Developable cubics in $\mathbb{P}^4$ and the Lefschetz locus in GOR(1, 5, 5, 1)

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Developable cubics in $\mathbb{P}^4$

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $\mathbb{K} = \mathbb{C}$ if you want. An irreducible projective variety $X \subset \mathbb{P}^N$ over $\mathbb{K}$ is called developable if it has degenerate Gauss map. It means that $X = \overline{\cup L_\alpha}$ is covered by linear spaces where the tangent space is constant. Cones are trivial examples.
An hypersurface $X = V(f) \subset \mathbb{P}^N$ is developable iff $\text{hess}_f(x) = 0$ for all $x \in X$.

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Developable three-fold were studied by several authors, a complete classification up to isomorphism, using tools of algebraic geometry, was done by Mezzetti and Tommasi. There are 7 classes. Hypersurfaces of $\mathbb{P}^4$ is a particular case. We restrict ourselves to cubic hypersurfaces $X = V(f) \subset \mathbb{P}^4$ to find a classification up to projective transformations. Not all possible developable hypersurfaces can occur in degree 3.

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There are three classes, up to projective transformations, of classical examples of developable cubic hypersurfaces $X = V(f) \subset \mathbb{P}^4$ that are not cones. They correspond to hyperplane sections of the secant variety of the Veronese surface of $\mathbb{P}^5$.

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Developable cubics in $\mathbb{P}^4$

Let $\mathcal{V} = \mathcal{V}_2(\mathbb{P}^2) \subset \mathbb{P}^5$ be the Veronese surface, the image of the Veronese embedding $\mathcal{V}_2 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

Let V be a $\mathbb{K}$-vector space of dimension three, $\mathcal{V}_2$ is given by sending a linear form $[\ell] \in \mathbb{P}^2 = \mathbb{P}(V^*)$ to its square $\mathcal{V}_2([\ell]) = [\ell^2] \in \mathbb{P}(S^2V^*) = \mathbb{P}^5$.

It is well known that $\mathcal{V}^* = S\mathcal{V} \subset \mathbb{P}^5$, the dual of the Veronese surface, is projectively equivalent to its secant variety.

It is a cubic hypersurface $S = S\mathcal{V} = V(g) \subset \mathbb{P}^5$.

The fibers of the Gauss map $\mathcal{G} : S \to \mathcal{V}$ are planes, therefore, every hyperplane section of S, $S \cap H \subset H = \mathbb{P}^4$, is a developable cubic in $\mathbb{P}^4 = H$.

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Developable cubics in $\mathbb{P}^4$

The pullback of $H \subset \mathbb{P}^5$ via $\mathcal{V}_2 : \mathbb{P}^2 \to \mathbb{P}^5$, $C = \mathcal{V}_2^*(H) \subset \mathbb{P}^2$ is a conic.

1. If C is a smooth conic, then $H^\perp$ corresponds to a general point. In this case $\mathcal{V}_2(C) = H \cap \mathcal{V} = C_4 \subset \mathbb{P}^4$ is a rational normal quartic and $X = S(C_4)$.
2. If $C = L_1 \cup L_2$ is a pair of lines, then $H^\perp$ corresponds to a point $P \in S$. in this case $\mathcal{V}_2(C) = H \cap \mathcal{V} = C_1 \cup C_2 \subset \mathbb{P}^4$ is the union of two lines sharing a point.
3. If $C = 2L$ is a double line, then $H^\perp$ corresponds to a point $P \in S$. In this case $\mathcal{V}_2(C) = H \cap \mathcal{V} = 2C'$ is a double conic. In this case X is a tangent band over C' and $\text{hess}_f = 0$. It is a geometric reinterpretation of Perazzo's and Gordan-Noether's work.

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Developable cubics in $\mathbb{P}^4$

Theorem (Ferrer, Fassarella, -)

Let $X \subset \mathbb{P}^4$ be an irreducible cubic hypersurface. Assume that X is not a cone. Then X is developable if and only if X is protectively equivalent to a hyperplane section of the secant variety of the Veronese surface $\mathcal{V} \subset \mathbb{P}^5$.

The Lefschetz locus in GOR(1, 5, 5, 1)

The scheme Gor(T) parametrizing AG algebras with Hilbert vector T was described by Iarrobino and Kanev using catalecticants. Diesel, and Boij also studied this parameter spaces.

In our very particular case, by Macaulay-Matlis duality, the scheme GOR$(1, N+1, N+1, 1)$ can be identified with the parameter space of degree 3 homogeneous polynomials $f \in \mathbb{K}[x_0, \ldots, x_N]$, up to scalars, such that A_f has Hilbert vector Hilb$(A_f) = (1, N+1, N+1, 1)$.

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There is an identification

$$\mathrm{GOR}(1, N+1, N+1, 1) \simeq \mathbb{P}^{\nu(N)} \setminus \mathcal{C}_N$$

where $\nu(N) = \binom{N+3}{3} - 1$ and $\mathcal{C}_N$ is the parameter space of cubic cones in $\mathbb{P}^N$. In particular

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By Maeno-Watanabe criterion, an AG algebra A of socle degree 3 has the SLP if and only if its dual generator f satisfies $\mathrm{hess}_f \neq 0$. Up to a projective transformation there is only one such cubic in 5 variables.

$$f = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2 \in \mathbb{K}[x, y, z, u, v]_3.$$

Proposition

Let A be a standard graded Artinian Gorenstein $\mathbb{K}$-algebra of codimension 5 and socle degree 3. Assume that A does not satisfy the SLP. Then A is isomorphic to the following algebra

$$\frac{\mathbb{K}[X_0, X_1, X_2, X_3, X_4]}{\langle (X_0, X_1, X_2)^2, X_0X_4, X_2X_3, X_1X_3 - X_2X_4, X_0X_3 - X_1X_4, (X_3, X_4)^3 \rangle}.$$

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The Lefschetz locus in GOR(1, 5, 5, 1)

Proposition

Let A be a standard graded Artinian Gorenstein $\mathbb{K}$-algebra of Hilbert vector $\mathrm{Hilb}(A) = (1, N+1, N+1, 1)$. *If $N \leq 3$, then A has the SLP. If $N = 4$, then the possible Jordan types of A are: either $\mathfrak{J}_A = 4^1 \oplus 2^4$ if A has the SLP or $\mathfrak{J}_A = 4^1 \oplus 2^3 \oplus 1^2$ if it fails SLP.*

The Lefschetz locus in GOR$(1,5,5,1)$

Theorem (Ferrer, Fassarella, -)

The space GOR$(1,5,5,1)$ *parametrizing AG algebras with Hilbert vector* $(1,5,5,1)$ *coincides with* $\mathbb{P}^{34} \setminus \mathcal{C}_4$, *where* $\mathcal{C}_4$ *is the space of cubic cones in* $\mathbb{P}^4$. *The locus* $\mathcal{C}_4$ *is a projective variety of dimension* 23 *and degree* 1365, *birational to a projective bundle over* $\mathbb{P}^4$. *Moreover, the following assertions hold true.*

1. *The locus of algebras failing SLP coincides with* $\mathcal{J}^2 \setminus \mathcal{C}_4$, *where* $\mathcal{J}^2$ *is a projective variety of dimension* 18 *and degree* 29960 *which is birational to a projective bundle over the Grassmannian* $\mathbb{G}(2,4)$.
2. *The intersection* $\mathcal{J}^2 \cap \mathcal{C}_4$ *is a divisor in* $\mathcal{J}^2$ *of degree* 116420.