# Hyper-Twistors and Higher-Dimensional Space-Times

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2 September 2019

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# Overview

We present an an approach to the unification of space-time physics and quantum theory motivated by twistor theory.

We take the view that classical space-time itself is not to be regarded as a primary object which is then subjected to some form of quantization procedure.

The central object is a mathematical structure that we call a "quantum space-time".

Intuitively, this structure can be regarded as the space of all space-time-point valued quantum operators.

That is to say, each point in the infinite-dimensional quantum space-time corresponds to a quantum operator with the property that its expectation, in any given quantum state, is a space-time point.

The space of all such operators has a rich structure that appears to contain many of the elements one needs both for a characterization of the causal structure of relativistic space-time as well as a representation of the phenomena of quantum theory.

# Cartan spinors and Hyperspinors

In four-dimensional space-time there is a local isomorphism between the Lorentz group $SO(1,3)$ and the spin transformation group $SL(2,\mathbb{C})$:

$$SL(2,\mathbb{C}) \sim SO(1,3)$$

$SL(r,\mathbb{C})$ hyperspinors $\qquad$ $SO(N,\mathbb{C})$ Cartan spinors

In higher dimensions this relation bifurcates, and we are left with two different concepts of "spinor"— one for $SL(r,\mathbb{C})$, and one for $SO(N,\mathbb{C})$.

In what follows we shall take the hyperspinor approach. So-called hyperspinors were introduced by Finkelstein, who showed that these lead in a natural way to models for higher-dimensional space-times. The dimension of such a Finkelstein hyperspace is a perfect square, and the resulting geometry is pseudo-Finslerian, with a rich "chronometric" structure, generalizing the usual causal relations of Minkowski space.

# Spinors and space-time

Two-component spinors are connected with both (a) quantum mechanics, and (b) the causal structure of space-time.

This point has often been mentioned by Roger Penrose as a motivating factor in various aspects of his work.

In particular, we have the basic relation

$$2\det(x^{\mathbf{AA'}}) = t^2 - x^2 - y^2 - z^2.$$

In more detail, we introduce a separation matrix defined by

$$r^{\mathbf{AA'}} = x^{\mathbf{AA'}} - y^{\mathbf{AA'}}.$$

It follows that

$$2\det(r^{\mathbf{AA'}}) = \epsilon_{\mathbf{AB}}\epsilon_{\mathbf{A'B'}}r^{\mathbf{AA'}}r^{\mathbf{BB'}} = g_{\mathrm{ab}}r^{\mathrm{a}}r^{\mathrm{b}},$$

where

$$\epsilon_{\mathbf{AB}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The canonical decomposition of a space-time interval follows as a consequence:

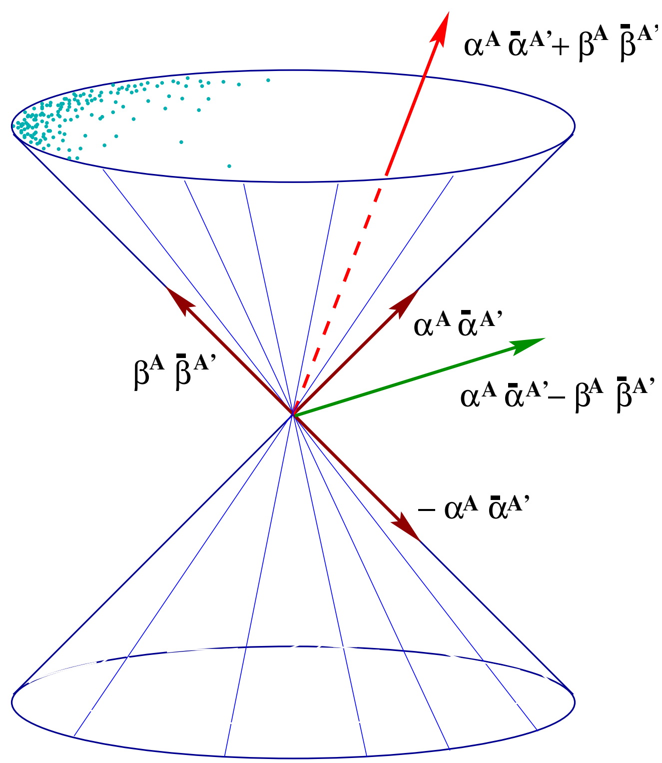
(i) $g_{ab}r^a r^b = 0$

$$r^{\mathbf{AA}'} = \alpha^{\mathbf{A}}\bar{\alpha}^{\mathbf{A}'} \quad \text{future-pointing null separation}$$
$$r^{\mathbf{AA}'} = -\alpha^{\mathbf{A}}\bar{\alpha}^{\mathbf{A}'} \quad \text{past-pointing null separation}$$

(ii) $g_{ab}r^a r^b \neq 0$

$$r^{\mathbf{AA}'} = \alpha^{\mathbf{A}}\bar{\alpha}^{\mathbf{A}'} + \beta^{\mathbf{A}}\bar{\beta}^{\mathbf{A}'} \quad \text{future-pointing time-like separation}$$
$$r^{\mathbf{AA}'} = \alpha^{\mathbf{A}}\bar{\alpha}^{\mathbf{A}'} - \beta^{\mathbf{A}}\bar{\beta}^{\mathbf{A}'} \quad \text{space-like separation}$$
$$r^{\mathbf{AA}'} = -\alpha^{\mathbf{A}}\bar{\alpha}^{\mathbf{A}'} - \beta^{\mathbf{A}}\bar{\beta}^{\mathbf{A}'} \quad \text{past-pointing time-like separation}$$

# Causal relations in Minkowski space




Canonical forms of the spinor decomposition.

# Hyperspinors

For a typical hyperspinor we write

$$\alpha^A \in \mathbb{C}^r, \qquad A = 1, 2, \ldots, r.$$

The complex conjugation operation is given by $\alpha^A \longrightarrow \bar{\alpha}^{A'}$ (component by component).

The inner product between a hyperspinor $\alpha^A$ and a dual hyperspinor $\beta_A$ is given by $\alpha^A \beta_A$.

We also have the totally antisymmetric hyperspinor with $r$ indices:

$$\varepsilon_{AB\cdots C} = \varepsilon_{[AB\cdots C]}.$$

The dimension of the associated space-time (or "hyperspace") is $N = r^2$.

For a typical point in the hyperspace we write

$$x^{AA'} \qquad (A, A' = 1, 2, \ldots, r).$$

The Finkelstein chronometric form, which is a symmetric tensor with $r$ hyperspace indices, is then defined by

$$g_{\mathrm{ab}\cdots\mathrm{c}} = g_{AA'BB'\ldots CC'} = \varepsilon_{AB\cdots C}\,\varepsilon_{A'B'\cdots C'}.$$

For a typical space-time interval we write

$$r^{\mathrm{a}} = x^{\mathrm{a}} - y^{\mathrm{a}} = x^{AA'} - y^{AA'}.$$

For the metric of the space-time interval we have

$$\Delta(r) = g_{\mathrm{ab}\cdots\mathrm{c}}\, r^{\mathrm{a}}\, r^{\mathrm{b}} \cdots r^{\mathrm{c}} = \varepsilon_{AB\cdots C}\,\varepsilon_{A'B'\cdots C'}\, r^{AA'}\, r^{BB'} \cdots r^{CC'}.$$

It follows that

$$\frac{1}{r!}\,\Delta(r) = \det\,(r^{AA'}),$$

and that for a "degenerate" separation (vanishing determinant) we have

$$\varepsilon_{AB\cdots C}\ \varepsilon_{A'B'\cdots C'}\ r^{AA'} r^{BB'} \cdots r^{CC'} = 0.$$

# Causal structures on hyperspace

The chronometric form defines a family of causal relations on hyperspace.

As an example, let us consider the case $r = 3$, for which $\Delta = g_{\rm abc} r^{\rm a} r^{\rm b} r^{\rm c}$. Then we have the following:

(i) $g_{\rm abc} r^{\rm c} = 0$

$$r^{AA'} = 0 \;\; \text{zero separation}$$

(ii) $g_{\rm abc} r^{\rm b} r^{\rm c} = 0$ and $g_{\rm abc} r^{\rm c} \neq 0$

$$\begin{aligned} r^{AA'} &= \alpha^A \bar{\alpha}^{A'} && \text{future-pointing null separation} \\ r^{AA'} &= -\alpha^A \bar{\alpha}^{A'} && \text{past-pointing null separation} \end{aligned}$$

(iii) $\Delta = 0$ and $g_{\rm abc} r^{\rm b} r^{\rm c} \neq 0$

$$\begin{aligned} r^{AA'} &= \alpha^A \bar{\alpha}^{A'} + \beta^A \bar{\beta}^{A'} && \text{degenerate time-like future-pointing} \\ r^{AA'} &= \alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'} && \text{degenerate space-like} \\ r^{AA'} &= -\alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'} && \text{degenerate time-like past-pointing} \end{aligned}$$

(iv) $\Delta \neq 0$ and $g_{\mathrm{ab}} r^{\mathrm{a}} r^{\mathrm{b}} \neq 0$

$$
\begin{array}{ll}
r^{AA'} = \alpha^A \bar{\alpha}^{A'} + \beta^A \bar{\beta}^{A'} + \gamma^A \bar{\gamma}^{A'} & \text{future pointing time-like} \\
r^{AA'} = \alpha^A \bar{\alpha}^{A'} + \beta^A \bar{\beta}^{A'} - \gamma^A \bar{\gamma}^{A'} & \text{semi-space-like} \\
r^{AA'} = \alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'} - \gamma^A \bar{\gamma}^{A'} & \text{semi-space-like} \\
r^{AA'} = -\alpha^A \bar{\alpha}^{A'} - \beta^A \bar{\beta}^{A'} - \gamma^A \bar{\gamma}^{A'} & \text{past pointing time-like}
\end{array}
$$

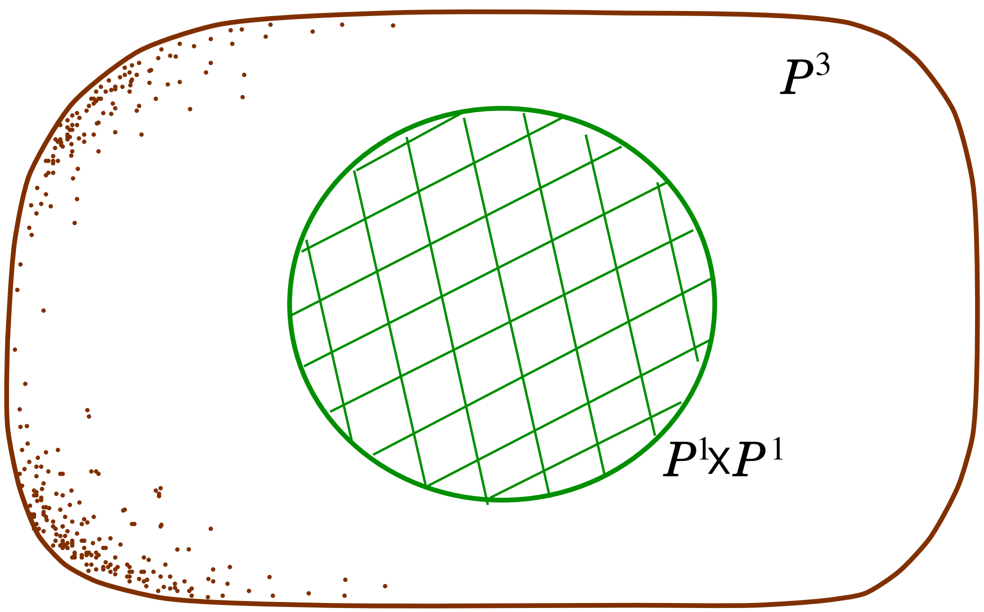
Clearly a similar scheme can be worked out in detail for any value of $r \geq 1$.

# Null directions

## Complex null directions for 4-dimensional space-time

For $r = 2$ we obtain a quadric of null directions in the space $P^3$ of all directions at a point in space-time:

$$g_{\mathrm{ab}} z^{\mathrm{a}} z^{\mathrm{b}} = 0 \Longleftrightarrow z^{AA'} = \alpha^A \beta^{A'} \implies \mathbb{Q}^2 = \mathbb{P}^1 \times \mathbb{P}^1.$$

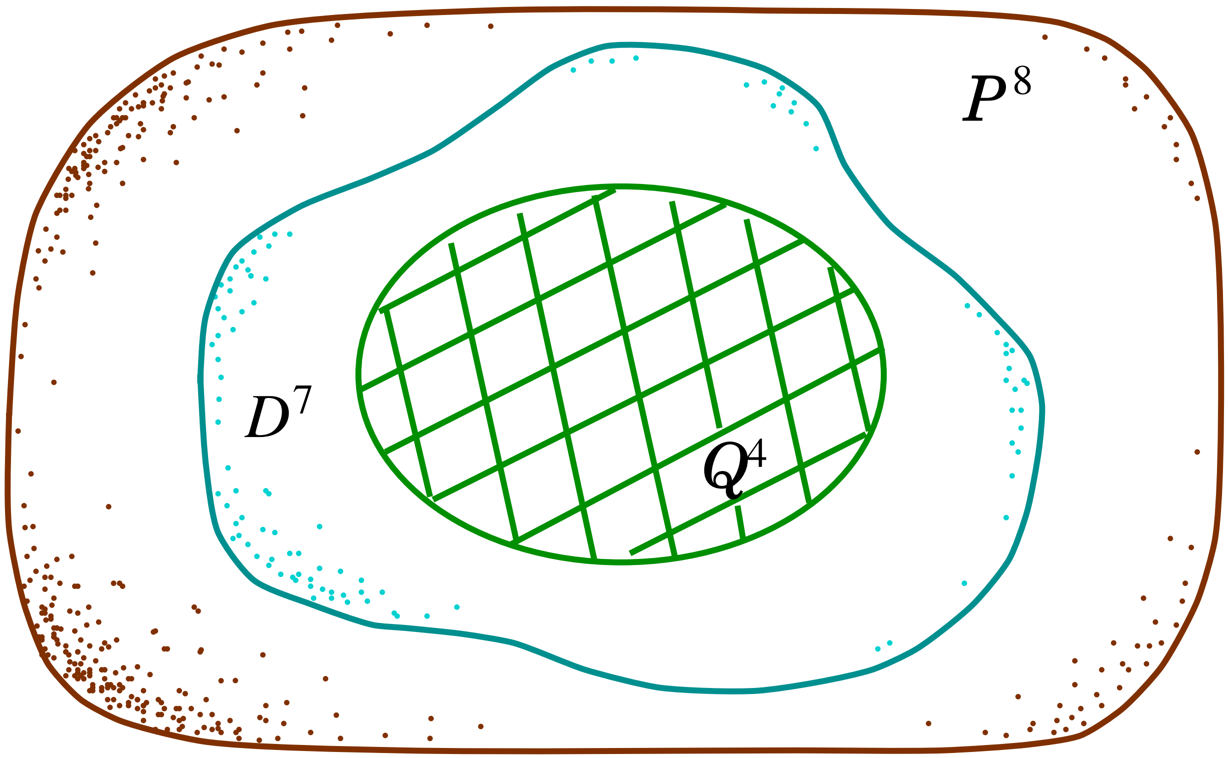



Space of null directions at a point of space-time.

## Complex null directions for 9-dimensional space-time

For $r = 3$ the space of degenerate directions is a cubic surface $\mathcal{D}^7$ in $P^8$, which contains as a subvariety the space $P^2 \times P^2$ of totally null directions :

$$\{g_{\rm abc} z^{\rm a} z^{\rm b} z^{\rm c} = 0\} = \mathcal{D}^7 \qquad \{g_{\rm abc} z^{\rm a} z^{\rm b} = 0\} = Q^4 = P^2 \times P^2.$$

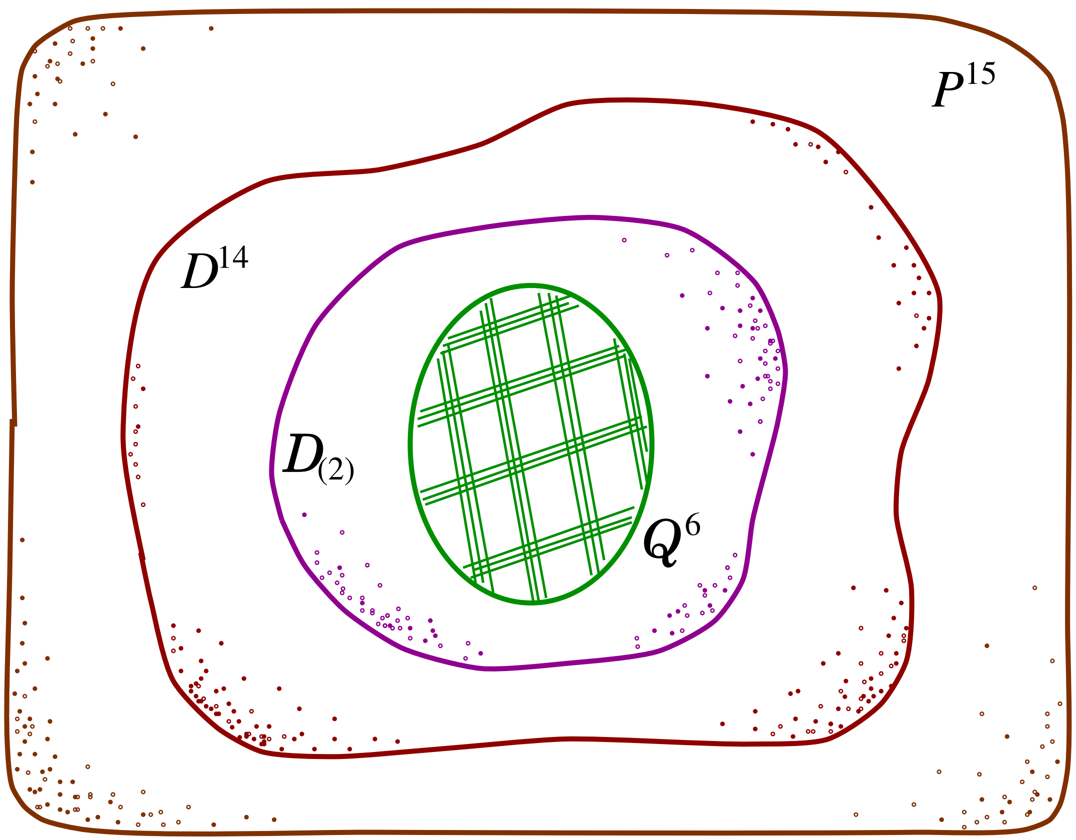



Space of degenerate directions at a point of 9-dimensional space-time.

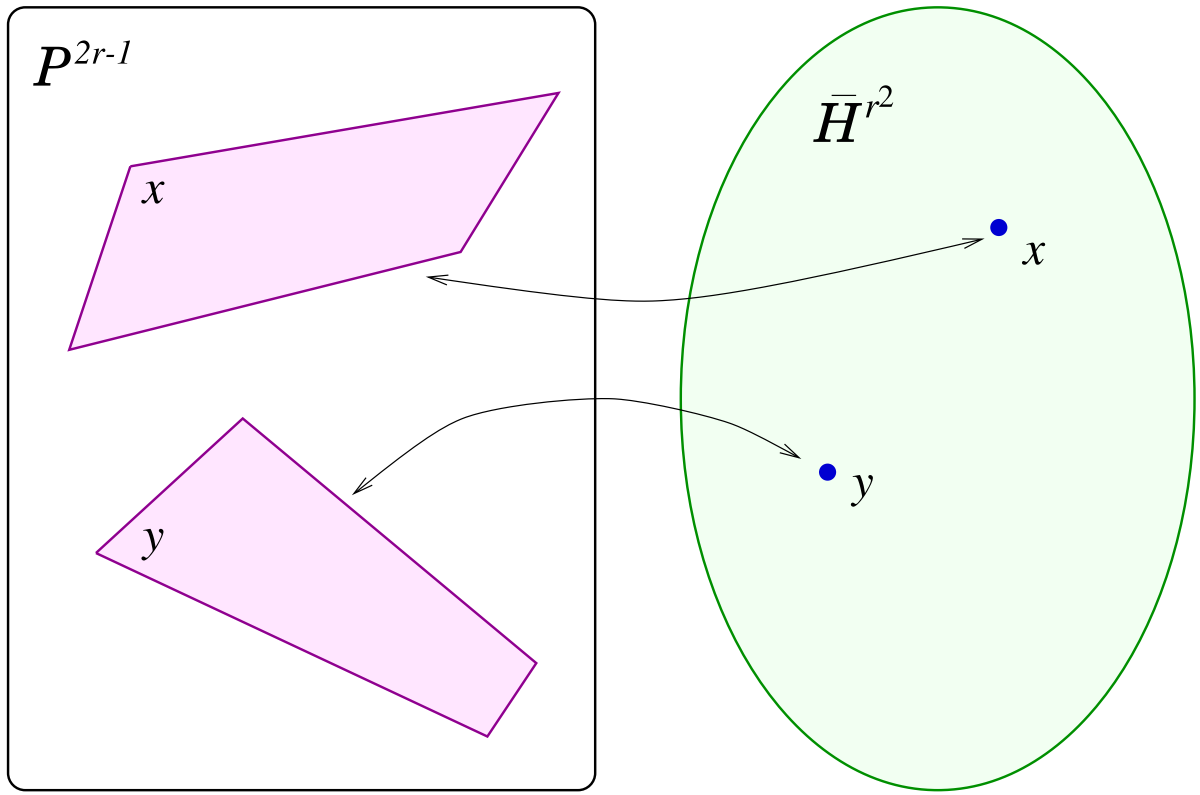
# Complex null directions (16-dimensional space-time)

For $r = 4$ the space of degenerate directions is a 14-dimensional quartic surface in $P^15$. The totally null directions form a subvariety $Q^6 = P^3 \times P^3$.

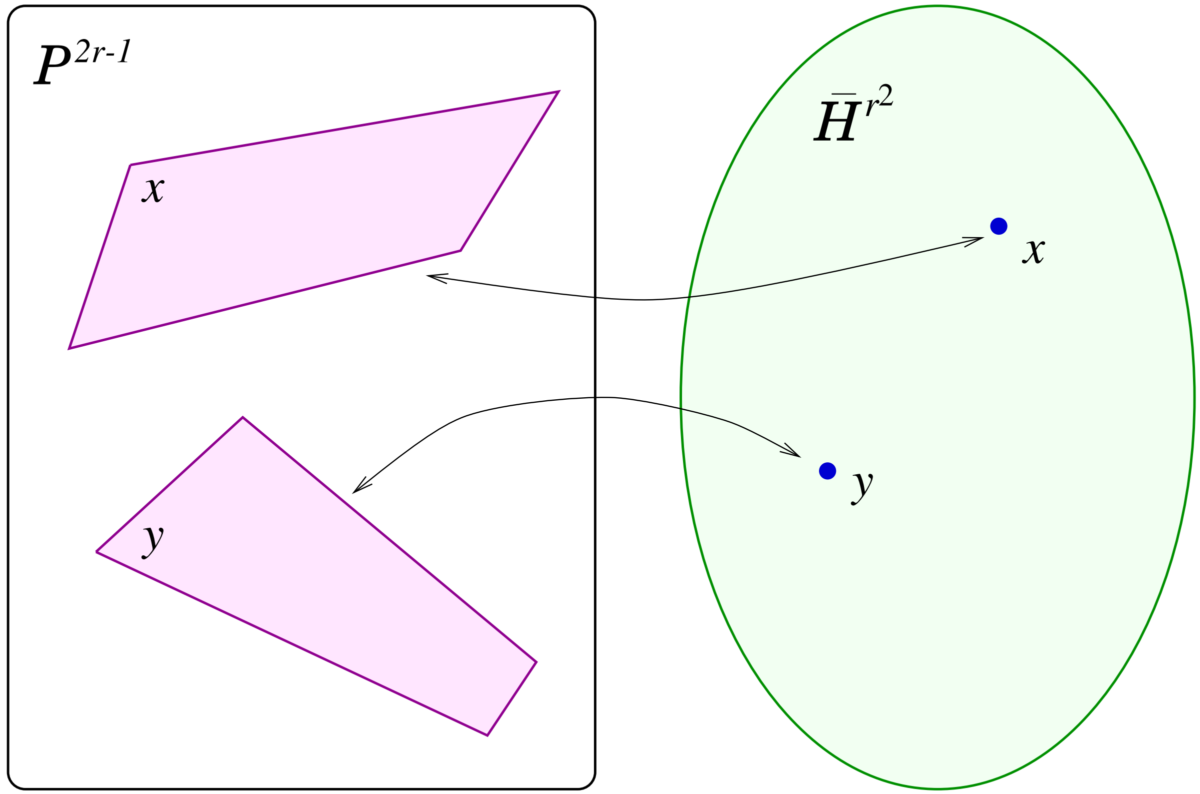
$$g_{\mathrm{abcd}} z^{\mathrm{a}} z^{\mathrm{b}} z^{\mathrm{c}} z^{\mathrm{d}} = 0, \quad g_{\mathrm{abcd}} z^{\mathrm{a}} z^{\mathrm{b}} z^{\mathrm{c}} = 0, \quad g_{\mathrm{abcd}} z^{\mathrm{a}} z^{\mathrm{b}} = 0.$$



# Klein representation for hyperspace




The aggregate of all $(r-1)$-planes in $\mathbb{P}^{2r-1}$ forms a manifold of dimension $r^2$. This space is the complexified, compactified quantum space-time $\mathcal{CH}^{r^2}$. Each point of $\mathcal{CH}^{r^2}$ corresponds to an $(r-1)$-plane in $\mathbb{P}^{2r-1}$.




This picture suggests that we can introduce the concept of a "hypertwistor", defined by analogy with the usual twistor construction by

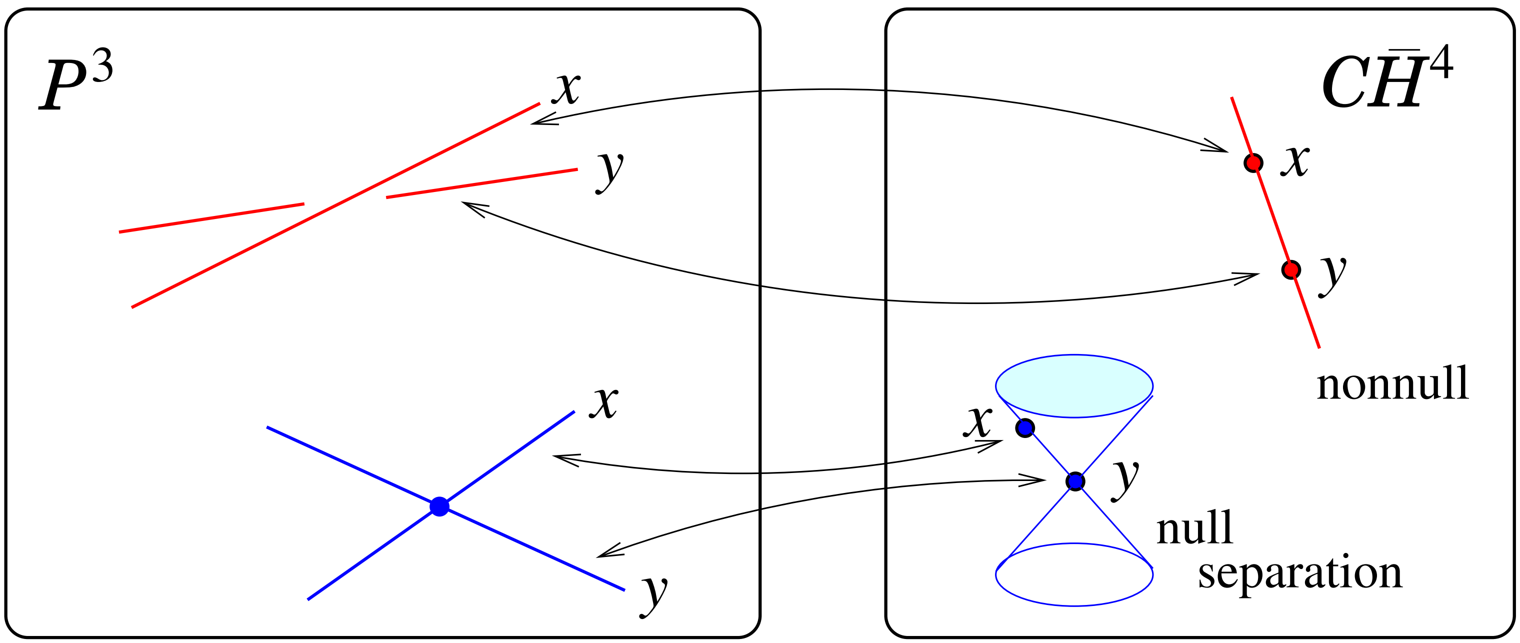
$$\mathbb{P}^{2r-1} \sim (\omega^A, \pi_{A'}),$$

where $(\omega^A, \pi_{A'})$ is a pair of hyperspinors. The equation for an $(r-1)$-plane $\mathbb{P}^{r-1}$ in $\mathbb{P}^{2r-1}$ is then determined for fixed $x^{AA'}$ by the relation

$$\omega^A = \mathrm{i}x^{AA'}\pi_{A'}.$$

# Causal structure via twistor geometry

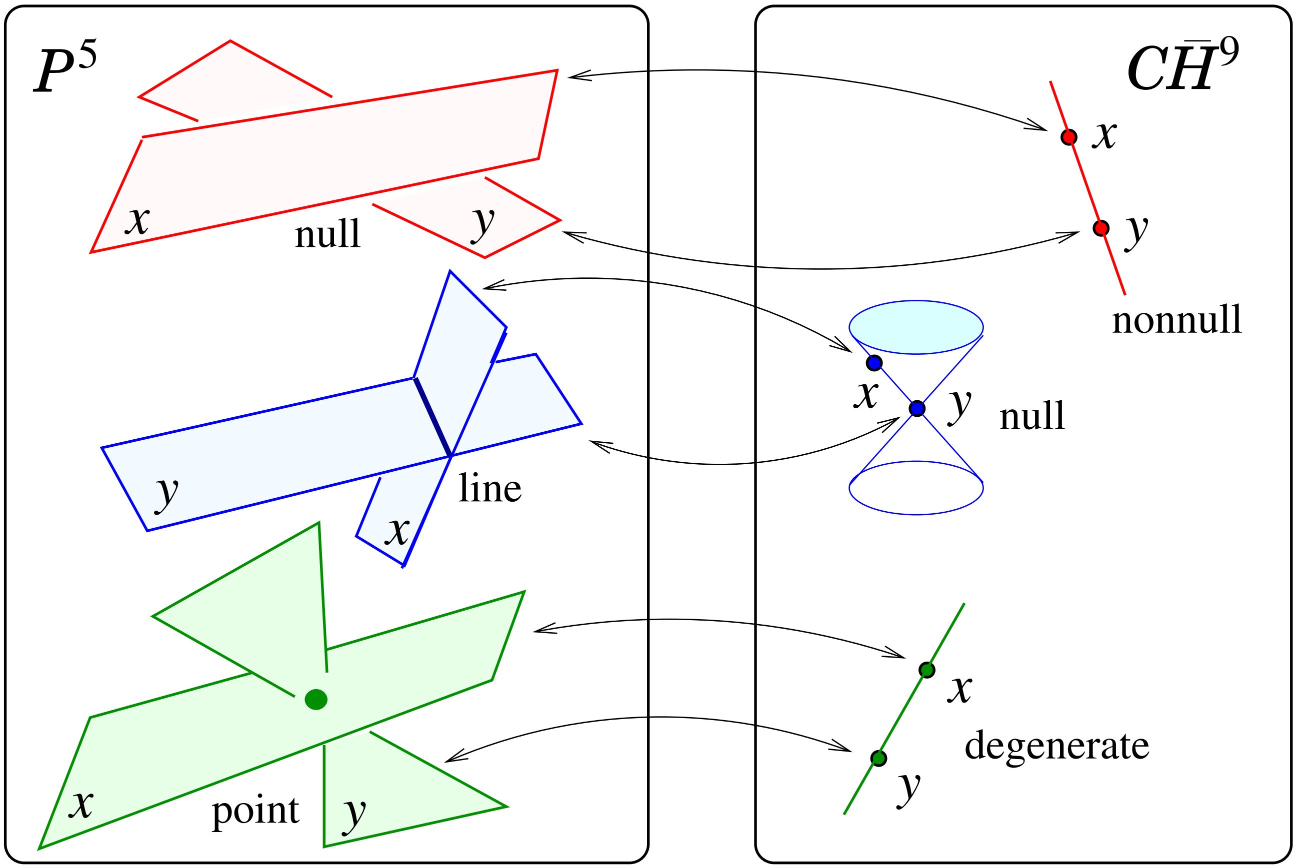
The intersection of a pair of $(r-1)$-planes implies degenerate separation of the corresponding points in hyperspace. For $r=2$, we have:




Two points in $\mathcal{C}\bar{\mathcal{H}}^4$ are null-separated if and only if the corresponding lines in $\mathbb{P}^3$ intersect.

More generally, the intersection of a pair of $(r-1)$-planes in $P^{2r-1}$ implies a degenerate separation for the associated hyperspace points.

For $r=3$, for example, we have:



# Breakdown of symmetry

Let $r = 2n$ be even and set

$$A = \mathbf{A}i, \qquad \mathbf{A} = 1, 2 \quad i = 1, 2, \ldots, n$$

where $\mathbf{A}, \mathbf{A}'$ are ordinary two-component spinor indices, and $i$ can be regarded as a Hilbert space index representing "internal" degrees of freedom.

Thus when symmetry is broken a hyperspinor can be regarded as a "multiplet" of ordinary spinors.

It follows that hyperspace geometry admits an "embedding" of the ordinary four-dimensional space-time geometry, and we can take the view that the extra degrees of freedom are "quantum" in nature.

The points of hyperspace, after symmetry breaking, can be regarded as space-time-point valued quantum operators. We have

$$x^{AA'} = x^{\mathbf{A}i\mathbf{A}'}{}_j = x^{\mathbf{A}\mathbf{A}'i}{}_j = x^{\mathbf{a}i}{}_j$$

where

$$\mathbf{a} = 0, 1, 2, 3 \qquad i = 1, 2, \ldots, n$$

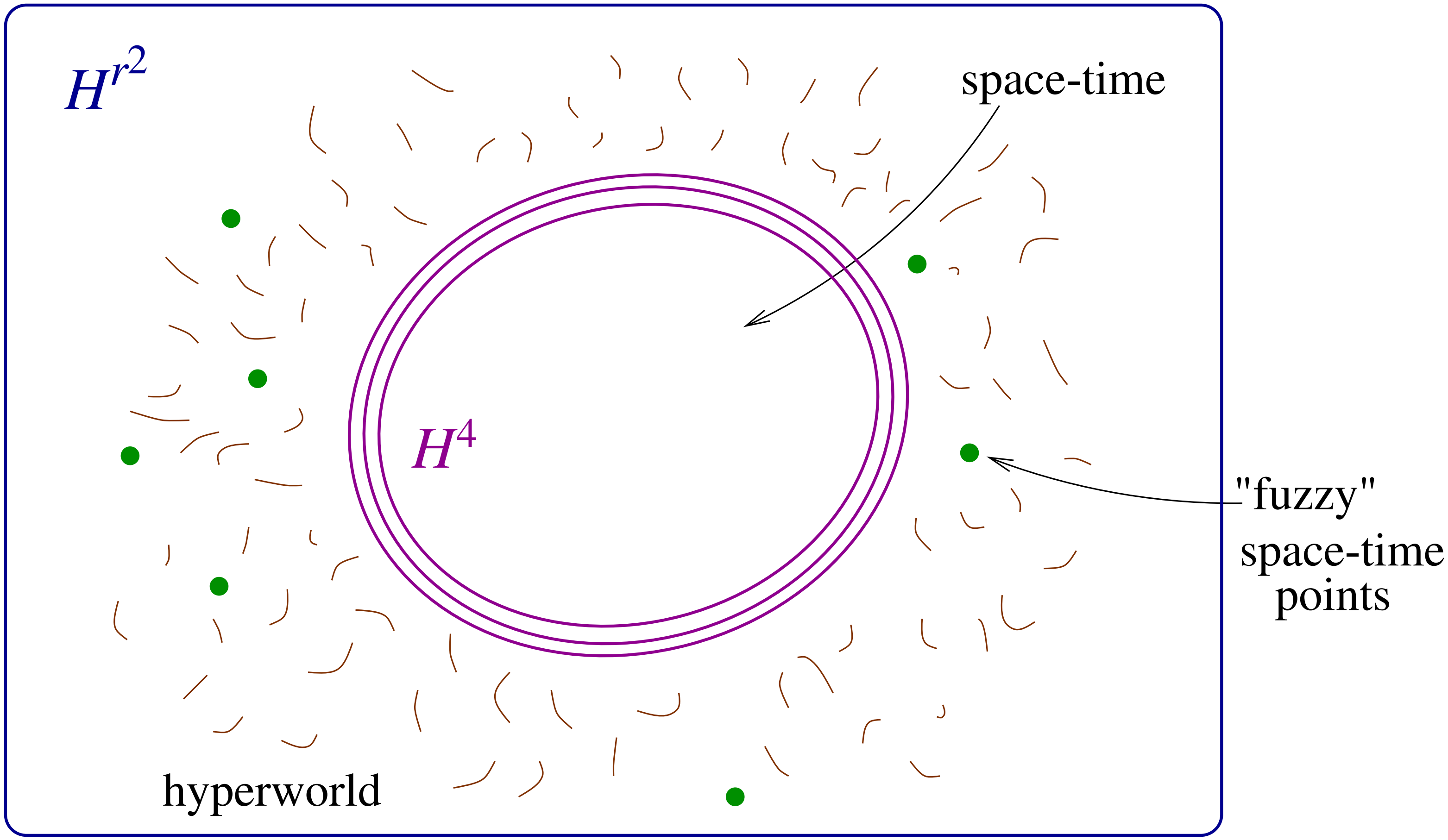
In particular, the breakdown of symmetry is associated with the identification of a Segre variety $P^1 \times P^{r-1}$ in the projective hyper spin space $P^{2r-1}$.

The points of “ordinary” space-time correspond to those points of hyperspace for which the four-dimensional points of space-time are “disentangled” from the internal degrees of freedom in hyperspace.

Thus ordinary four-dimensional space-time is embedded in a larger hyperspace of dimension $4r^2$.

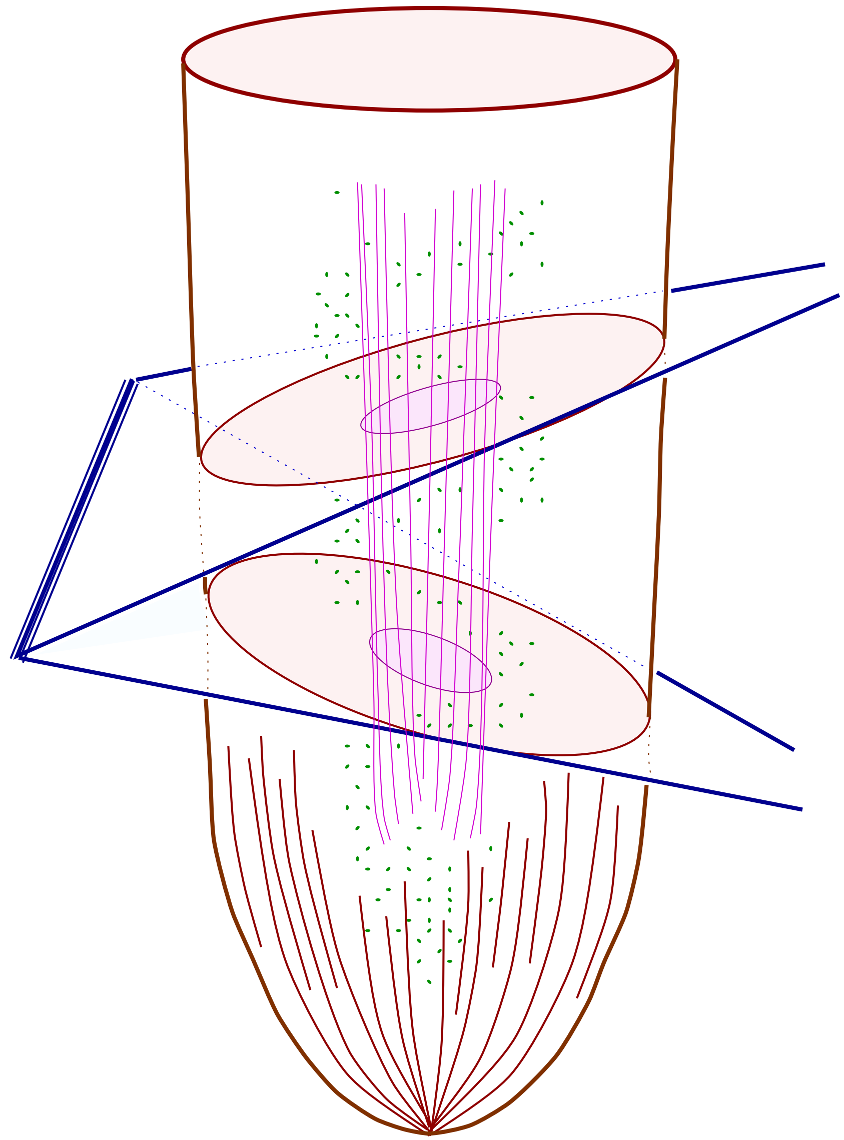
The associated hyper-twistors can then be interpreted as multiplets of standard Penrose twistors.

# 4-dimensional space-time embedded in a higher-dimensional hyperspace




Quantum space-time.

# Cosmological phase transition and matter formation



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## Cosmologies

The structures of conformally-flat space-times can be described by properties of a quadratic surface $\Omega$ in $\mathbb{P}^5$:

$$\Omega_{ij}X^iX^j = 0,$$

Here the reality structure is determined by the signature :
$\Omega_{ij} = \mathrm{diag}(+1, +1, -1, -1, -1, -1)$, which implies that $\Omega \sim S^1 \times S^3$.

The points $Q \in \Omega$ that are null-separated from $P$ constitutes the null cone of $P$.

If $\Omega_{ij}X^iX^j < 0$, $X$ is a time-like hypersurface, with the topology $S^1 \times S^2$.
If $\Omega_{ij}X^iX^j > 0$, $X$ is a space-like hypersurface, with the topology $S^3$.

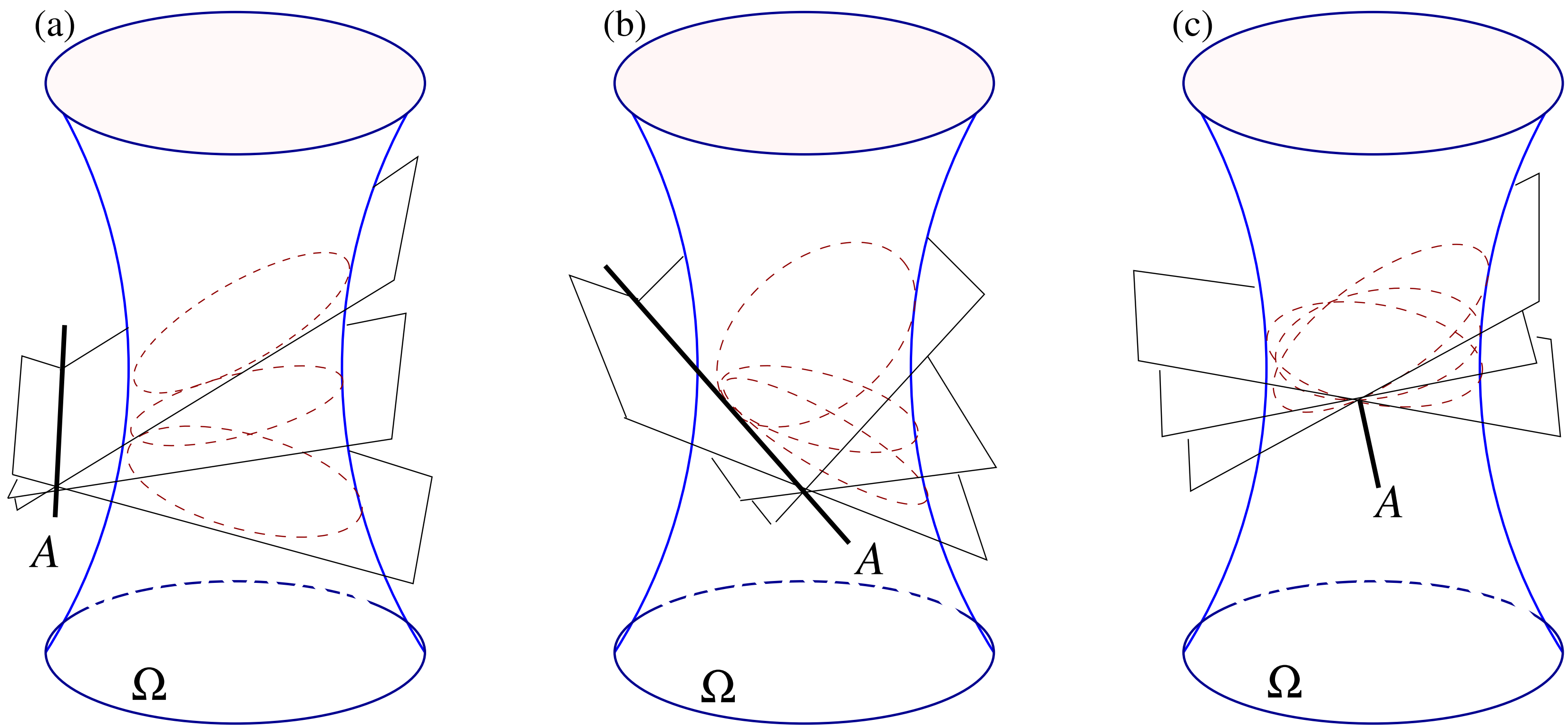
The structure of Minkowski space arises from the choice of a point at infinity $I \in \Omega$ and its associated null cone $\mathfrak{I}$ at infinity.

The space $\Omega - \mathfrak{I}$, topologically $= \mathbb{R}^4$, is real Minkowski space.

The linear transformations on $\mathbb{P}^5$ that leave $\Omega$ invariant constitute the pseudo-orthogonal group $SO(2,4)$.

The subgroup that leaves $I$ fixed is the Poincaré group of the Minkowski space.

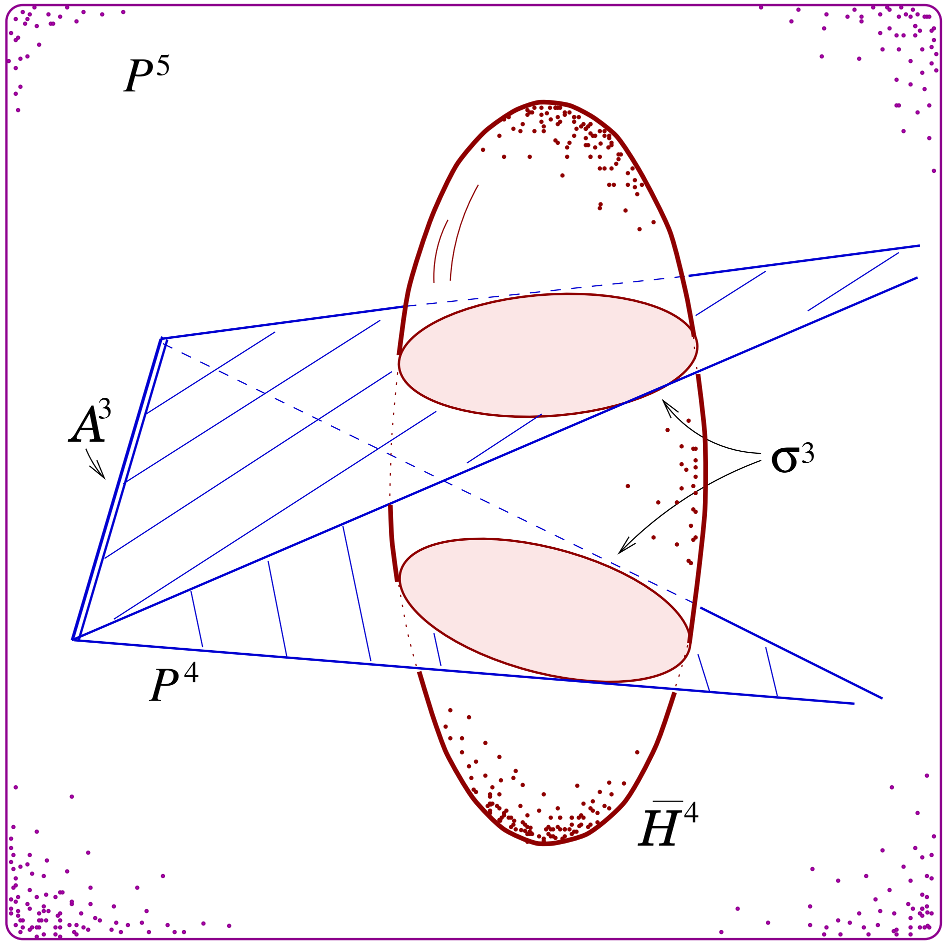
# FRW cosmologies




Pencil of hyperplanes hinged on a axis $A$.

(a) $k = 1$, $\mathbb{A} \cap \Omega = \emptyset$, (b) $k = 0$, $\mathbb{A} \cap \Omega = \text{point}$, (c) $k = -1$, $\mathbb{A} \cap \Omega = S^2$ (Hurd 1985, Penrose & Rindler 1986).

# FRW $(k = 1)$ cosmology




Pencil of hyperplanes hinged on an axis $A^3$.

# 16-dimensional hypercosmology