Expecting the unexpected

Workshop on Lefschetz Properties in Algebra, Geometry and Combinatorics II

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Main goal: describe some results from

- Giuseppe Favacchio, Elena Guardo, Brian Harbourne, JM (in progress) [FGHM]

And also (briefly)

- Brian Harbourne, JM, Uwe Nagel, Zach Teitler, to appear in Michigan J. Math. [HMNT].

- David Cook II, Brian Harbourne, JM, Uwe Nagel, Compositio Math. 2018. [CHMN]

- Brian Harbourne, JM, Halszka Tutaj-Gasińska, preprint 2019. [HMT]

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If $h^1(\mathcal{I}_X(t)) = 0$ (e.g. if X is ACM of dimension ≥ 1), it is equivalent to have

$$h^0(\mathcal{I}_{X \cup A_m}(t)) \cdot h^1(\mathcal{I}_{X \cup A_m}(t)) \neq 0.$$

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Let $L_1, \ldots, L_r$ be distinct linear forms in $[R]_1$, and let X be the set of points in $\mathbb{P}^n$ dual to the union of the hyperplanes defined by the L_i.

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Fix any integers $t \geq m > 1$. Then the following are equivalent:

(a) *X admits an unexpected hypersurface of degree t with a general point P of multiplicity m;*

(b) *$R/(L_1^t, \ldots, L_r^t)$ fails SLP in degree $i = m - 1$ with range $k = t - m + 1$.*

Combined with a different result in [HMNT] we have:

Corollary (HMNT)

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There exists an ideal $I = (L_1^t, \ldots, L_e^t)$ (for suitable e) for which

$$\times L^{t-m+1} : [R/I]_{m-1} \to [R/I]_t$$

fails to have maximal rank iff one of the following holds.

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(i) *We have $n = 2$ and $t > m > 2$.*

(ii) *We have $n \geq 3$ and $t \geq m \geq 2$.*

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In particular, when $t = m$ this algebra fails WLP.

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(We'll have other connections to WLP later in this talk.)

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That is, what situations guarantee that the fat point will fail to impose the “expected” number of conditions?

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Main Question: When can you **expect the unexpected**?

That is, what situations guarantee that the fat point will fail to impose the “expected” number of conditions?

This was the motivating question for [FGHM], but initial results had been obtained earlier.

For example, here's an older result:

Theorem (HMNT)

Assume $X \subset \mathbb{P}^n$ *is a*

- *reduced,*
- *equidimensional,*
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subvariety. Let P be a general point. Then the cone over X with vertex P is an unexpected hypersurface for X of degree $t = d$ and multiplicity $m = d$.

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(ii) Now start taking points of Z away. With a little care we can even assume that the points impose independent conditions on hypersurfaces of degree d, still keeping $[I_X]_d = [I_Z]_d$.

(iii) In some situations we can even have $[I_X]_d \subsetneq [I_Z]_d$ but the cone over X is still unexpected for Z.

(iv) If $X \subset \mathbb{P}^n$ has codimension ≥ 3, the cones with vertex a general linear space of dimension

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But now the number of conditions you expect to be imposed depends on the Hilbert polynomial of a fat linear space.

Repeat: if X is a non-degenerate codimension 2 variety of degree d, then that alone is enough to guarantee that at least for degree $t = d$ and $m = d$, X admits an unexpected hypersurface.

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Question 2

Fix t, m. Suppose X is a finite set of points. What properties of X force the conclusion that X admits an unexpected hypersurface of degree t and multiplicity m?

Specifically, what is the geometry of X that forces this behavior?

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Question 3

In general, are there conditions that force X not to have any unexpected hypersurfaces?

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Are there more general conditions on the Hilbert function that force X not to admit any unexpected hypersurfaces?

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Question 4

Are there more general conditions on the Hilbert function that force X not to admit any unexpected hypersurfaces?

Or (much more interestingly) are there conditions on the Hilbert function that force X to admit unexpected hypersurfaces of some sort?

Analogy

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This motivates Question 4.

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But there are such Hilbert functions, if you bring in some geometric assumptions.

Idea: Hilbert functions of finite sets of points can force relevant curves to exist (example coming). If we know the curve is not degenerate then we can say something.

We illustrate with an example from the paper.

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- This curve has to be a twisted cubic, C, by LGP.
- $\Longrightarrow$ exactly 5 points of X lie off C, and they're in LGP . Call this set X_1.
- With vertex a general point P, the cone over C plus a quadric cone containing X_1 gives a surface with $t = m = 5$.
- $\dim[I_X]_5 = 35 = \binom{m-1+3}{3}$ so this surface is unexpected.

Conclusion: by making a geometric assumption (LGP), we are able to exhibit Hilbert functions that force the existence of an unexpected hypersurface.

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The relevance of non-degeneracy will be explained shortly.

The *AV*-sequences

Notation

Let $t \geq m$ be positive integers and let $P \in \mathbb{P}^n$ be a general point. Let $X \subset \mathbb{P}^n$ be a subvariety.

$$\mathrm{adim}(X, t, m) = \dim[I_X \cap I_P^m]_t \quad \text{(actual dimension)}$$

$$\mathrm{vdim}(X, t, m) = \dim[I_X]_t - \binom{m+n-1}{n} \quad \text{(virtual dimension)}$$

$$\mathrm{edim}(X, t, m) = \max\{0, \mathrm{vdim}(X, t, m)\} \quad \text{(expected dimension)}$$

Remark

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So if $\operatorname{adim}(X,t,m) = \operatorname{vdim}(X,t,m)$, then there is no unexpected hypersurface of degree t and multiplicity m at the general point P.

Definition

Let $X \subset \mathbb{P}^n$ be a subvariety. Let $j \geq 0$ be a non-negative integer. Then we define the AV-sequence

$$AV_{X,j}(m) = \text{adim}(X, m+j, m) - \text{vdim}(X, m+j, m) \quad \text{for} \quad m \geq 1.$$

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It turns out that there are often nice formulas for the AV sequences.

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Lemma

Let $X \subset \mathbb{P}^n$ be a subvariety. Fix any $j \geq 0$ and $m \geq 1$. Then

$$\begin{aligned} AV_{X,j}(m) &= \dim\left[R/(I_X + I_P^m)\right]_{m+j} \\ &= h^1(\mathcal{I}_{X \cup A_m}(m+j)). \end{aligned}$$

Recall Notation

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These are both useful, but the problem is that the ideals themselves ($I_X + I_P^m$ and $\mathcal{I}_{X \cup A_m}$) change with m.

These are formulas for the individual terms of the AV-sequence, not for the whole sequence.

Theorem

Fix X and any integer $j \geq 0$.

Set $J = gin(I_X) : x_0^{j+1}$ (using the lexicographic order and $x_0 > x_1 > \dots$).

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This gives a huge restriction on what sequences can be *AV* sequences.

Here's an important consequence of this theorem:

Corollary

Let $X \subset \mathbb{P}^n$ be any degenerate variety of any dimension. Then X admits no unexpected hypersurfaces of any kind.

More precisely, $AV_{X,j}(m) = 0$ for all $m \geq 0$ and $j \geq 0$.

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&\Longrightarrow \quad AV_{X,j}(m) = 0 \text{ for all } m \text{ and } j. \quad \square
\end{aligned}
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Remark

Question 4 asked (in particular) if there is any condition on the Hilbert function that guarantees that X will not admit any unexpected hypersurfaces.

Degeneracy can be read off from the Hilbert function, so that part of Question 4 has an affirmative answer.

A nice formula for curves in $\mathbb{P}^3$ when $j = 0$:

Theorem

Let $X \subset \mathbb{P}^3$ be a reduced, equidimensional curve of degree e and arithmetic genus g. Then

$$AV_{X,0}(t) = \binom{e-1}{2} - g \quad \text{for all } t \geq e.$$

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Now a bit more about curves in $\mathbb{P}^3$.

$AV_{X,1}$-sequences for smooth ACM curves in $\mathbb{P}^3$

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Definition

A Stanley-Iarrobino (SI)-sequence is a finite sequence of positive integers of the form

$$\underline{h} = (1, h_1, h_2, \dots, h_{e-2}, h_{e-1}, h_e)$$

satisfying

1. *$\underline{h}$ is an O-sequence (i.e. obeys Macaulay's bound).*
2. *$\underline{h}$ is symmetric.*
3. *The first half of $\underline{h}$ is differentiable.*

E.g. $(1, 3, 6, 7, 8, 7, 6, 3, 1)$.

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Fact: (Harima) *The SI-sequences are precisely the Hilbert functions of artinian Gorenstein algebras with the WLP.*

Conjecture

Let $X \subset \mathbb{P}^3$ be a smooth ACM curve not lying on a quadric surface. Then the sequence $AV_{X,1}$ is an SI-sequence!!

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Let $X \subset \mathbb{P}^3$ be a smooth ACM curve not lying on a quadric surface. Then the sequence $AV_{X,1}$ is an SI-sequence!!

Remarks

1. The conjecture is supported by a huge amount of computer evidence.

2. We have not been able to find any artinian Gorenstein algebra, with or without WLP, whose Hilbert function is $AV_{X,1}$ for a given smooth curve X in $\mathbb{P}^3$.

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3. All the assumptions (curve in $\mathbb{P}^3$, ACM, not on a quadric, smooth) are needed. We have counterexamples otherwise.

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6. For X a smooth ACM surface in $\mathbb{P}^4$, the first difference of the $AV_{X,1}$-sequence seems to be an SI-sequence.

   Specifically, it's the SI-sequence you'd get from a general hyperplane section $X \cap H$ in $H = \mathbb{P}^3$.

   So there seems to be a lot of geometry floating around.

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What's mostly missing is the symmetry (and length).

Thank you.