangle}\delta^4\left(\sum_i k_i\right)$$

- Interpreted in twistor space [Nair 1986].
- as holomorphic twistor-string [Witten 2003]
- Full YM tree S-matrix, [Witten, Roiban, Spradlin, Volovich 2004].
- Gravity MHV n-point in 4d [Berends, Giele and Kuijf 1986], determinant formula [Hodges 2012],
- Full gravity tree S-matrix [Cachazo, Skinner 2012], from gravity twistor-string 2013 [Skinner 2013].
- Full YM and gravity tree S-matrix, all dims [Cachazo, He, Yuan 2013].
- from Ambitwistor-string [M. & Skinner 2013]

Ambitwistor spaces: spaces of complex null geodesics.



- Extends Penrose/Ward's gravity/Yang-Mills twistor constructions to non-self-dual fields.
- Yang-Mills Witten and Isenberg, et. al. 1978, 1985.
- Conformal and Einstein gravity LeBrun [1983,1991] Baston & M. [1987] .

Ambitwistor Strings:

- Strings in ambitwistor space [M. & Skinner 2013] give CHY formulae for gauge and gravity tree S-Matrices (all dims).
- Models for Einstein-YM, DBI, BI, NLS, etc. following CHY formulae, [Casali, Geyer, M., Monteiro, Roehrig 1506.08771].
- Loop amplitudes from higher genus [Adamo, Casali, Skinner 2013] proved and obtained efficiently from the Riemann sphere [Geyer, M., Monteiro, Tourkine, 1507.00321, 1511.06315, 1607.08887].

Provide string theories at $\alpha' = 0$ for field theory amplitudes.

Ambitwistors from chiral bosonic strings at $\alpha' = 0$

Bosonic ambitwistor string action:

- Σ Riemann surface, coordinate $\sigma \in \mathbb{C}$
- Complexify space-time (M, g), coords $X \in \mathbb{C}^d$, g hol.
- $(X, P) : \Sigma \to T^*M$, $\quad P \in K$, holomorphic 1-forms on Σ.

$$S_B = \int_\Sigma P_\mu \bar{\partial} X^\mu - e\, P^2/2\,.$$

Underlying geometry:

- Lagrange multiplier e enforces $P^2 = 0$,
- e is also worldsheet gauge field for Hamiltonian flow of P^2:

$$\delta(X, P, e) = (\alpha P, 0, 2\bar{\partial}\alpha).$$

Target reduces to

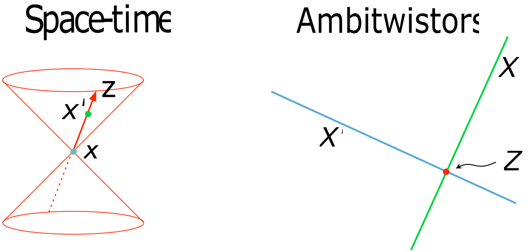
$$\mathbb{A} = T^*M|_{P^2=0}/\{\text{gauge}\}.$$

This is *Ambitwistor space*, space of complexified light rays.
It is holomorphic symplectic with potential $\theta = P_\mu dX^\mu$.

The geometry of space of complex light rays

Ambitwistor space $\mathbb{A}$ is space of complexified light rays.

- Light rays primary, an event $x \leftrightarrow$ its lightcone $X \subset \mathbb{A}$.
- Space-time $M =$ space of such $X \subset \mathbb{A}$.



Space-time geometry is encoded in complex structure of $\mathbb{A}$.

Theorem (LeBrun 1983 following Penrose 1976)

Complex structure of $\mathbb{A}$ determines $(M, [g])$. Correspondence stable under deformations of $P\mathbb{A}$ that preserve $\theta = P_\mu \mathrm{d}X^\mu$.

Quantize bosonic ambitwistor string:

- $(X, P) : \Sigma \to T^*M$,

$$S_B = \int_\Sigma P_\mu(\bar\partial + \tilde{e}\partial)X^\mu - e\,P^2/2\,.$$

- Gauge fix $\tilde{e} = e = 0$, $\rightsquigarrow$ ghosts & BRST Q
- Introduce vertex operators $V_i \leftrightarrow$ field perturbations.

Amplitudes are computed as correlators of vertex ops

$$\mathcal{M}_n = \langle V_1 \ldots V_n \rangle$$

For gravity add type II worldsheet susy $S_{\Psi_1} + S_{\Psi_2}$ where

$$S_\Psi = \int_\Sigma \Psi_\mu \bar\partial \Psi^\mu + \chi P \cdot \Psi\,.$$

From deformations of $\mathbb{A}$ to the scattering equations

Gravitons $\leftrightarrow$ vertex operators $V_i =$ def'm of action $\delta S = \int_\Sigma \delta\theta$.

- θ determines complex structure on $P\mathbb{A}$ via $\theta \wedge d\theta^{d-2}$. So:
- Deformations of complex structure $\leftrightarrow [\delta\theta] \in H^1_{\bar\partial}(P\mathbb{A}, L)$.

Proposition

For perturbation $\delta g_{\mu\nu} = \mathrm{e}^{ik\cdot x}\epsilon_\mu\epsilon_\nu$ of flat space-time

$$\delta\theta = \bar\delta(k\cdot P)\mathrm{e}^{ik\cdot X}(\epsilon\cdot P)^2$$

Proof: Penrose transform.

Ambitwistor repn $\Rightarrow \bar\delta(k\cdot P) \Rightarrow$ scattering equs.

Proposition

CHY formulae for massless tree amplitudes e.g. YM & gravity arise from appropriate choices of worldsheet matter.

Ambitwistors in 4 dimensions

In $d = 4$, (super) Twistor space $\mathbb{T} := \mathbb{C}^{4|\mathcal{N}}$ and $\mathbb{A} = T^*\mathbb{PT}$.

- Solve $P^2 = 0$ with 2 cpt spinors:

$$P_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}\,, \qquad \alpha = 0, 1, \quad \dot\alpha = \dot 0, \dot 1\,.$$

- Replace $x^{\alpha\dot\alpha}$ by $(\mu^\alpha, \tilde\mu^{\dot\alpha})$

$$\mu^{\dot\alpha} = -ix^{\alpha\dot\alpha}\lambda_\alpha\,, \qquad \tilde\mu^\alpha = ix^{\alpha\dot\alpha}\tilde\lambda_{\dot\alpha}$$

  invariant under $x^{\alpha\dot\alpha} \to x^{\alpha\dot\alpha} + \alpha\lambda^\alpha\tilde\lambda^{\dot\alpha}$.

- Define

$$Z = (\lambda_\alpha, \mu^{\dot\alpha}) \in \mathbb{T}\,, \quad W = (\tilde\mu^\alpha, \tilde\lambda_{\dot\alpha}) \in \mathbb{T}^*$$

  Incidence $\rightsquigarrow Z \cdot W = 0$, where $Z \cdot W := \lambda_\alpha\tilde\mu^\alpha + \mu^{\dot\alpha}\tilde\lambda_{\dot\alpha}$.

- Thus:

$$\mathbb{A} = \{(Z, W) \in \mathbb{T} \times \mathbb{T}^* | Z \cdot W = 0\}/\{Z \cdot \partial_Z - W \cdot \partial_W\}\,.$$

The symplectic potential

$$\Theta := P \cdot dx = iW \cdot \mathrm{d}Z - iZ \cdot \mathrm{d}W$$

on $\mathbb{A}$ gives action

$$S_{4d} = i \int_\Sigma W \cdot \bar\partial Z - Z \cdot \bar\partial W + aZ \cdot W + S_m \,.$$

- a is Lagrange multiplier $\rightsquigarrow Z \cdot W = 0$ and gauges phase.
- $S_m =$ theory dependent worldsheet action for extra fields, e.g., current algebra for gauge theory.
- At $\mathcal{N} = 4$ similar to Witten/Berkovits twistor-string.
- Take $Z, W \in K^{1/2}$, (reduce to $Z \cdot W = 0$ globally).
- The worldsheet OPE encodes Poisson structure

$$Z(\sigma)^I W(\sigma')_J \sim \frac{\delta^I_J}{\sigma - \sigma'} + \dots \qquad (1)$$

Key proposal: S-matrix for field theory Φ (e.g., gauge or gravity)

$$\mathcal{S}[\Phi_{in}] = \int D[WZ\ldots]\mathrm{e}^{iS_{4d}^{\Phi_{in}}}$$

Miracle: RHS is extremely simple perturbatively.

Set $\Phi_{in} = \sum_i \epsilon_i \Phi_i$, where Φ_i are linear perturbations

$$S_{4d}^{\Phi_{in}} = S_{4d}^{\Phi=0} + \sum_i \epsilon_i \int V_i$$

V_i are *vertex operators* represented by $\frac{\delta\theta}{\delta\Phi}(\Phi_i) \in H^1(\mathbb{A},\ldots)$.
For amplitude must compute:

$$\mathcal{M}(\Phi_1,\ldots,\Phi_n) = \int D[WZ\ldots]V_1\ldots V_n\,\mathrm{e}^{iS_{4d}^{\Phi=0}} =: \langle V_1\ldots V_n\rangle$$

Gauge theory amplitudes in 4d

Take $S_m =$ current algebra (WZW, free fermions etc.) $\rightsquigarrow$

- $J \in \mathfrak{g} \otimes \Omega^1_\Sigma$ Lie algebra valued current from S_m.
- Gauge theory wave fns given by cohomology classes on $\mathbb{PT}$ ($+$ helicity) and $\mathbb{PT}^*$ ($-$ helicity) $\times$ colour $t_i \in \mathfrak{g}$

$$f_i \in H^1(\mathbb{PT}, \mathcal{O}), \qquad \tilde{f}_i \in H^1(\mathbb{PT}^*, \mathcal{O}),$$

- $\rightsquigarrow$ YM vertex ops

$$V_i = \int_\Sigma f_i(Z) J \cdot t_i , \qquad \widetilde{V}_i = \int_\Sigma \tilde{f}_i(W) J \cdot t_i,$$

- For an N^{k}MHV amplitude

$$\begin{aligned} \mathcal{A}(1, \ldots, n) &= \langle V_1 \ldots V_k \tilde{V}_{k+1} \ldots \tilde{V}_n \rangle \\ &= \int D[ZW] V_1 \ldots V_k \tilde{V}_{k+1} \ldots \tilde{V}_n \, e^{iS_{4d}} . \end{aligned}$$

Yang-Mills amplitude formulae

- Momentum eigenstates $k_i = \lambda_i \tilde{\lambda}_i$ arise from:

$$V_i = \int \frac{\mathrm{d}s_i}{s_i} \bar{\delta}^2(\lambda_i - s_i \lambda(\sigma_i)) \mathrm{e}^{is_i[\mu\, \tilde{\lambda}_i]} J \cdot t_i \,, \qquad \widetilde{V}_i = \text{tilded version}$$

- Take exponentials into action $\rightsquigarrow$ sources for $\lambda, \tilde{\lambda}$

$$\bar{\partial}\lambda = \sum_{r=k+1}^{n} s_r \lambda_r \delta^2(\sigma - \sigma_r)\,, \qquad \bar{\partial}\tilde{\lambda} = \sum_{i=1}^{k} s_i \tilde{\lambda}_i \delta^2(\sigma - \sigma_i)$$

- With hgs coords $\sigma_a = \frac{1}{s}(1, \sigma)$ on $\mathbb{CP}^1$, solutions are

$$\lambda(\sigma) = \sum_{r=k+1}^{n} \frac{\lambda_r}{(\sigma\, \sigma_r)}\,, \qquad \tilde{\lambda}(\sigma) = \sum_{i=1}^{k} \frac{\tilde{\lambda}_i}{(\sigma\, \sigma_i)}\,,$$

- Amplitude reduces to residue formula

$$\mathcal{A}(1, \dots, n) = \int \prod_{i=1}^{k} \bar{\delta}^2(\tilde{\lambda}_i - \tilde{\lambda}(\sigma)) \prod_{i=k+1}^{n} \bar{\delta}^2(\lambda_i - \lambda(\sigma_i)) \prod_{i=1}^{n} \frac{\mathrm{d}^2 \sigma_i}{(\sigma_i\, \sigma_{i+1})}$$

- Delta functions localize integrals on scattering equs

- Introduce skew *infinity twistors*

$$\langle Z_1 Z_2\rangle = \langle \lambda_1\,\lambda_2\rangle\,, \quad [W_1\,W_2] = [\tilde\lambda_1\,\tilde\lambda_2]\,.$$

- include Fermions $(\rho, \tilde\rho) \in \mathbb{T}\times\mathbb{T}^*\otimes K^{1/2}\Sigma$.
- gauge currents from an $SL(1|2)$.
- Vertex ops from $h \in H^1(\mathbb{PT}, \mathcal{O}(2))$ and conjugates are

$$V_h = [W, \partial]h + \rho\cdot\partial[\tilde\rho,\partial]h, \qquad \widetilde{V}_{\tilde h} = \langle Z, \tilde\partial\rangle \tilde h + \tilde\rho\cdot\tilde\partial\langle\rho,\tilde\partial\rangle\tilde h\,.$$

- Amplitude formulae: replace $\prod \frac{1}{(i\,i{+}1)}$ by $\det{}'\,\mathcal{H}$

$$\mathbb{H}_{ij} = \frac{\langle i\,j\rangle}{(i\,j)}, \quad i,j\le k\,, \qquad \widetilde{\mathbb{H}}_{ij} = \frac{[i\,j]}{(i\,j)}\,, \quad i,j>k\,,$$

$$\mathcal{H}_{ij} = \begin{pmatrix}\mathbb{H} & 0\\ 0 & \widetilde{\mathbb{H}}\end{pmatrix}\,, \qquad \mathcal{H}_{ii} = -\sum_{l\neq i}\mathcal{H}_{il}\,.$$

$$\mathcal{M}(1,\dots,n) = \int \det{}'\mathcal{H}\prod_{i=1}^n \mathrm{d}^2\sigma_i \prod_{i=1}^k \bar\delta^2(\tilde\lambda_i - \tilde\lambda(\sigma)) \prod_{i=k+1}^n \bar\delta^2(\lambda_i - \lambda(\sigma_i))$$

Amplitude formulae

$$\mathcal{M}(1,\ldots,n) = \int \det{}'\mathcal{H} \prod_{i=1}^{n} \mathrm{d}^2\sigma_i \prod_{i=1}^{k} \bar\delta^2(\tilde\lambda_i - \tilde\lambda(\sigma)) \prod_{i=k+1}^{n} \bar\delta^2(\lambda_i - \lambda(\sigma_i))$$

- Formulae proved by BCFW recursion.
- Valid for any amount of SUSY unlike original twistor-strings.
- Provides gauge and gravity theories in ambitwistor space without formal neighbourhoods.
- Using an identification due to George Sparling, string can be understood as being at null infinity $\mathscr{I}$ as $\mathbb{A} = T^*\mathscr{I}$.
- At $\mathscr{I}$ BMS symmetries act geoetrically and obtain soft theorems from supertranslations etc..

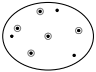
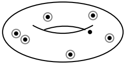
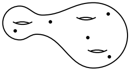
Ambitwistor strings with combinations of matter

CGMMRS 1506.08771

S^l \ S^r	S_Ψ	S_{Ψ_1,Ψ_2}	$S_{\rho,\Psi}^{(\tilde{m})}$	$S_{CS,\Psi}^{(\tilde{N})}$	$S_{CS}^{(\tilde{N})}$
S_Ψ	E				
S_{Ψ_1,Ψ_2}	BI	Galileon			
$S_{\rho,\Psi}^{(m)}$	*EM* $\mathrm{U}(1)^m$	DBI	*EMS* $\mathrm{U}(1)^m \times \mathrm{U}(1)^{\tilde{m}}$		
$S_{CS,\Psi}^{(N)}$	EYM	ext. DBI	*EYMS* $\mathrm{SU}(N) \times \mathrm{U}(1)^{\tilde{m}}$	*EYMS* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$	
$S_{CS}^{(N)}$	YM	Nonlinear σ	*EYMS* $\mathrm{SU}(N) \times \mathrm{U}(1)^{\tilde{m}}$	*gen. YMS* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$	*Biadjoint Scalar* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$

Table: Theories arising from the different choices of matter models.

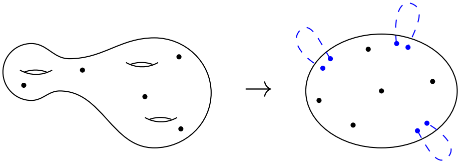
The string paradigm gives

$\mathcal{M}_n =$  $+$  $+ \ldots +$  $+ \ldots$

Can we make sense of this at 1 loop, i.e., on a torus?

- Yes! for critical models with vanishing anomalies, i.e., type II super-gravity, 10d. [Adamo,Casali, Skinner 2013]
- Integration by parts gives description on nodal spheres. Applies in all dims to generic gauge & gravity theories .

  [Geyer, M., Monteiro, Tourkine].



  Now proved at two loops [Geyer,Monteiro].
- Can now reproduce divergences of quantum gravity!

Further thoughts 1

Divergences?!?

- The ultraviolet divergences arise in (P, x) model.
- Can twistors compactify ultraviolet momenta?

Too much complexification?

- For real, non-analytic metric g, can still have P_μ or $\mathscr{I}$ complex, but x real.
- Then need $\Sigma = \mathbb{CP}^1 = D^+ \cup D^-$ with $\partial D^\pm \subset$ real slice.
- $D^\pm$ lives in positive/negative frequency region.
- WKB analysis of Fadeev's inverse scattering seems to give this picture.

Dichotomy: Are we quantizing many particles or one field?

Can compute amplitudes as quantization of:

1. many particles forming graphs of interactions,
   [Resonant of spin networks, but different rules;
   does worldline field theory connect to loops?]
2. Nonlinear field.

Strings provide systematic intermediate approach.
Key proposal: S-matrix for gravity metric g

$$\Psi[g] = \int D[WZ\ldots]\mathrm{e}^{iS^g_{ambitwistor-string}}$$

Can we complexify spin networks to holomorphic spin networks on Riemann surfaces?

Thanks for the collaborations,

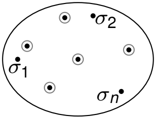
Happy 70th George!!!

The scattering equations

Take n null momenta $k_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, $k_i^2 = 0$, $\sum_i k_i = 0$,

- define $P : \mathbb{CP}^1 \to \mathbb{C}^d$

$$P(\sigma) := \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i}\,, \qquad \sigma, \sigma_i \in \mathbb{CP}^1$$



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- Solve for $\sigma_i \in \mathbb{CP}^1$ with the n *scattering equations* [Fairlie 1972]

$$\mathrm{Res}_{\sigma_i}\left(P^2\right) = k_i \cdot P(\sigma_i) = \sum_{j=1}^{n} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0\,.$$

$$\Rightarrow \quad P^2 = 0 \ \forall \sigma.$$

- For Mobius invariance $\Rightarrow P \in \mathbb{C}^d \otimes K$, $K = \Omega^{1,0}\mathbb{CP}^1$
- There are $(n-3)!$ solutions.

Arise in large α' strings [Gross-Mende 1988] & twistor-strings [Roiban, Spradlin, Volovich, Witten 2004].

Amplitude formulae for massless theories.

Proposition (Cachazo, He, Yuan 2013,2014)

Massless tree amplitudes in d-dims are integrals/residue sums:

$$\mathcal{M}_n = \delta^d\left(\sum_i k_i\right)\int_{(\mathbb{CP}^1)^n} \frac{\mathcal{I}^l\mathcal{I}^r\prod_i \bar\delta(k_i\cdot P(\sigma_i))}{\mathrm{Vol}\ \mathrm{SL}(2,\mathbb{C})\times\mathbb{C}^3}$$

where $\mathcal{I}^{l/r} = \mathcal{I}^{l/r}(\epsilon_i^{l/r}, k_i, \sigma_i)$ *depend on the theory.*

- polarizations ϵ_i^l for spin 1, $\epsilon_i^l\otimes\epsilon_i^r$ for spin-2 ($k_i\cdot\epsilon_i = 0\dots$).
- Introduce skew $2n\times 2n$ matrices $M = \begin{pmatrix} A & C \\ -C^t & B\end{pmatrix}$,

  $$A_{ij} = \frac{k_i\cdot k_j}{\sigma_i-\sigma_j},, \quad B_{ij} = \frac{\epsilon_i\cdot\epsilon_j}{\sigma_i-\sigma_j}, \quad C_{ij} = \frac{k_i\cdot\epsilon_j}{\sigma_i-\sigma_j}, \qquad \text{for } i\neq j$$

  and $A_{ii} = B_{ii} = 0$, $C_{ii} = \epsilon_i\cdot P(\sigma_i)$.
- For YM, $\mathcal{I}^l = Pf'(M)$, $\mathcal{I}^r = \prod_i \frac{1}{\sigma_i-\sigma_{i-1}}$.
- For GR $\mathcal{I}^l = Pf'(M^l)$, $\mathcal{I}^r = Pf'(M^r)$.

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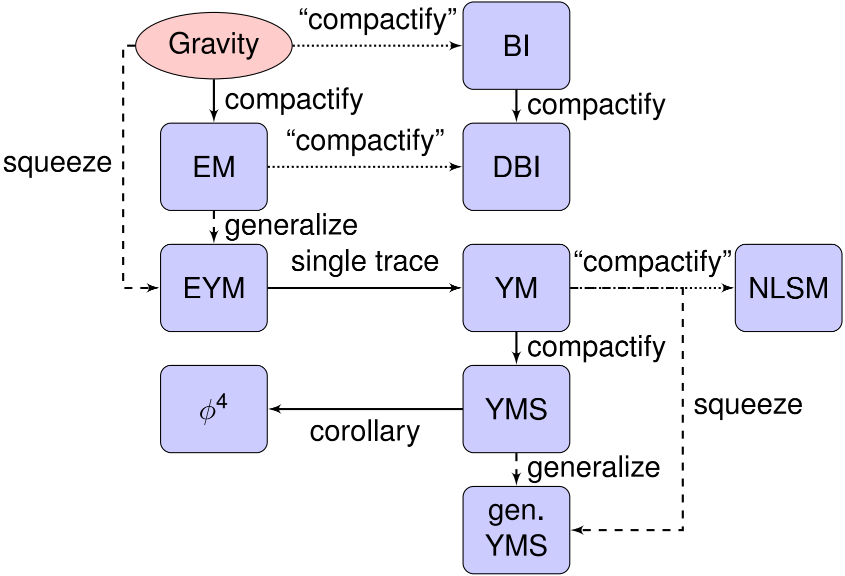

Figure: Theories studied by CHY and operations relating them.