Symbolic bounded remainder sets

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Prime numbers, determinism and pseudorandomness–CIRM

Bounded remainder sets

Kronecker sequences and toral translations

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in [0,1]^d$ with $1, \alpha_1, \cdots, \alpha_d$ $\mathbb{Q}$-linearly independent. Consider the sequence in $[0,1]^d$

$$(\{n\alpha_1\}, \ldots, \{n\alpha_d\})_n$$

associated with the translation over $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

$$R_\alpha \colon \mathbb{T}^d \mapsto \mathbb{T}^d, \ x \mapsto x + \alpha$$

One has

$$(\{n\alpha_1\}, \ldots, \{n\alpha_d\}) = R_\alpha^n(0)$$

Bounded remainder sets

Discrepancy

$$\Delta_N = \sup_{B \text{ box}} \left|\mathrm{Card}\left\{0 \leq n < N; R_\alpha^n(0) \in B\right\} - N \cdot \mu(B)\right|$$

Bounded remainder set A measurable set X for which there exists $C > 0$ s.t. for all N

$$|\mathrm{Card}\{0 \leq n \leq N; R_\alpha^n(0) \in X\} - N\mu(X)| \leq C$$

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Bounded ergodic deviations for ergodic sums associated with the function $\mathbf{1}_X$ for $(\mathbb{T}^d, R_\alpha)$

$$\left|\sum_{0 \leq n \leq N-1} \mathbf{1}_X(R_\alpha^n(0)) - N\mu(X)\right| \leq C \text{ for all } N$$

Bounded remainder sets for toral translations

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$$|\mathrm{Card}\{0 \leq n \leq N; R_\alpha^n(0) \in X\} - N\mu(X)| \leq C$$

[Kesten'66] $d = 1$ Intervals that are bounded remainder sets are the intervals with length in $\mathbb{Z} + \alpha\mathbb{Z}$

[Liardet'87] $d \geq 2$ There are no nontrivial boxes that are bounded remainder sets

[Grepstad-Lev,Haynes-Kelly-Koivusalo] Any parallelotope in $\mathbb{R}^d$ spanned by vectors $v_1, \cdots, v_d$ belonging to $\mathbb{Z}\alpha + \mathbb{Z}^d$ is a bounded remainder set for the translation by $\alpha = (\alpha_1, \cdots, \alpha_d)$ on $\mathbb{T}^d$, with $1, \alpha_1, \cdots, \alpha_d$ linearly independent.

Rauzy's program

- How to subdivide bounded remainder sets into smaller ones? How to get multiscale bounded remainder sets?

- How to construct bounded remainder sets via symbolic codings of Kronecker sequences? via subshifts with pure discrete spectrum?

Symbolic bounded remainder set

The shift T acts on $\mathcal{A}^{\mathbb{Z}}$ as $T((u_n)_n) = (u_{n+1})_n$
A subshift (X, T) is a closed shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$

Let (X, T, μ) be a minimal and uniquely ergodic subshift

A bounded remained set is a measurable set A for which there exists $C > 0$ such that for all N

$$\left| \sum_{0 \leq n \leq N-1} \mathbf{1}_A \circ T^n(x) - N\mu(A) \right| \leq C$$

Example Take a cylinder $[v] = \{u \in X,\ u_0 \cdots u_{|v|-1} = v\}$

$\rightsquigarrow$ balance and frequency results

Topological approach

- Consider a strictly ergodic subshift (X, T, μ)
- The cylinder $[v]$ is a bounded remainder set (X, T) iff the ergodic sums $\sum_{n=0}^{N-1} f_v(T^n(u))$ for $f_v = \mathbf{1}_{[v]} - \mu[v]$ are bounded

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- Theorem [Gottschalk-Hedlund] Let X be a compact metric space and $T \colon X \to X$ be a minimal homeomorphism. Let $f \colon X \to \mathbb{R}$ be a continuous function. Then f is a coboundary

$$f = g - g \circ T$$

for a continuous function g if and only if there exists x and there exists $C > 0$ such that for all N

$$|\sum_{n=0}^{N} f(T^n(x))| < C$$

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$$|\sum_{n=0}^{N} f(T^n(x))| < C$$

- The cylinder $[v]$ is a BRS iff $\mathbf{1}_{[v]} - \mu_v$ is a coboundary

Bounded remainder sets and spectrum

$[v]$ is a bounded remainder set iff $\mathbf{1}_{[v]} - \mu[v]$ is a coboundary

$$f_v = \mathbf{1}_{[v]} - \mu_v \quad \rightsquigarrow \quad f_v = g - g \circ T$$

$$\exp(2i\pi\, g \circ T) = \exp(2i\pi\mu[v]) \exp(2i\pi g)$$

$\exp(2i\pi g)$ is a continuous eigenfunction associated with the eigenvalue $\exp(2i\pi\mu[v]) \rightsquigarrow$ Topological rotation factor

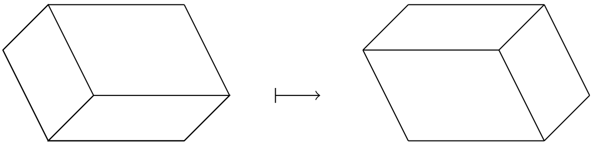
$$\begin{array}{ccc} X & \underset{\text{T}}{\longrightarrow} & X \\ \downarrow & & \downarrow \\ G & \overset{R}{\longrightarrow} & G \end{array}$$

Hypercubic billiards

Billiard

Consider a subshift generated by a billiard in the d-dimensional hypercube with slope $(\alpha_1, \cdots, \alpha_d)$, with $(\alpha_1, \cdots, \alpha_d)$ linearly independent over $\mathbb{Q}$. It is minimal and uniquely ergodic.

- Trajectories are coded according to the type of face they hit $\rightsquigarrow$ coding words in $\{1, 2, \cdots, d\}^{\mathbb{Z}}$.
- These coding words code translations on the torus $\mathbb{T}^{d-1}$ represented by a domain exchange acting on the following hexagonal fundamental domain.



[Arnoux-Mauduit-Shiokawa-Tamura'94, Baryshnikov'95]

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Sturmian case $d = 2$. Cylinders are bounded remainder sets.

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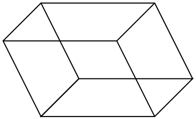
Cubic case and beyond $d \geq 3$ [B.-Bedaride-Julien]
Letter cylinders are bounded remainder sets but cylinders associated with factors of length at least 2 are not bounded remainder sets.

"Proof" Assume that w has bounded discrepancy. Let μ stand for the invariant measure. Then, $\mu[w]$ is an additive eigenvector and $\mu[w] \in \langle \alpha_1, \cdots, \alpha_d \rangle$. However, the areas of the zones that correspond to factors of length large enough do not belong to $\langle \alpha_1, \cdots, \alpha_d \rangle$.

Factors of length 2 $\rightsquigarrow$ 

Bounded remainder sets for toral translations

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in [0,1]^d$, $R_\alpha \colon \mathbb{T}^d \to \mathbb{T}^d$, $x \mapsto x + \alpha$

[Kesten'66] $d = 1$ Intervals that are bounded remainder sets are the intervals with length in $\mathbb{Z} + \alpha\mathbb{Z}$.
Cylinders are bounded remainder sets for Sturmian words.

[Grepstad-Lev'15,Haynes-Kelly-Koivusalo'17] Any parallelotope in $\mathbb{R}^d$ spanned by vectors $v_1, \cdots, v_d$ belonging to $\mathbb{Z}\alpha + \mathbb{Z}^d$ is a bounded remainder set for the translation by $\alpha = (\alpha_1, \cdots, \alpha_d)$ on $\mathbb{T}^d$, with $1, \alpha_1, \cdots, \alpha_d$ linearly independent.
Letter cylinders are bounded remainder sets for hypercubic billiard words.

Toward self-similarity

Kronecker sequences $\rightsquigarrow$ Multidimensional continued fractions

$\rightsquigarrow$ Products of nonnegative matrices

$\rightsquigarrow$ Products of substitutions

$\rightsquigarrow$ Symbolic bounded remainder sets $\rightsquigarrow$ Fractals

Based on the Substitution/Induction correspondence and on the fact that:

> "Induced of toral translations with respect to bounded remainder sets are translations" [Rauzy'84,Ferenczi'92]

Dynamical dimension group

Let (X, T) be a minimal subshift

- Coboundaries $\beta\colon C(X,\mathbb{Z}) \to C(X,\mathbb{Z}),\ f \mapsto f \circ T - f$
- Dimension group $H(X, T) = C(X,\mathbb{Z})/\beta C(X,\mathbb{Z})$

Thue-Morse substitution

$a \mapsto ab,\ b \mapsto ba$

$H(X, T) = \mathbb{Z}[1/2]$

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\mathbf{1}_{aa} - 1/6 \notin \beta C(X, T)$

[aa] is not a BRS

Fibonacci substitution

$a \mapsto ab,\ b \mapsto a$

$H(X, T) = \mathbb{Z}^2$

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$1_{[w]} \in \langle \mathbf{1}_{[0]}, \mathbf{1}_{[1]} \rangle$

Cylinders are BRS

From letters to factors

Theorem [B.-Cecchi-Durand-Leroy-Perrin-Petite] Let (X, S) be a primitive unimodular proper S-adic subshift. Any function $f \in C(X, \mathbb{Z})$ is cohomologuous to some integer linear combination of the form

$$\sum_{a \in \mathcal{A}} \alpha_a \mathbf{1}_{[a]} \in C(X, \mathbb{Z})$$

Moreover, the classes $[\mathbf{1}_{[a]}]$, $a \in \mathcal{A}$, are $\mathbb{Q}$-independent.

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Moreover, the classes $[\mathbf{1}_{[a]}]$, $a \in \mathcal{A}$, are $\mathbb{Q}$-independent.

- Holds for interval exchanges or Arnoux-Rauzy sequences.

- If letter cylinders are bounded remainder sets, then all cylinders are bounded remainder sets. Moreover, every $f \in C(X, \mathbb{R})$ is balanced for (X, T), i.e., there exists a constant $C_f > 0$ such that

$$|\sum_{i=0}^{n} f(T^i x) - f(T^i y)| \leq C_f \text{ for all } x, y \in X \text{ and for all } n.$$

Substitutions and bounded remainder sets

Let σ be a primitive substitution.

Theorem [Adamczewski]

- If σ is a Pisot substitution, then letter cylinders are bounded remainder sets in X_σ.
- Conversely, if letter cylinders are bounded remainder sets in X_σ, then the Perron–Frobenius eigenvalue of M_σ is the unique eigenvalue of M_σ that is larger than 1 in modulus, and all possible eigenvalues of modulus one of M_σ are roots of unity.

The Pisot substitution conjecture

Substitutive structure + Algebraic assumption (Pisot)

= Order

Order $\equiv$ discrete spectrum $\equiv$ translation on a compact group

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The Pisot substitution conjecture If σ is a Pisot irreducible substitution, then (X_σ, T) has discrete spectrum

Dates back to the 80's [Bombieri-Taylor, Rauzy,Thurston] and proved for two-letter alphabets [Host, Hollander-Solomyak]

Beyond the Pisot conjecture

Classical multidimensional continued fraction algorithms generate bounded remainder sets

Take your favorite continued fraction algorithm A (Jacobi-Perron, Brun, (Cassaigne)-Selmer, Arnoux-Rauzy, etc.)

Theorem [B.-Steiner-Thuswaldner]
For almost every $(\alpha, \beta) \in [0, 1]^2$, the translation by (α, β) on the torus $\mathbb{T}^2$ admits a symbolic model: the S-adic system provided by the multidimensional continued fraction algorithm A applied to (α, β) is measurably conjugate to the translation by (α, β). Moreover, the geometric realization of cylinders provides bounded remainder sets for the translation by (α, β).

See also [N. Pytheas Fogg: Andrieu, Bedaride, Bertazzon, Cassaigne, Mercat, Monteil]

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Continued fractions and codings of translations

Let A be a d-dimensional multidimensional continued fraction algorithm satisfying Lagarias' conditions. If moreover the following holds:

- The Pisot condition holds $\lambda_1(A) > 0 > \lambda_2(A)$.
- The set of admissible matrices is described by a graph with finitely many vertices (sofic).
- Cylinders have positive measure.

Then it is possible to associate an S-adic shift such that a.e. this S-adic shift is measurably conjugate to a translation on the torus $\mathbb{T}^{d-1}$; in particular, its measure-theoretic spectrum is purely discrete and cylinders give bounded remainder sets.

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Lagarias' conditions

(H1) Ergodicity The map T admits an ergodic invariant probability measure μ that is absolutely continuous with respect to Lebesgue measure.

(H2) Covering Property The map T is piecewise continuous with non-vanishing Jacobian almost everywhere.

(H3) Semi-Weak convergence This is a mixing condition for T which implies weak convergence.

(H4) Boundedness This is log-integrability of the cocycle A which is necessary in order to apply the Oseledets Theorem.

(H5) Partial quotient mixing The expectation of the number n for which $A^{(n)}(\mathbf{x})$ becomes a strictly positive matrix is finite.

Higher-dimensional case

Numerical experiments indicate that classical multidimensional continued fraction algorithms seem to cease to be strongly convergent for high dimensions. The only exception seems to be the Arnoux-Rauzy algorithm which, however, is defined only on a set of measure zero [B.-Steiner-Thuswaldner]

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d	$\lambda_2(A_B)$	$1 - \frac{\lambda_2(A_B)}{\lambda_1(A_B)}$	d	$\lambda_2(A_B)$	$1 - \frac{\lambda_2(A_B)}{\lambda_1(A_B)}$
2	-0.11216	1.3683	7	-0.01210	1.0493
3	-0.07189	1.2203	8	-0.00647	1.0283
4	-0.04651	1.1504	9	-0.00218	1.0102
5	-0.03051	1.1065	10	$+0.00115$	0.9943
6	-0.01974	1.0746	11	$+0.00381$	0.9799

Table: Heuristically estimated values for the second Lyapunov exponent and the uniform approximation exponent of the Brun Algorithm