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For Christian – 7/11/2019

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equivalently (Weyl's criterion)

$$B(\Lambda) = \{x \in \mathbb{R},\ \forall k \neq 0,\ \frac{1}{N}\sum_{n\leq N} e(kt_n x) \to 0.\}$$

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3. For intermediate growth rate ? For example, the Furstenberg sequence :

$$(s_n) = \{2^j 3^k, j \geq 1, k \geq 1\} \text{ re-arranged in increasing order?}$$

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Comments. 1. No information on the associated sequence Λ.
2. No such result if we impose Λ to be an increasing sequence.

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We need some clarification...

Reminds on substitutions

Let ζ be a substitution on a finite alphabet A of cardinality s (a map $A \to A^*$ extended by concatenation). A is identified to $\{1, 2, \ldots, s\}$.

1. $M := M(\zeta)$ is the $s \times s$-matrix with entries

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4. The eigenvalues of M are algebraic integers.
   If M is primitive, M admits a simple positive and dominant eigenvalue θ, which is a Perron number, i.e. $|\theta_j| < \theta$ (θ_j other eigenvalues of M) ;
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Example. $1 \to 2,\ 2 \to 346,\ 3 \to 15,\ 4 \to 1,\ 5 \to 2,\ 6 \to 5$ irreducible but not primitive.

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Back to Christian's result. *Let ζ be an irreducible substitution on A admitting an infinite fixed point u. Let $\Lambda := \Lambda(u, a)$ be the increasing sequence obtained by indexing the appearances of the letter a in u. Then*

$$B = B(\Lambda) \text{ for some } u, a \Longleftrightarrow \mathbb{R}\backslash B \text{ is a finite real field extension of } \mathbb{Q}$$

(generated by the θ_j involved in Λ).

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2. Algebraic description of $W_i = \{\alpha \in \mathbb{R}, \ \lim_{n\to\infty} e(\ell_n(i)\alpha) = 1\}$.
Observe : $\ell_n(i) = \sum_{\theta\in\Theta_i} P_i(n)\theta^n$. Christian invokes a famous theorem of Pisot :

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He gets this improvement (and the implication) :

Theorem (Pisot family)

Let $\theta_1, \dots, \theta_r$ be distinct algebraic numbers, $|\theta_i| \geq 1$ for $1 \leq i \leq r$; let $P_i \in \mathbb{Z}[X]$, and $\alpha_i \in \mathbb{R}$, $1 \leq i \leq r$, not all zero, with $|| \sum_{1\leq i\leq r} \alpha_i P_i(n)\theta_i^n|| \to 0$. Then the θ_i's are algebraic integers, every conjugate of the θ_i's (different from θ_i) belongs to the unit open disk and $\alpha_i \in \mathbb{Q}(\theta_i)$ for every i.

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If F is a finite real field extension of $\mathbb{Q}$ of degree s, there exists a Pisot number θ of degree s such that $F = \mathbb{Q}(\theta)$.

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Let us fix this θ.

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If θ is a Perron number ($|\theta_j| < \theta$), there exists $M = (m_{i,j}) \in \mathbb{N}^{s \times s}$, $M \geq 0$ and M primitive such that θ and its conjugates are the set of eigenvalues of M.

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2. Finally ζ defined on $A = \{a_1, \ldots, a_s\}$ by

$$\zeta(a_i) = a_1^{m_{i,1}} a_2^{m_{i,2}} \cdots a_s^{m_{i,s}} \qquad \forall a_i \in A,$$

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Examples. 1. Fibonacci : $\zeta(a) = ab$, $\zeta(b) = a$, $u = \zeta^\infty(a)$; here,
$\ell_n(a) = f_{n+1}$, $\ell_n(b) = f_n$ and $W = \mathbb{Q}(\sqrt{5})$.

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2. $\zeta(a) = abb$, $\zeta(b) = ba$, $u = \zeta^\infty(a)$; here, $W = \mathbb{Q}(\sqrt{2})$, since

$$\ell_n(a) = ((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1})/2, \ \ell_n(b) = ((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1})/2\sqrt{2}.$$

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$$H^\infty(\Lambda) = \{x \in \mathbb{T},\ ||t_n x|| \to 0\};$$

and more generally, for $p \geq 1$

$$H^p(\Lambda) = \{x \in \mathbb{T},\ \sum_n ||t_n x||^p < \infty\}.$$

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3. More widely, role of the lacunarity, of the arithmetic properties of Λ ?

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- If Λ is the Furstenberg sequence, $W(\Lambda)$ is uncountable (it contains Liouville numbers of the form $\sum_{n\geq 1} \varepsilon_n 6^{-n!}$, $\varepsilon_n = 0$ and 1 *i.o.*). What is its dimension ?

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Theorem

Let $\Lambda = (t_n)$ an increasing sequence of integers with $t_n | t_{n+1}$. Then $W(\Lambda)$ supports a Rajchman measure.

Link with rigid sequences

Definition (/Proposition)

A sequence of integers (n_k) *is* *rigid* *if there exists a weak mixing system* (X, T, μ) *such that*

$$||f \circ T^{n_k} - f||_{L^2(\mu)}, \quad \forall f \in L^2(\mu).$$

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- Prime numbers sequence is not rigid ($W(\Lambda)$ countable – Vinogradov).

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1. Is the sequence $n_k = 2^k + 3^k$ rigid ?
2. Is the Furstenberg sequence sequence rigid ?
3. We can define $B(\Lambda)$, $W(\Lambda)$, and $H^\infty(\Lambda)$ for a real sequence Λ. The quoted theorem of Pisot says that $H^\infty((\theta^n)) = \mathbb{Q}(\theta)$ as soon as θ is algebraic. What about θ transcendental ?
4. Does $W((t_n))$ support a Rajchman measure for some (t_n) without the divisibility property ? Which lacunarity condition gives the result ?
5. Erdòs proved : there exists infinitely many prime numbers in $(q_n(\alpha))$ for a.e. α. In which proportion ?
6. Automatic sequence of integers (Christian). If Λ consists of integers, the 2-adic representation of which obeys the language of parenthesis, is it true that $B(\Lambda) \supset \mathbb{R} \backslash \mathbb{Q}$?

THANK YOU Christian!