Workshop on Differential Geometry and Nonassociative Algebras
CIRM, Marseille November 12th, 2019

Symplectic Jacobi Jordan algebras

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Definition

A finite dimension algebra $(J, .)$ is said to be Jacobi Jordan algebra if it is commutative and satisfies the Jacobi identity :

$$J(x, y, z) := x.(y.z) + y.(z.x) + z.(x.y) = 0, \ \ \forall x, y, z \in J.$$

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Proposition : D. Burde, A. Fialowski : 2014

Every Jacobi-Jordan algebra $(J, .)$ is a nilpotent Jordan algebra such that $x^3 = 0, \ \forall x \in J$.

Example

Let $(J, .)$ be an anti-associative algebra. Consider the second bilinear product $\circ$ in J defined by $x \circ y := x.y + y.x, \ \ \forall x, y, z \in J$. A simple computation proves that $(J, \circ)$ is a Jacobi-Jordan algebra.

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Let $(J_4, .)$ be the 4-dimensional algbera defined by
$e_1.e_1 = e_2, \quad e_1.e_3 = e_3.e_1 = e_4$, where $\{e_1, e_2, e_3, e_4\}$ is a basis of J_4.
$(J_4, .)$ is a Jacobi-Jordan algebra.

Admissible Jacobi-Jordan algebras

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Definition

Let $(J, .)$ be an algebra. We consider the new product $\circ$ defined as above on the vector space J by :

$$x \circ y := x.y + y.x, \ \forall x, y, z \in J.$$

The algebra $(J, .)$ is called Jacobi-Jordan admissible algebra if $(J, \circ)$ is a Jacobi-Jordan algebra. In this case the product "." will be called also Jacobi-Jordan admissible product.

Admissible Jacobi-Jordan algebras

Definition

An algebra $(J, .)$ is said to be left (resp. right) skew-symmetric if $Aasso(x, y, z) = -Aasso(y, x, z)$, (resp. $Aasso(x, y, z) = -Aasso(x, z, y)$), for all $x, y, z \in J$.

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Proposition

Any right (resp. left) skew-symmetric algebra is an admissible Jacobi-Jordan algebras.

Derivation and antiderivation

$D \in \mathrm{Hom}(J, J)$. D is called a derivation (resp. anti-derivation) of the algebra $(J, .)$ if for any $x, y \in J$ we have

$$D(x.y) := D(x).y + x.D(y), \quad (resp. \;\; D(x.y) := -D(x).y - x.D(y)).$$

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Examples

Let $(J, .)$ be a Jacobi-Jordan algebra.

1. For all $x \in J$, the left multiplication map $L_x : J \to J$, defined by $L_x(y) := xy, \ \ \forall y \in J$, belong to $\mathrm{Ader}(J)$.
2. If $D, D' \in \mathrm{Ader}(J)$, then $\{D, D'\} := DD' + D'D$, is an anti-derivation if and only if

$$\{D, D'\}(xy) = D(x)D'(y) + D'(x)D(y), \ \forall x, y \in J.$$

Admissible pair

Definition

An admissible pair of a Jacobi-Jordan algebra $(J, .)$ is a pair (D, A_0) where $D \in \mathrm{Ader}(J)$ and $A_0 \in \mathrm{Ker}D$ such that $D^2 = -\frac{1}{2}L_{A_0}$. The set of all admissible pairs of $(J, .)$ is denoted by $\mathrm{Padm}(J)$.

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Example : Let $x \in J$. Since $L_x \in \mathrm{Ader}(J)$ and $J(x, x, y) = 0$, then $L_x^2 = -\frac{1}{2}L_{x^2}$ and $L_x(x^2) = 0$. Thus, $(D = L_x, A_0 = x^2)$ is an admissible pair of J.

Definition

A symplectic form on a Jacobi-Jordan algebra $(J, .)$ is a skewsymmetric nondegenerate bilinear form ω satisfying

$$\omega(x.y, z) + \omega(y.z, x) + \omega(z.x, y) = 0, \ \ \forall x, y, z \in J.$$

A Jacobi-Jordan algebra is called symplectic if it is endowed with a such form.

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Example

Let $(J_4, .)$ be the 4-dimensional Jacobi-Jordan algbera defined above by $e_1.e_1 = e_2, \ \ e_1.e_3 = e_3.e_1 = e_4$, where $\{e_1, e_2, e_3, e_4\}$ is a basis of the $\mathbb{K}-$vector space J_4. The skew-symmetric bilinear form ω defined on J_4 by $\omega(e_1, e_4) = 1$ *and* $\omega(e_3, e_2) = 2$ is a symplectic form on J_4.

No sympletic structure :

Example :

Let $(J, .)$ be the 4-dimensional Jacobi-Jordan algebra defined by $e_1 e_1 = e_2$, where $\{e_1, e_2, e_3, e_4\}$ is a basis of the $\mathbb{K}-$vector space J. For any skew-symmetric bilinear form ω on J satisfying $\omega(x.y, z) + \omega(y.z, x) + \omega(z.x, y) = 0, \ \forall x, y, z \in J$

$$we \ \ have \ \ \omega(e_2, e_1) = \omega(e_2, e_2) = \omega(e_2, e_3) = \omega(e_2, e_4) = 0.$$

So, there is no symplectic structure on J.

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$$we \ \ have \ \ \omega(e_2, e_1) = \omega(e_2, e_2) = \omega(e_2, e_3) = \omega(e_2, e_4) = 0.$$

So, there is no symplectic structure on J.

Remark :

By similar computation as the previous example, we prove that the algebra J_4 given in a previous example is the unique not null symplectic Jacobi Jordan algebra of dimension 4.

Admissible Jacobi-Jordan algebra from symplectic one :

Proposition

Let $(J, ., \omega)$ be a symplectic Jacobi-Jordan Algera. Consider the product defined by

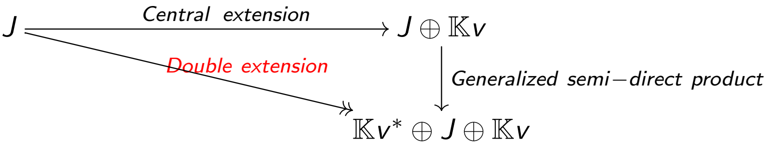
$$\omega(x \odot y, z) = \omega(x, y.z), \ \ \forall x, y, z \in J.$$

This product satisfies :

(i) $x.y = \ x \odot y \ + \ y \odot x, \ \ \forall x, y, \in J.$
This means that $(J, \odot)$ is an admissible Jacobi-Jordan algebra

(ii) The vector space J endowed with this new product $\odot$ is a right skew-symmetric algebra.

Double extension



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Central extension

Proposition

Let $(J_1, .)$ be a Jacobi-Jordan algebra, $\mathbb{K}v$ a one dimensional vector space and $\varphi : J_1 \times J_1 \longrightarrow \mathbb{K}$ a bilinear form. On the vector spaces $J_2 = J_1 \oplus \mathbb{K}v$ we define the following product :

$$(x + \alpha v) \circ (y + \beta v) = x.y + \varphi(x, y)v, \ \forall x, y \in J_1, \alpha, \beta \in \mathbb{K}.$$

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$$(x + \alpha v) \circ (y + \beta v) = x.y + \varphi(x, y)v, \ \forall x, y \in J_1, \alpha, \beta \in \mathbb{K}.$$

The product above is a Jacobi-Jordan algebra if and only if φ satisfies

$$\sum_{\circlearrowleft} \varphi(x, yz) = 0, \ \ \textit{for all } x, y, z \in J_1.$$

In this case, We say that $(J_2, \circ)$ is the central extension of $(J_1, .)$ by means of φ .

Generalized semi-direct product

Proposition

Let $(J_1, .)$ be a Jacobi-Jordan algebra, $\mathbb{K}v$ a one dimensional vector space, $D : J_1 \to J_1$ a linear map and $A_0 \in J_1$.
On the vector spaces $J_2 = J_1 \oplus \mathbb{K}v$ we define the following product :

$$(x + \alpha v) \bullet (y + \beta v) = x.y + \beta D(x) + \alpha D(y) + \alpha\beta A_0,$$

$\forall x, y \in J_1, \alpha, \beta \in \mathbb{K}.$

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$\forall x, y \in J_1, \alpha, \beta \in \mathbb{K}$. The product above is a Jacobi-Jordan algebra if and only if $(D.A_0)$ is an admissible pair of $(J_1, .)$.
In this case, We say that $(J_2, \bullet)$ is a generalized semi-direct product of $(J_1, .)$ by means of $(D.A_0)$.

Theorem : Double extension

Let $(J, ., \omega)$ be a symplectic Jacobi-Jordan algebra and $(D, A_0) \in Padm(J)$ such that $A_0 \in (Im(D))^{\perp}$. Let $\mathbb{K}e$ and $\mathbb{K}e^*$ two dual linear spaces
The vector space $\tilde{J} := \mathbb{K}e \oplus J \oplus \mathbb{K}e^*$ endowed with the product $\star$:

$$\begin{cases} e \star e := A_0, \;\; e^* \star X = X \star e^* := 0, \;\; \forall X \in \tilde{J} \\ e \star x = x \star e := D(x) + \frac{1}{2}\omega(A_0, x)e^*, \\ x \star y := xy + \omega((D - D^*)(x), y)e^*, \;\; \forall x, y \in J, \end{cases}$$

and the skew-symmetric bilinear form $\tilde{\omega}$:

$$\tilde{\omega}_{|_{J\times J}} = \omega, \;\; \tilde{\omega}(e, e^*) = 1,$$
$$\tilde{\omega}(e, J) = \tilde{\omega}(e^*, J) = \{0\}, \;\; \tilde{\omega}(e, e) = \tilde{\omega}(e^*, e^*) = 0,$$

is a symplectic Jacobi-Jordan algebra.

Proof : Let us consider the bilinear form φ of J defined by

$$\varphi(x,y) = \omega((D-D^*)x,y), \ \forall x,y \in J.$$

Since D is an antiderivation and ω is symplectic, then

$$\sum_{\circlearrowleft} \varphi(xy,z) = \sum_{\circlearrowleft} \omega((D-D^*)(xy),z)$$
$$= -\omega(yD(z),x) - \omega(D(z)x,y) + \omega(D(x)y,Z) + \omega(xD(y),z) = 0,$$

for all $x,y,z \in J$. Thus, φ is a symmetric 2- cocycle of J.
It follows that we can consider the central extension $J_1 = J \oplus \mathbb{K}e^*$ of J by means of φ. Now, Let $(\tilde{D}, A_0) \in End(J_1) \times J_1$ be the pair defined by : $\tilde{D}(x+\alpha e^*) := D(x) + \frac{1}{2}\omega_1(x,A_0)e^*, \quad \forall x \in J, \alpha \in \mathbb{K}$.
We have to check that $(\tilde{D}, \tilde{A}_0)$ is an admissible pair of J_1. After that, we can consider the generalized semi-direct product $\tilde{J} = \mathbb{K}e \oplus J_1$ of J_1 by means of $(\tilde{D}, A_0)$.

Example : Let us consider the algebra J_4 given above, defined by $e_1.e_1 = e_2, \quad e_1.e_3 = e_3.e_1 = e_4$. It is clear that

$$Ann(J_4) = Vect\{e_2, e_4\}.$$

Let D be the antiderivation of J_4 defined by

$$D(e_1) = \alpha e_3 + \beta e_4, \qquad D(e_2) = -2\alpha e_4, \qquad D(e_3) = D(e_4) = 0$$

and let $A_0 := \gamma e_4$. We can verify that the pair (D, A_0) is admissible. Thus, we can consider the double extension of J_4. We obtain the new Jacobi-Jordan algebra $\tilde{J}_4 := \mathbb{K}e \oplus J_4 \oplus \mathbb{K}e^*$ with the product defined by

$$e \star e = \gamma e_4, \quad e \star e_1 = \alpha e_3 + \beta e_4 - \frac{1}{2}\gamma e^*, \quad e \star e_2 = -2\alpha e_4,$$
$$e_1 \star e_1 = e_2 - 2\beta e^*, \quad e_1 \star e_3 = e_4.$$

A symplectic structure $\tilde{\omega}$ on $\tilde{J}_4$ is defined by

$$\tilde{\omega}(e_1, e_4) = 1, \quad \tilde{\omega}(e_2, e_3) = 2, \quad \tilde{\omega}(e, e^*) = 1.$$

Symplectomorphism :

Definition

A symplectomorphism between two symplectic Jacobi-Jordan algebras $(\tilde{J}, \star, \tilde{\omega})$ and $(\tilde{J}', \bullet, \tilde{\omega}')$ is an isomorphism $\Phi : (\tilde{J}, \star) \to (\tilde{J}', \bullet)$ of Jacobi-Jordan algebras such that $\tilde{\omega}'(\Phi(X), \Phi(Y)) = \tilde{\omega}(X, Y), \ \forall X, Y \in \tilde{J}$.

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Now, For a symplectic Jacobi-Jordan algebra $(J, ., \omega)$, we consider $(\tilde{J} := \mathbb{K}e \oplus J \oplus \mathbb{K}e^*, \star, \tilde{\omega})$ (resp. $(\tilde{J}' = \mathbb{K}e' \oplus J \oplus \mathbb{K}e'^*, \bullet, \tilde{\omega}')$) the symplectic double extension of $(J, ., \omega)$ by means of(D, A_0) (resp. (D', A_0')). We will investigate how (D, A_0) and (D', A_0') are connected with each other when $(\tilde{J}, \star, \tilde{\omega})$ and $(\tilde{J}', \bullet, \tilde{\omega}')$ are symplectomorphic.

Theorem

Under the conditions above, the two double extensions $(\tilde{J}, \star, \tilde{\omega})$ and $(\tilde{J}', \bullet, \tilde{\omega}')$ are symplectomorphic if and only if there exists a symplectomorphism Φ_0 of $(J, ., \omega)$, $x_0 \in J$ and $\gamma \in \mathbb{K} - \{0\}$ such that

$$\begin{aligned} \Phi_0(A_0) &= \gamma^{-2} A'_0 + 2\gamma^{-1} D'(x_0) + x_0.x_0 && (1) \\ L_{x_0} &= \Phi_0 \circ D \circ \Phi_0^{-1} - \gamma^{-1} D'. && (2) \end{aligned}$$

In this cas, there exists a symplectomorphism $\Phi : (\tilde{J}, \star, \tilde{\omega}) \Longleftrightarrow (\tilde{J}', \bullet, \tilde{\omega}')$ defined by :

$$\begin{cases} \Phi(e) = \gamma^{-1} e' + x_0 + \beta e'^*, \text{ where } \beta \in \mathbb{K}, \\ \Phi(e^*) = \gamma e'^*, \\ \Phi(x) = \Phi_0(x) + \gamma\omega(\Phi_0(x), x_0) e'^*, \quad \forall x \in J, \end{cases}$$

Theorem

Let $(\tilde{J}, \star, \tilde{\omega})$ be a symplectic Jacobi-Jordan algebra such that $\tilde{J} \neq \{0\}$. Then, $(\tilde{J}, \star, \tilde{\omega})$ is a symplectic double extension of a symplectic Jacobi-Jordan algebra $(J, ., \omega)$.

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Corollary

Every symplectic Jacobi-Jordan algebra $(J \neq \{0\}, \omega)$ is obtained from the algebra $\{0\}$ by a finite sequence of symplectic double extensions of symplectic Jacobi-Jordan algebras.

Definition

Let $(J, .)$ be a Jacobi-Jordan algebra. A bilinear form B on J is said to be an associative scalar product on $(J, .)$ if B a nondegenerate, symmetric and associative (or invariant) bilinear. B is associative (or invariant) means $B(x.y, z) = B(x, y.z), \forall\, x, y, z \in J$.
A Jacobi-Jordan algebra $(J, .)$ is said to be pseudo-Euclidean algebra if it is endowed with an invariant scalar product.

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A Jacobi-Jordan algebra $(J, .)$ is said to be pseudo-Euclidean algebra if it is endowed with an invariant scalar product.

Example : Let $J_4 := \text{Span}\{e_1, e_2, e_3, e_4\}$ be the 4-dimentional Jacobi-Jordan algbera defined by : $e_1.e_1 = e_2$, $e_1.e_3 = e_3.e_1 = e_4$. The symmetric bilinear form defined on J by :
$B(e_1, e_4) = B(e_2, e_3) = 1$, is an invariant scalar product. Moreover, we have shown that J_4 has a symplectic structure

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The symmetric bilinear form defined on J by :
$B(e_1, e_4) = B(e_2, e_3) = 1$, is an invariant scalar product. Moreover, we have shown that J_4 has a symplectic structure
Remark : J_4 is the unique $4-$dimentional Jacobi-Jordan which is simultaneously symplectic and pseudo-Euclean.

Not symplectic nor pseudo-Euclidean Jacobi-Jordan algebra :

Example : Let $J := \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ be the 4-dimensional Jacobi-Jordan algbera defined by : $e_1.e_1 = e_2, \quad e_3.e_3 = e_2$. For any symmetric invariant bilinear form B and for any skew-symmetric bilinear form ω which satisfies $\omega(x.y, z) + \omega(y.z, x) + \omega(z.x, y) = 0, \forall\, x, y, z \in J$, we have :

$$B(e_2, e_1) = B(e_2, e_2) = B(e_2, e_3) = B(e_2, e_4) = 0$$

and

$$\omega(e_2, e_1) = \omega(e_2, e_2) = \omega(e_2, e_3) = \omega(e_2, e_4) = 0.$$

So, there is neither pseudo-Euclidean structure nor symplectic structure on $(J, .)$.

Pseudo-Euclidean but not symplectic Jacobi-Jordan algebra :

Example :
We have shown that the 4-dimensional Jacobi-Jordan algbera $J := \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ with the product defined defined by : $e_1.e_1 = e_2$ have no symplectic structure on L. In fact, for any skew-symmetric bilinear form ω on J satisfying

$$\omega(x.y, z) + \omega(y.z, x) + \omega(z.x, y) = 0, \ \ \forall x, y, z \in J$$

we have

$$\omega(e_2, e_1) = \omega(e_2, e_2) = \omega(e_2, e_3) = \omega(e_2, e_4) = 0.$$

But the bilinear form B defined by : $B(e_1, e_4) = B(e_2, e_3) = 1$ is a pseudo-Euclidean structure on this Jacobi-Jordan algebra $(J, .)$.

Theorem

Let $(J, ., B)$ be a pseudo-euclidean Jacobi-Jordan algebra, $(D, A_0) \in adm(J)$ such that $D^2 = 0$ and D is symmetric with respect to B, $A_0 \in \mathrm{Ann}(J)$ and $B(A_0, A_0) = 0$.
If $\mathbb{K}e$ is a one-dimensional linear space and $\mathbb{K}e^*$ its dual Linear space, then the vector space $\tilde{J} := \mathbb{K}e \oplus J \oplus \mathbb{K}e^*$ endowed with the product $\diamond$ define by :

$$\begin{cases} e \diamond e := A_0 + \lambda e^*, \quad \text{where}\, \lambda \in \mathbb{K}, \\ e \diamond x = x \diamond e := D(x) + B(A_0, x)e^*, \quad \forall x \in J, \\ x \diamond y := x.y + B(D(x), y)e^*, \quad \forall x, y \in J, \\ e^* \diamond x = x \diamond e^* := 0, \quad \forall x \in J, \end{cases}$$

is a Jacobi-Jordan algebra.

Moreover, the symmetric bilinear form $\tilde{B}$ defined by :

$$\tilde{B}_{|_{J\times J}} = B,\ \tilde{B}(e,e^*) = 1,\ \tilde{B}(e,J) = \tilde{B}(e^*,J) = \{0\},$$

$$\tilde{B}(e,e) = \tilde{B}(e^*,e^*) = 0,$$

is an associative scalar product on $(\tilde{J},\diamond,\tilde{B})$.

The pseudo-euclidean Jacobi-Jordan algebra $(\tilde{J},\diamond,\tilde{B})$ is called a double extension of $(J,.,B)$ by means of (D,A_0,λ).

Definition

An isometry between two pseudo-euclidean Jacobi-Jordan algebras $(\tilde{J}, \diamond, \tilde{B})$ and $(\tilde{J}', \triangleright, \tilde{B}')$ in an isomorphism $\Phi : (\tilde{J}, \diamond) \to (\tilde{J}', \triangleright)$ of Jacobi-Jordan algebras such that
$\tilde{B}'(\Phi(X), \Phi(Y)) = \tilde{B}(X, Y), \ \forall X, Y \in \tilde{J}$.

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Theorem

Two double extensions $(\tilde{J}, \diamond, \tilde{B})$ and $(\tilde{J}', \triangleright, \tilde{B}')$ of $(J, ., B)$ are isometric if and only if there exists an isometry Φ_0 of $(J, ., B)$, $p_0 \in J$ and $t \in \mathbb{K} \backslash \{0\}$ satisfying

$$\begin{cases} D' = t(\Phi_0 \circ D \circ \Phi_0^{-1} - L_{p_0}), \\ p_0 . p_0 = 0, \\ D'(p_0) = t(\Phi_0 \circ D \circ \Phi_0^{-1}(p_0). \end{cases}$$

Let $(J, ., \omega, B)$ be a symplectic pseudo-euclidean Jacobi-Jordan algebra. Let (D, A_0) be a special admissible such that D is symmetric with respect to B, $A_0 \in \mathrm{Ann}(J)$ and $B(A_0, A_0) = 0$. Let δ be the invertible derivation of $(J, .)$ such that $\omega(x, y) = B(\delta(x), y), \forall\, x, y \in J$. Assume that there exists $\mu \in \mathbb{K} \backslash \{0\}$ and $d_0 \in J$ satisfying

$$[D, \delta] = \ \mu D - L_{d_0}, \ D(d_0) = \frac{1}{2}\delta(A_0) + \mu A_0.$$

The symplectic double extension $(\tilde{J}, \star, \tilde{\omega})$ of $(J, ., \omega)$ by means of (D, A_0) is endowed with the pseudo-Euclidean structure $\tilde{B}$, (where $(\tilde{J}, \diamond := \star, \tilde{B})$ is the double extension of $(J, ., B)$ by means $(D, A_0, 0)$).

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The symplectic double extension $(\tilde{J}, \star, \tilde{\omega})$ of $(J, ., \omega)$ by means of (D, A_0) is endowed with the pseudo-Euclidean structure $\tilde{B}$, (where $(\tilde{J}, \diamond := \star, \tilde{B})$ is the double extension of $(J, ., B)$ by means $(D, A_0, 0)$). In addition, the derivation Δ of $(\tilde{J}, \star)$ which satisfies $\tilde{\omega}(X, Y) = \tilde{B}(\Delta(X), Y)$. is defined by :

$$\Delta(e^*) = \mu e^*, \ \ \Delta(e) = -\mu e + d_0 + \nu e^*, \ \ \Delta(x) = \delta(x) + \omega(x, d_0)e^*,$$

Let $(J, ., \omega, B)$ be a symplectic pseudo-euclidean Jacobi-Jordan algebra , $(\tilde{J} := \mathbb{K}e \oplus J \oplus \mathbb{K}e^*, \star, \tilde{\omega}, \tilde{B})$ (resp. $(\tilde{J}' = \mathbb{K}e' \oplus J \oplus \mathbb{K}e'^*, \bullet, \tilde{\omega}'\tilde{B}'))$ the double extension of $(J, ., \omega, B)$ by means of (D, A_0) (resp. $(D', A_0'))$. Let δ, Δ, Δ' be the invertible derivations satisfying
$\omega = B(\delta(.), .)$, $\tilde{\omega} := \tilde{B}(\Delta(.), .)$, $\tilde{\omega}' := \tilde{B}'(\Delta'(.), .)$.
A symplecto-isometry from $(\tilde{J}, \star, \tilde{\omega}, \tilde{B})$ to $(\tilde{J}', \bullet, \tilde{\omega}'\tilde{B}')$ is a Jacobi-Jordan isomorphism $\Phi : (\tilde{J}, \star) \to (\tilde{J}', \bullet)$ such that

$$\tilde{\omega}'(\Phi(X), \Phi(Y)) = \tilde{\omega}(X, Y) \text{ and } \tilde{B}'(\Phi(X), \Phi(Y)) = \tilde{B}(X, Y).$$

Let $(\tilde{J} := \mathbb{K}e \oplus J \oplus \mathbb{K}e^*, \star, \tilde{\omega}, \tilde{B})$ and $(\tilde{J}' = \mathbb{K}e' \oplus J \oplus \mathbb{K}e'^*, \bullet, \tilde{\omega}'\tilde{B}')$ be two sympletomorphic double extensions of J.
Let Δ and Δ' be the invertible derivations relating respcectively the two structure of the double extensions $\tilde{J}$ and $\tilde{J}'$ which are defined respectively by :

$$\Delta(e^*) = \mu e^*, \Delta(e) = -\mu e + d_0 + \nu e^*, \Delta(x) = \delta(x) + \omega(x, d_0)e^*,$$

$$\Delta'(e'^*) = \mu' e'^*, \Delta'(e') = -\mu' e' + d_0' + \nu' e'^*, \Delta'(x) = \delta(x) + \omega(x, d_0')e'^*.$$

The two double extension above are symplecto-isometric if and only if

$$\begin{cases} (i)\, \mu = \mu', \\ (ii)\, \delta \circ \Phi_0 = \Phi_0 \circ \delta, \\ (iii)\, \Phi_0(d_0) - \gamma^{-1} d_0' = \delta(x_0) + \mu x_0, \\ (iv)\, \omega(\Phi_0(d_0) - x_0, d_0' - \gamma x_0) = \gamma b - \gamma^{-1} b' - \beta(\mu + \mu'). \end{cases}$$

Theorem

Let $(\tilde{J} \neq \{0\}, \star, \tilde{\omega}, \tilde{B})$ be a symplectic pseudo-euclidean Jacobi-Jordan algebra and let Δ be the invertible derivation of $(\tilde{J}, \star)$ such that $\tilde{\omega}(X, Y) = \tilde{B}(\Delta(X), Y), \ \forall X, Y \in \tilde{J}$. If either Δ admits an eigenvector in $\mathrm{Ann}(\tilde{J})$ or $\mathbb{K}$ is algebraically closed, then $(\tilde{J}, \star, \tilde{\omega}, \tilde{B})$ is a double extension of a symplectic pseudo-euclidean Jacobi-Jordan algebra $(J, ., \omega, B)$.

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Corollary

If $\mathbb{K}$ is algebraically closed, then every non-zero symplectic pseudo-euclidean Jacobi-Jordan algebra $(J \neq \{0\}, B)$ is obtained from the algebra $\{0\}$ by a finite sequence of double extensions of a symplectic pseudo-euclidean Jacobi-Jordan algebra.

Thank you for your attention