Triality

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Workshop on Differential Geometry
and Nonassociative Algebras

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Wikipedia: In mathematics, triality is a relationship among three vector spaces, analogous to the duality relation between dual vector spaces. Most commonly, it describes those special features of the Dynkin diagram D_4 and the associated Lie group Spin_8 ... arising because the group has an outer automorphism of order three. There is a geometrical version of triality, analogous to duality in projective geometry. ... one finds a curious phenomenon involving 1-, 2-, and 4-dimensional subspaces of 8-dimensional space, historically known as "geometric triality".

Geometric duality

Points $\longleftrightarrow$ Hyperplanes

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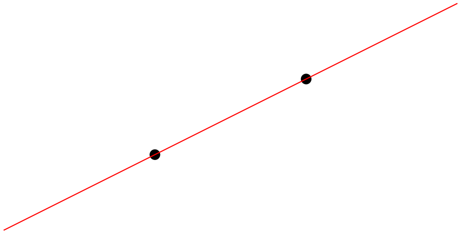
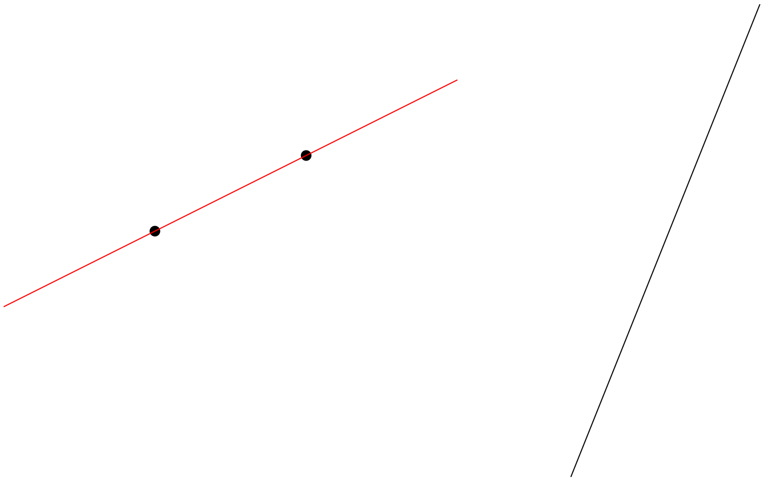


Geometric duality

Points ⟷ Hyperplanes



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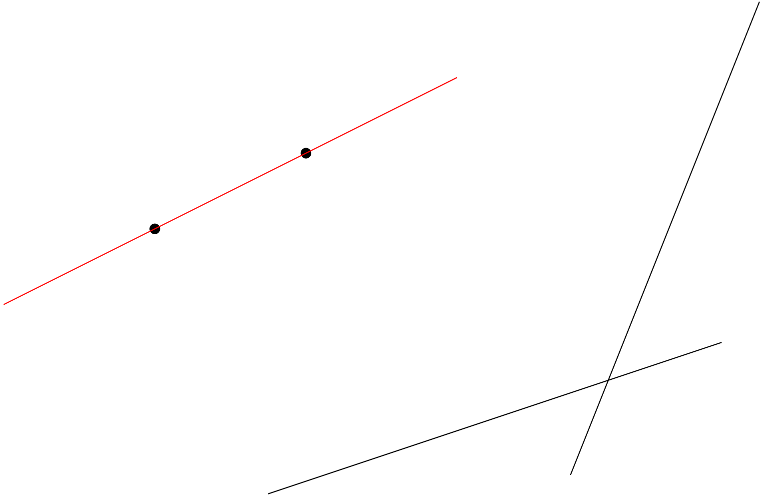

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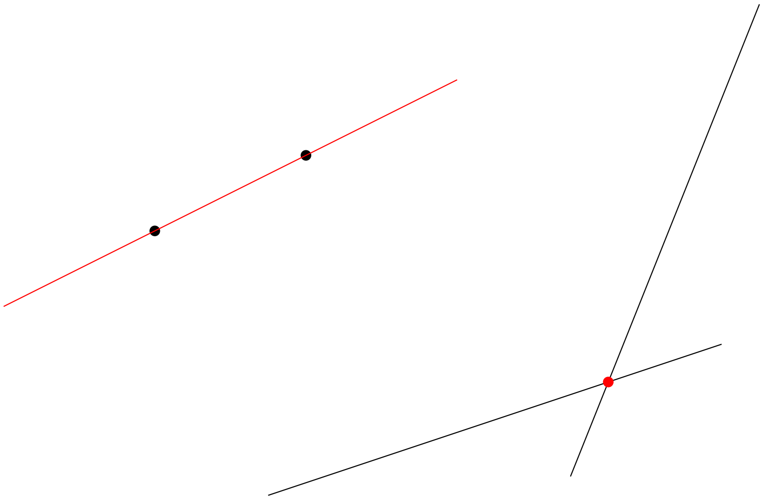

Geometric Triality

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  $U_i := \{\text{isotropic subspaces of dimension } i\}$, $i = 1, 2, 3, 4$.

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  U_1 : points; U_2 : lines; U_3 : planes; U_4 : "solids".

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- Two solids are of the **same kind** if their intersection (as vector subspaces) is of even dimension.
  It turns out that two solids are of the same kind if and only if they belong to the same orbit under the action of the special orthogonal group. There are exactly two kinds of solids.

Incidence relations

On the set of points and solids there is a natural incidence relation:

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- Two points are incident if they lie on a line (inside Q).
- Two solids of the same kind are incident if their intersection is not trivial.
- Two solids of different kinds are incident if their intersection is a plane.
- A point is incident with a solid if it lies in it.

Geometric Triality

Theorem (Eduard Study 1913)

- *The variety of solids of a fixed kind in Q is a quadric isomorphic to Q.*
- *Any proposition in the geometry of Q (about incidence relations) remains true if the concepts of points, solids of one kind, and solids of the other kind, are cyclically permuted.*

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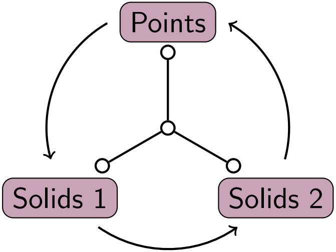

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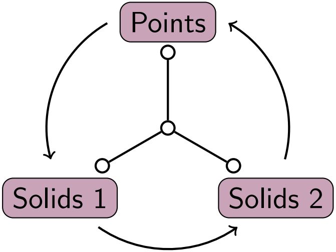

Élie Cartan (1925): *On peut dire que le principe de dualité de la géometrie projective est remplacé par un principe de trialité.*

Composition algebras

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Definition

- A **composition algebra** over a field is a triple $(C, \cdot, n)$ where
  - C is a vector space,
  - $x \cdot y$ is a bilinear multiplication $C \times C \to C$,
  - $n : C \to \mathbb{F}$ is a multiplicative ($n(x \cdot y) = n(x)n(y)\ \forall x, y \in C$) nonsingular quadratic form.

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- The unital composition algebras are called **Hurwitz algebras**.

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- The eight-dimensional Hurwitz algebras are termed **octonion (or Cayley) algebras**.

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Theorem

- *Hurwitz algebras are isomorphic iff their norms are isometric.*
- *For each dimension 2, 4, or 8, there is a unique, up to isomorphism, Hurwitz algebra with isotropic norm:*
  - $\mathbb{F} \times \mathbb{F}$ *with* $n\big((\alpha, \beta)\big) = \alpha\beta$,
  - $\mathrm{Mat}_2(\mathbb{F})$ *with* $n = \det$,
  - *The algebra of Zorn matrices (or* **split Cayley algebra***):*

$$\mathcal{C}_s = \left\{ \begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} : \alpha, \beta \in \mathbb{F}, u, v \in \mathbb{F}^3 \right\}, \quad \textit{with}$$

$$\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} \cdot \begin{pmatrix} \alpha' & u' \\ v' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + (u \mid v') & \alpha u' + \beta' u - v \times v' \\ \alpha' v + \beta v' + u \times u' & \beta\beta' + (v \mid v') \end{pmatrix},$$

$$n\left(\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix}\right) = \alpha\beta - (u \mid v).$$

Octonions and Geometric Triality

Let $(\mathcal{C}_s, \cdot, n)$ be the split Cayley algebra and identify our quadric Q with $\{\mathbb{F}x : 0 \neq x \in \mathcal{C}_s,\ n(x) = 0\}$.

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Theorem (Felix Vaney 1929)

- *The solids of the two kinds are precisely the subspaces:*

$$S_a^1 := \{x \in \mathcal{C}_s : a \cdot x = 0\}, \qquad S_a^2 := \{x \in \mathcal{C}_s : x \cdot a = 0\},$$

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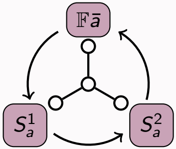
- *The cyclic permutation*



  is a 'geometric triality' (it preserves incidence relations).

Trialitarian automorphisms

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Answer

Yes: **Okubo algebras**.

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On the vector space $\mathfrak{sl}_3(\mathbb{F})$ consider the multiplication:

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy)1,$$

and norm: $$n(x) = -\frac{1}{2} \operatorname{tr}(x^2).$$

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In particular, $(\mathfrak{sl}_3(\mathbb{F}), *, n)$ is a *composition algebra*.

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- $P_8(\mathbb{F})$ makes sense in characteristic 2, because $\mathrm{tr}(x^2)$ 'is a multiple of 2' if $\mathrm{tr}(x) = 0$.
- Okubo and Osborn (1981) gave an 'ad hoc' definition of $P_8(\mathbb{F})$ over fields of characteristic 3 by means of its multiplication table.

Okubo algebras

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In order to define Okubo algebras over arbitrary fields consider the Pauli matrices:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

in $\mathrm{Mat}_3(\mathbb{C})$, which satisfy

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For $i, j \in \mathbb{Z}/3\mathbb{Z}$, $(i, j) \neq (0, 0)$, define

$$x_{i,j} := \frac{\omega^{ij}}{\omega - \omega^2} x^i y^j.$$

$\{x_{i,j} : (i, j) \neq (0, 0)\}$ is a basis of $\mathfrak{sl}_3(\mathbb{C})$.

Okubo algebras

$$
\begin{aligned}
x_{i,j} * x_{k,l} &= \omega x_{i,j} x_{k,l} - \omega^2 x_{k,l} x_{i,j} - \frac{\omega - \omega^2}{3} \operatorname{tr}(x_{i,j} x_{k,l}) 1 \\
&= \begin{cases} x_{i+k,j+l} \\ 0 \\ -x_{i+k,j+l} \end{cases} \qquad \big(x_{0,0} := 0\big)
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Besides, $n(x_{i,j}) = 0$ for any i, j, and

$$n(x_{i,j}, x_{k,l}) = \begin{cases} 1 & \text{for } (i,j) = -(k,l), \\ 0 & \text{otherwise.} \end{cases}$$

Okubo algebras

Thus, the $\mathbb{Z}$-module

$$\mathcal{O}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\,\{x_{i,j} : -1 \leq i, j \leq 1,\ (i,j) \neq (0,0)\}$$

is closed under $*$, and n restricts to a nonsingular multiplicative quadratic form on $\mathcal{O}_{\mathbb{Z}}$.

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Definition

Let $\mathbb{F}$ be an arbitrary field. Then

$$\mathcal{O}_{\mathbb{F}} := \mathcal{O}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F},$$

with the induced multiplication and nonsingular quadratic form, is called the **split Okubo algebra** over $\mathbb{F}$.

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Symmetric composition algebras

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A composition algebra $(\mathcal{C}, *, n)$ is said to be **symmetric** if the polar form of its norm is associative:

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Markus Rost, around 1994, realized that this is the right class of algebras to deal with the phenomenon of triality.

Examples

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- Given any Hurwitz algebra $(\mathcal{B}, \cdot, n)$, the algebra $(\mathcal{B}, \bullet, n)$, where

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  is called the associated **para-Hurwitz** algebra (Okubo-Myung 1980).

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  Para-Hurwitz algebras are symmetric.

Theorem (Okubo-Osborn 1981, E.-Pérez-Izquierdo 1996)

Any eight-dimensional symmetric composition algebra is either a para-Hurwitz algebra or an Okubo algebra.

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Sketch of proof

- If $(\mathcal{C}, *, n)$ is a symmetric composition algebra over $\mathbb{F}$, there is a field extension $\mathbb{K}/\mathbb{F}$ of degree ≤ 3 such that $(\mathcal{C}_{\mathbb{K}}, *, n)$ contains a nonzero idempotent. Hence we may assume that there exists $0 \neq e \in \mathcal{C}$ with $e * e = e$. Then $n(e) = 1$.

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  Then $(\mathcal{C}, \cdot, n)$ is a Hurwitz algebra with unity $1 = e$, and the map $\tau : x \mapsto e * (e * x) = n(e, x)e - x * e$ is an automorphism of both $(\mathcal{C}, *, n)$ and of $(\mathcal{C}, \cdot, n)$, such that $\tau^3 = \mathrm{id}$.

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- If $\tau = \mathrm{id}$, $(\mathcal{C}, *, n)$ is para-Hurwitz, otherwise it may be either para-Hurwitz or Okubo.

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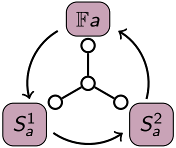
- The cyclic permutation  is a geometric triality.

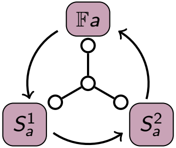
Symmetric compositions and geometric triality

Let $(\mathcal{C}, *, n)$ be a symmetric composition algebra with isotropic norm and identify our quadric Q with $\{\mathbb{F}x : 0 \neq x \in \mathcal{C},\ n(x) = 0\}$.

- The solids of the two kinds are precisely the subspaces

$$S_a^1 := \{x \in \mathcal{C} : a * x = 0\}, \qquad S_a^2 := \{x \in \mathcal{C} : x * a = 0\},$$

  for $0 \neq a \in \mathcal{C}$, $n(a) = 0$.

- The cyclic permutation 

is a geometric triality.

- Any geometric triality is given in this way. The one attached to the para-Cayley algebra coincides with the familiar one, related to the split Cayley algebra. The ones attached to Okubo algebras constitute the other type in Tits' classification.

Classification of Okubo algebras

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Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F} \neq 3$, containing a primitive cubic root of 1. By restriction we obtain a natural isomorphism

$$\mathbf{PGL}_3 \simeq \mathbf{Aut}\big(\mathrm{Mat}_3(\mathbb{F})\big) \to \mathbf{Aut}\big((\mathfrak{sl}_3(\mathbb{F}), *, n)\big)$$

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Theorem (E.-Myung 1991, 1993)

The map

$$\begin{Bmatrix} \textit{Isomorphism classes of} \\ \textit{central simple degree 3} \\ \textit{associative algebras} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \textit{Isomorphism classes} \\ \textit{of Okubo algebras} \end{Bmatrix}$$

$$[\mathcal{A}] \mapsto [(\mathcal{A}_0, *, n)]$$

is bijective.

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Theorem (E.-Myung 1991, 1993)

The map

$$\left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{pairs } (\mathcal{B},\sigma)\textit{, where } \mathcal{B} \textit{ is a simple}\\ \textit{degree 3 associative algebra}\\ \textit{over } \mathbb{K} \textit{ and } \sigma \textit{ a } \mathbb{K}/\mathbb{F}\textit{-involution}\\ \textit{of the second kind}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\textit{Isomorphism classes}\\ \textit{of Okubo algebras}\end{array}\right\}$$

$$[(\mathcal{B},\sigma)] \mapsto [(\mathrm{Skew}(\mathcal{B},\sigma)_0, *, n)]$$

is bijective.

Classification of Okubo algebras (char $\mathbb{F} = 3$)

Theorem (Chernousov-E.-Knus-Tignol 2013)

Let $(\mathcal{O}, *, n)$ *be the split Okubo algebra over a field* $\mathbb{F}$ *(*char $\mathbb{F} = 3$*).*

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Theorem (Chernousov-E.-Knus-Tignol 2013)

Let $(\mathcal{O}, *, n)$ be the split Okubo algebra over a field $\mathbb{F}$ (char $\mathbb{F} = 3$).

- ***Aut**$(\mathcal{O}, *, n)$ is not smooth:* dim **Aut**$(\mathcal{O}, *, n) = 8$ *while* $\mathfrak{Der}(\mathcal{O}, *, n)$ *is a simple (nonclassical) Lie algebra of dimension* 10*.*

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- $\mathbf{Aut}(\mathcal{O}, *, n) = \mathbf{HD}$, *where* $\mathbf{H} = \mathbf{Aut}(\mathcal{O}, *, n)_{\text{red}}$ *and* $\mathbf{D} \simeq \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$.

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- *The map*

$$H^1(\mathbb{F}, \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \to H^1\big(\mathbb{F}, \mathbf{Aut}(\mathcal{O}, *, n)\big)$$

induced by the inclusion $\mathbf{D} \hookrightarrow \mathbf{Aut}(\mathcal{O}, *, n)$, *is surjective.*

Classification of Okubo algebras (char $\mathbb{F} = 3$)

Recall that $\mathcal{O}$ is spanned by elements $x_{i,j}$, $(i,j) \neq (0,0)$ (indices modulo 3). It is actually generated by $x_{1,0}$ and $x_{0,1}$. Given $0 \neq \alpha, \beta \in \mathbb{F}$, the elements

$$x_{1,0} \otimes \alpha^{\frac{1}{3}},\ x_{0,1} \otimes \beta^{\frac{1}{3}} \in \mathcal{O} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$$

generates, by multiplication and linear combinations over $\mathbb{F}$, a twisted form of $(\mathcal{O}, *, n)$. Denote it by $\mathcal{O}_{\alpha,\beta}$.

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Corollary

The following map is surjective:

$$\begin{array}{ccc} \mathbb{F}^\times/(\mathbb{F}^\times)^3 \times \mathbb{F}^\times/(\mathbb{F}^\times)^3 & \longrightarrow & \left\{ \begin{array}{c} \textit{Isomorphism classes} \\ \textit{of Okubo algebras} \end{array} \right\} \\ \big(\alpha(\mathbb{F}^\times)^3, \beta(\mathbb{F}^\times)^3\big) & \mapsto & [\mathcal{O}_{\alpha,\beta}] \end{array}$$

Classification of Okubo algebras (char $\mathbb{F} = 3$)

Theorem (E. 1997)

- *Any Okubo algebra over $\mathbb{F}$ (char $\mathbb{F} = 3$) is isomorphic to $\mathcal{O}_{\alpha,\beta}$ for some $0 \neq \alpha, \beta \in \mathbb{F}$.*
- *For $0 \neq \alpha, \beta \in \mathbb{F}$, let*

$$S_{\alpha,\beta} := \mathit{span}_{\mathbb{F}^3}\left\{\alpha^{\pm 1}, \beta^{\pm 1}, \alpha^{\pm 1}\beta^{\pm 1}\right\}.$$

Then $\mathcal{O}_{\alpha,\beta}$ is either isomorphic or antiisomorphic to $\mathcal{O}_{\gamma,\delta}$ if and only if $S_{\alpha,\beta} = S_{\gamma,\delta}$.

Symmetric composition algebras and triality

Let $(\mathcal{C}, *, n)$ be an eight-dimensional symmetric composition algebra. Write

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Therefore, the map $x \mapsto \begin{pmatrix} 0 & L_x \\ R_x & 0 \end{pmatrix}$ extends to an isomorphism of algebras with involution

$$\Phi : \big(\mathfrak{Cl}(\mathcal{C}, n), \tau\big) \longrightarrow \big(\mathsf{End}(\mathcal{C} \oplus \mathcal{C}), \sigma_{n \perp n}\big)$$

Spin group

Consider the *spin group*:

$$\mathrm{Spin}(\mathcal{C}, n) = \left\{ u \in \mathfrak{Cl}(\mathcal{C}, n)^{\times}_{\bar{0}} : u \cdot \mathcal{C} \cdot u^{-1} \subseteq \mathcal{C},\ u \cdot \tau(u) = 1 \right\}.$$

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The natural and the two half-spin representations are linked!

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Theorem

Let $(\mathcal{C}, *, n)$ be an eight-dimensional symmetric composition algebra. Then:

$$\begin{aligned}\mathrm{Spin}(\mathcal{C}, n) \simeq \{(f_0, f_1, f_2) \in \mathrm{O}^+(\mathcal{C}, n)^3 :& \\ f_0(x * y) = f_1(x) * f_2(y) \quad \forall x, y \in \mathcal{C}\}&\end{aligned}$$

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The cyclic symmetry on the right hand side induces an outer automorphism of order 3 (*trialitarian automorphism*) of $\mathrm{Spin}(\mathcal{C}, n)$.

The Principle of Triality

Theorem

Let $(\mathcal{C}, *, n)$ be an eight-dimensional symmetric composition algebra. Then, for any $f_0 \in \mathrm{O}'(\mathcal{C}, n)$, there are elements $f_1, f_2 \in \mathrm{O}'(\mathcal{C}, n)$, unique up to scalar multiplication of both by -1, such that (f_0, f_1, f_2) is a related triple.

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Remark

All this is functorial, and we get three exact sequences

$$1 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow \mathbf{Spin}(\mathcal{C}, n) \longrightarrow \mathbf{O}^+(\mathcal{C}, n) \longrightarrow 1.$$

Trialitarian automorphisms

Theorem (Chernousov,Knus,Tignol,E. 2012-2015)

- *A simply connected simple group of type* $^1D_4$ *admits trialitarian automorphisms if and only if it is isomorphic to* $\mathbf{Spin}(n)$ *for a 3-fold Pfister form; i.e., the norm of an eight-dimensional composition algebra.*

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- *The groups of type* $^2D_4$ *and* $^6D_4$ *do not admit trialitarian automorphisms.*
- *The trialitarian automorphisms of the groups of type* $^3D_4$ *are related too to symmetric composition algebras.*

Application: Freudenthal Magic Square

Local principle of triality

Theorem

Let $(\mathcal{C}, *, n)$ *be an eight-dimensional symmetric composition algebra. Then, for any* $d_0 \in \mathfrak{so}(\mathcal{C}, n)$*, there are unique elements* $d_1, d_2 \in \mathfrak{so}(\mathcal{C}, n)$ *such that* $d_0(x * y) = d_1(x) * y + x * d_2(y)$*, for any* $x, y \in S$*. Moreover,*

- *The map* $\theta : \mathfrak{tri}(\mathcal{C}, *, n) \to \mathfrak{tri}(\mathcal{C}, *, n)$*,* $(d_0, d_1, d_2) \mapsto (d_1, d_2, d_0)$*, is a Lie algebra automorphism.*
- *Any of the projections* $\mathfrak{tri}(\mathcal{C}, *, n) \to \mathfrak{so}(\mathcal{C}, n)$*,* $(d_0, d_1, d_2) \mapsto d_i$*, is an isomorphism of Lie algebras.*

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- *The map $\theta : \mathfrak{tri}(\mathcal{C}, *, n) \to \mathfrak{tri}(\mathcal{C}, *, n)$, $(d_0, d_1, d_2) \mapsto (d_1, d_2, d_0)$, is a Lie algebra automorphism.*
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The Lie algebra

$$\mathfrak{tri}(\mathcal{C}, *, n) = \{(d_0, d_1, d_2) \in \mathfrak{so}(\mathcal{C}, n)^3 : \\ d_0(x * y) = d_1(x) * y + x * d_2(y) \quad \forall x, y \in \mathcal{C}\}$$

is called the **triality Lie algebra** of $(\mathcal{C}, *, n)$.

Application: Freudenthal Magic Square

Symmetric construction (E. 2004)

Let $(\mathcal{C}, *, n)$ and $(\mathcal{C}', \star, n')$ be two symmetric composition algebras over a field $\mathbb{F}$ of characteristic $\neq 2$. One can construct a Lie algebra as follows:

$$\mathfrak{g} = \mathfrak{g}(\mathcal{C}, \mathcal{C}') = \big(\mathfrak{tri}(\mathcal{C}) \oplus \mathfrak{tri}(\mathcal{C}')\big) \oplus \Big(\oplus_{i=0}^{2} \iota_i(\mathcal{C} \otimes \mathcal{C}')\Big),$$

with bracket given by:

- the Lie bracket in $\mathfrak{tri}(\mathcal{C}) \oplus \mathfrak{tri}(\mathcal{C}')$, which thus becomes a Lie subalgebra of $\mathfrak{g}$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d_0', d_1', d_2'), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d_i'(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((x * y) \otimes (x' \star y')\big)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = \ldots$

Application: Freudenthal Magic Square

Symmetric construction

$\mathfrak{g}(\mathcal{C}, \mathcal{C}')$		$\dim \mathcal{C}'$: 1	2	4	8
$\dim \mathcal{C}$	1	A_1	A_2	C_3	F_4
	2	A_2	$A_2 \oplus A_2$	A_5	E_6
	4	C_3	A_5	D_6	E_7
	8	F_4	E_6	E_7	E_8

Application: Freudenthal Magic Square

Symmetric construction: remarks

- In Freudenthal's approach to the Magic Square, each row corresponds to a different type of Geometry: Elliptic, Projective, Symplectic and 'Metasymplectic'.
  Tits construction (1966) of the Magic Square involves a Hurwitz algebra and a simple Jordan algebra of degree 3.
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- Different symmetric constructions have been given lately: Vinberg (1966), Allison-Faulkner (1996), Barton-Sudbery and Landsberg-Manivel (2003). They are equivalent to the construction above using para-Hurwitz algebras.
- The symmetric construction with Okubo algebras provides nice models of the exceptional algebras. They have been used in the study of gradings by abelian groups on these algebras: Aranda-Orna, Draper, Guido, Kochetov, Martín-González, ...

In characteristic 3 there exist nontrivial *symmetric composition superalgebras*.

These can be used to enlarge Freudenthal Magic Square with new simple Lie superalgebras (Cunha-E. 2007).

Most simple nonclassical modular contragredient Lie superalgebras appear in this *Magic Supersquare*.

The saying that God is the mathematician, so that, even with meager experimental support, a mathematically beautiful theory will ultimately have a greater chance of being correct, has been attributed to Dirac. Octonion algebra may surely be called a beautiful mathematical entity.

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Thank you!