Hessians, Higher Jacobians and Lefschetz Properties

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Preliminaries

Joint work A. Dimca amd R. Gondim, in press on Collectanea Mathematica.

- Let $\mathbb{K}$ be an algebraically closed field of characteristic zero;
- Let $R = \mathbb{K}[x_0, \ldots, x_n]$ be the polynomial ring with usual graduation;
- Let $I \subset R$ be an Artinian homogeneous ideal such that $I_1 = 0$.

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- Let $\mathbb{K}$ be an algebraically closed field of characteristic zero;
- Let $R = \mathbb{K}[x_0, \ldots, x_n]$ be the polynomial ring with usual graduation;
- Let $I \subset R$ be an Artinian homogeneous ideal such that $I_1 = 0$.

Then the Artinian graded $\mathbb{K}$-algebra $A = R/I = \bigoplus_{i=0}^{d} A_i$ is standard, i.e. generated in degree 1 as an algebra.

Definition

A standard graded Artinian algebra A as above is Gorenstein if and only if $h_d(A) = 1$ and the restriction of the multiplication of the algebra in complementary degree, that is, $A_k \times A_{d-k} \to A_d$ is a perfect paring for $k = 0, 1, \ldots, d$ (see [MW]). If $A_j = 0$ for $j > d$, then d is called the *socle degree* of A.

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The Hilbert vector of a graded Artinian Gorenstein $\mathbb{K}$-algebra A is symmetric, but it is not always unimodal, (see [BI]).

Example 1

The Hilbert vector in codimension 5:

$$\begin{pmatrix} 1 & 5 & 12 & 22 & 35 & 51 & 70 & 91 & 90 & 91 & 70 & 51 & 35 & 22 & 12 & 5 & 1 \end{pmatrix}.$$

- Let $Q = \mathbb{K}[X_0, \ldots, X_n]$ be the ring of differential operators of R, where $X_i := \frac{\partial}{\partial x_i}$ for $i = 0, \ldots, n$. We denote by $Q_k = Q[X_0, \ldots, X_n]_k$ the $\mathbb{K}-$vector space of homogeneous differential operators of R of degree k;
- let $f \in R$ be a homogeneous polynomial of degree $\deg f = d \geq 1$;
- for each integer k, with $d \geq k \geq 0$ there exist natural $\mathbb{K}$-bilinear maps $R_d \times Q_k \to R_{d-k}$ defined by differentiation:

$$(f, \alpha) \to f_\alpha := \alpha(f).$$

We define the *annihilator of f*

$$\mathrm{Ann}(f) := (\alpha \in Q | \alpha(f) = 0) \subset Q.$$

Remark

$$\text{Ann}(f)_1 \neq 0 \Leftrightarrow X = V(f) \subset \mathbb{P}^n \text{ is cone.}$$

We assume from now on that $V(f)$ is not a cone, and hence that $\text{Ann}(f)_1 = 0$.

We can define $A(f) = \frac{Q}{\text{Ann}(f)}$, it is a well known fact that it is a standard graded Artinian Gorenstein $\mathbb{K}$-algebra associated to f.

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From now on we denote $A(f)$ by A.

- Let $k \leq l$ be two integers;
- Let $L \in A_1$ be a linear form and $\bullet L^{l-k} : A_k \to A_l$ is the $\mathbb{K}$-vector spaces map;
- Let $\mathcal{B}_k = \{\alpha_1, \ldots, \alpha_r\}$ be a $\mathbb{K}$-linear basis of A_k;
- Let $\mathcal{B}_l = \{\beta_1, \ldots, \beta_s\}$ be a $\mathbb{K}$-linear basis of A_l.

Definition

We call mixed Hessian of f of mixed order (k, l) with respect to the basis $\mathcal{B}_k$ and $\mathcal{B}_l$ the matrix: $\text{Hess}_f^{(k,l)} := [\alpha_i \beta_j(f)]$. Moreover, we define $\text{Hess}_f^k = \text{Hess}_f^{(k,k)}$, $\text{hess}_f^k = \det(\text{Hess}_f^k)$ and $\text{hess}_f = \text{hess}_f^1$.

The following proposition is in [GZ]:

Proposition 1

For a generic L, one has the following:

- *the map* $\bullet L^d : A_0 \to A_d$ *is an isomorphism;*
- *the map* $\bullet L^{d-2} : A_1 \to A_{d-1}$ *is an isomorphism if and only if* $\mathrm{hess}_f \neq 0$.

Let $C : f = 0$ be a reduced curve of degree d in $X = \mathbb{P}^2(\mathbb{C})$ and $AR(f)$ be the graded R-module of Jacobian syzygies of f, i. e.

$$AR(f)_k := \left\{(a, b, c) \in R_k^3 \mid af_x + bf_y + cf_z = 0\right\}.$$

Its sheafification $E_C : \widetilde{AR}(f)$ is a rank two vector bundle on $\mathbb{P}^2$. In particular $E_C = T\langle C\rangle(-1)$, where $T\langle C\rangle$ is the sheaf of logarithmic vector fields along C.

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Definition

A curve C is free if the graded $R-$module $AR(f)$ is free, say with a basis ρ_1, ρ_2. If $\deg \rho_i = d_i$ $(i = 1, 2)$, the multiset of integers (d_1, d_2) is called the exponents of a free curve C. Equivalently we can say that E_C is free when it splits as a direct sum of two line bundles on X.

In general for a line L in V, the pair of integers (d_1^L, d_2^L) with $d_1^L \leq d_2^L$ such that $E_{C|L} \simeq \mathcal{O}_L(-d_1^L) \oplus \mathcal{O}_L(-d_2^L)$ is called the splitting type of E_C along L.

For generic line L_0, the corresponding splitting type $(d_1^{L_0}, d_2^{L_0})$ is constant.

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A line L in $\mathbb{P}^2$ is called a jumping line for E_C or, equivalently, for $T\langle C\rangle$, if $d_1^L < d_1^{L_0}$.

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A line L in $\mathbb{P}^2$ is called a jumping line for E_C or, equivalently, for $T\langle C\rangle$, if $d_1^L < d_1^{L_0}$.

Remark

For a free curve C there are no jumping line for E_C.

When the minimal resolution of $AR(f)$ is more complex, it is possible to talk about **nearly free** curve.

Definition

A curve C is nearly free if the graded R-module $AR(f)$ has a minimal system of generators of syzygies ρ_1, ρ_2 and ρ_3 of degree d_1, d_2 and d_3 respectively, such that $d_1 \leq d_2 = d_3$ and exists a relation $h\rho_1 + \ell_2\rho_2 + \ell_3\rho_3 = 0$, for $h \in R$ and linear independent forms $\ell_2, \ell_3 \in R$. The pair of the integers (d_1, d_2) is called exponents of the nearly free curve C.

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Remark

For a nearly free curve C, the jumping lines, if exist, give a line in the dual projective space $\mathbb{P}(R_1)$.

Let $C : f = 0$ be a reduced curve of degree d in $X = \mathbb{P}^2(\mathbb{C})$ and let $AR(f)$ be the graded R-module of the jacobian syzygies of f. We set $mdr(f) := \min\{k | AR(f)_k \neq 0\}$ the minimal degree of a jacobian syzygy for f and we assume $mdr(f) \geq 1$.

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Definition

The Jacobian module of the reduced curve C is the quotient module $N(f) = \hat{J}_f / J_f$, with J_f Jacobian ideal of f and $\hat{J}_f$ the saturation of the ideal J_f respect the maximal ideal $\mathbf{m} = (x, y, z)$ in R.

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The quotient module $N(f)$ coincides with cohomology local group associated to the Milnor algebra $H^0_{\mathbf{m}}(M(f))$.

A. Dimca and D. Popescu proved that $N(f)$ satisfies a property of type Lefschetz respect to the multiplication for generic linear forms. It implies:

$$0 \leq n(f)_0 \leq n(f)_1 \leq \cdots \leq n(f)_{\lceil \frac{T}{2} \rceil} \geq \cdots \geq n(f)_T \geq 0$$

where $T = 3d - 6$ and $n(f)_k = \dim N(f)_k$ for all k.

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where $T = 3d - 6$ and $n(f)_k = \dim N(f)_k$ for all k.
Setting $\sigma(C) := \min\{j : n(f)_j \neq 0\}$ and $\nu(C) = n(f)_{\lceil \frac{T}{2} \rceil}$, then it follows:

- C free $\Leftrightarrow \nu(C) = 0 \Leftrightarrow n(f)_k = 0 \quad \forall k \Leftrightarrow N(f) = 0 \Leftrightarrow J_f$ saturated ideal.
- C nearly free $\Leftrightarrow N(f) \neq 0$ and $n(f)_k \leq 1$ for all $k \Leftrightarrow \nu(C) = 1$.

Jacobian ideals and Milnor algebras

Let $R = \mathbb{C}[x_0, \ldots, x_n]$ be the polynomial graded ring in $n+1$ variables on $\mathbb{C}$ and $I \subseteq R$ an ideal such that $A = R/I = \oplus_{i=0}^{s} A_i$ is an Artinian graded ring on $\mathbb{C}$.

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It was proved by different authors (H. Brenner, T. Harima, A. Kaid, J.C. Migliore, U. Nagel and J. Watanabe) that a complete intersection in codimension 3 holds the WLP.

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R. Stanley proved that a complete intersection in the monomial case holds the WLP.

- Let $f \in R_d$ be a homogeneous polynomial of degree d such that the hypersurface $V = V(f) : f = 0 \subset \mathbb{P}^n$ is reduced.

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$J(f)$ is the Jacobian ideal of f, generated by the partial derivatives f_i, of f with respect to x_i for $i = 0, \dots, n$

$$J(f) := (f_0, \dots, f_n) = \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)$$

Remark

The Jacobian ideal of a smooth hypersurface is a complete intersection.

If X is nonsingular then $J(f)$ is generated by a regular sequence, and $M(f) := R/J(f)$ is a Gorenstein Artinian algebra.

It is called the graded Milnor (or Jacobian) algebra.

Assume now that $V \subset \mathbb{P}^n$ is singular, but reduced. In this case the Jacobian algebra is not of finite length, in particular it is not Artinian Gorenstein. It contains information on the structure of the singularities and on the global geometry of V.

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About the Milnor algebra we have proved the following propositions in [I].

Proposition 2

Let $V : f = 0$ be a hypersurface in $\mathbb{P}^n$ of degree $d > 2$, such that its singular locus V_s has dimension at most $n-3$. Then $M(f)$ has the WLP in degree $d-2$.

Sketch of proof

Assume: $\forall L \in R_1 \bullet L \colon M(f)_{d-2} \to M(f)_{d-1}$ is not injective.
Hence: $\exists \alpha = (\alpha_0 \ldots, \alpha_n) \in \mathbb{P}^n$ s.t. $P^1_\alpha(V) = \sum \alpha_i \frac{\partial f}{\partial x_i}$ is divisible by L.
Consider $Z = \left\{(L, \alpha) \in \mathbb{P}(R_1) \times \mathbb{P}^n : L \text{ divides } P^1_\alpha(V)\right\}$.

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Consider $Z = \left\{(L, \alpha) \in \mathbb{P}(R_1) \times \mathbb{P}^n : L \text{ divides } P^1_\alpha(V)\right\}$. We have:

- the projection $p_1 \colon Z \to \mathbb{P}(R_1)$ is surjective, hence $\dim Z \geq n$;
- the projection $p_2 \colon Z \to \mathbb{P}^n$ is surjective.

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- the projection $p_1\colon Z \to \mathbb{P}(R_1)$ is surjective, hence $\dim Z \geq n$;
- the projection $p_2\colon Z \to \mathbb{P}^n$ is surjective.

The generic member of a linear system is smooth outside the base locus, by Bertini theorem. The base locus of the linear system $P^1_\alpha(V)$ for $\alpha \in \mathbb{P}^n$ is exactly the singular locus V_s of V. It follows that for general α, the polar hypersurface $P^1_\alpha(V) = 0$ has a singular locus of dimension at most $n-3$. This implies that such a hypersurface $P^1_\alpha(V) = 0$ is irreducible, a contradiction with our assumption.

Proposition 3

Let $V : f = 0$ be a hypersurface in $\mathbb{P}^n$ of degree $d > 2$, such that its singular locus V_s has dimension at most $n - 3$. Then for every positive integer $k < d - 1$ $M(f)$ has the SLP in degree $d - k - 1$ at range k.

Sketch of proof

The proof is the same of Proposition (2), where we consider L^k instead of L.

Finally, more generally we can prove the following:

Theorem 4

Let $V : f = 0$ be a general hypersurface, then $M(f)$ has the SLP.

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We ask:

Question

For f generic we have proved that $M(f)$ has the SLP.

Is it true for any f with $V(f)$ smooth?

We have the following results in relation with this question (see [DGI]).

Proposition 5

Let $V : f = 0$ be a smooth surface in $\mathbb{P}^3$ of degree $d = 3$. Then $M(f)$ has the SLP.

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Scketch of proof

$$M(f) \text{ is AG } \overset{[MW]}{\Longrightarrow} M(f) \simeq Q/\operatorname{Ann}(g)$$

with g homogeneous polynomial. $\deg g =$ socle degree of $M(f) = (n+1)(d-2) = 4$ and $\mathrm{hess}_g \neq 0$, hence, by Proposition (1), $M(f)$ has the SLP.

Proposition 6

Let $V : f = 0$ be a smooth curve in $\mathbb{P}^2$ of even degree $d = 2d'$. Then the multiplication by the square of a generic linear form $\ell \in R_1$ induces an isomorphism

$$\ell^2 : M(f)_{3d'-4} \to M(f)_{3d'-2}.$$

In particular, when $d = 4$, the Milnor algebra $M(f)$ has the SLP.

Scketch of proof

$\ell^2 : M(f)_{3d'-4} \to M(f)_{3d'-2}$ is not an isomophism $\Rightarrow L : \ell = 0 \subset \mathbb{P}^2$ jumping line of the second kind for the rank 2 vector bundle $T\langle C\rangle$ on $\mathbb{P}^2$.

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Then the set of jumping lines of second kind is a curve in the dual projective plane $(\mathbb{P}^2)^\vee$ [H, Theorem 3.2.2]. When $d = 4$, this yields an isomorphism $\ell^2 : M(f)_2 \to M(f)_4$. The other isomorphisms necessary for SLP follow from Proposition (1) as in the proof of Proposition (5) above.

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- *If the hypersurface $V(f)$ is smooth, then* $\text{hess}_f \notin J(f)$.
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Question

Is is true that $\mathrm{hess}_f \in J(f)$ for any reduced, singular hypersurface $V(f)$?

Nothing we can say about the singular hypersurface, so we assume the question as a conjecture:

Conjecture 1

For any reduced, singular hypersurface $V : f = 0$, the hessian hess_f of f belongs to the Jacobian ideal J_f.

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Beyond the above general facts, valid for any graded R-module M of finite type, we have the following:

Conjecture 2

Let $V : f = 0$ be a degree d reduced singular hypersurface in $\mathbb{P}^n$ and let hess_f be the Hessian of the polynomial f. Let $T = \deg \mathrm{hess}_f = (n+1)(d-2)$. Then $N(f)_k = 0$ for $k \geq T$.

An important result is the following

Proposition 8

For any hypersurface $V : f = 0$, in $\mathbb{P}^n$ the hessian hess_f of f belongs to the saturation of the Jacobian ideal J_f.

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For any hypersurface $V : f = 0$, *in* $\mathbb{P}^n$ *the hessian* hess_f *of* f *belongs to the saturation of the Jacobian ideal* J_f.

In particular:

Conjecture (2) holds for $V \subset \mathbb{P}^n$	$\Rightarrow$	Conjecture (1) holds for $V \subset \mathbb{P}^n$.

$\exists \ell \in R_1$ s. t.
$\times \ell \colon M(f)_T \to M(f)_{T+1}$ is injective $\Rightarrow$

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The conjecture (1) holds for:

- any **free hypersurface** in $\mathbb{P}^n$, with $n \geq 2$;

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- any **free hypersurface** in $\mathbb{P}^n$, with $n \geq 2$;
- any **nearly free** surface in $\mathbb{P}^3$;

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The conjecture (1) holds for:

- any **free hypersurface** in $\mathbb{P}^n$, with $n \geq 2$;
- any **nearly free** surface in $\mathbb{P}^3$;
- any **generic hyperplane arrangement** $\mathcal{A}$ in $\mathbb{P}^n$, with $d = |\mathcal{A}| > n \geq 2$.

Higher order Jacobian ideals and Milnor algebras

Let $f \in R$ be an homogeneous polynomial.

Definition

The k-th order Jacobian ideal J^k is the homogeneous ideal generated by the k-th order partial derivatives of f.

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The k-th order Jacobian ideal J^k is the homogeneous ideal generated by the k-th order partial derivatives of f.

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The k-th order Milnor algebra of f is $M^k = M^k(f) = R/J^k$.

Higher order Jacobian ideals and Milnor algebras

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Theorem 9

The k-th order Milnor algebra $M^k(f)$ of a homogeneous polynomial f is Artinian if and only if the multiplicity of the projective hypersurface $V(f)$ (considered with its non-reduced structure if f is not reduced) at any point $p \in V(f)$ is at most k.

The Hilbert vector of $A(f)$, $M(f)$ and $M^2(f)$.

Let $f, f' \in R$ be two homogeneous polynomials and $M(f)$ and $M(f')$ the corresponding Milnor algebras.

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Let $f, f' \in R$ be two homogeneous polynomials and $M(f)$ and $M(f')$ the corresponding Milnor algebras.

$$M(f) \simeq M(f') \Leftrightarrow V(f) \simeq_{proj} V(f').$$

Remark

The claim fails for the second order Milnor algebra $M^2(f)$.

Example 2

The family of plane cubics $f_a = x_0^3 + x_1^3 + x_2^3 - 3ax_0x_1x_2$, where $a \neq 0$, $a^3 \neq 1$ and the Fermat cubic $f = x_0^3 + x_1^3 + x_2^3$ are not projectively equivalent, but $M^2(f_a) = R/(x_0, x_1, x_2)$ and $M^2(f)$ are the same.

We look at quartic curves in $\mathbb{P}^2$, i.e. $(n, d) = (2, 4)$. When $V(f)$ is not a cone, only dimension $h_2(A(f))$ has to be determined. It turns out that all the possible values $\{3, 4, 5, 6\}$ are obtained. We analyze two cases:

1. $V(f)$ smooth;
2. $V(f)$ singular.

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1. $V(f)$ smooth;
2. $V(f)$ singular.

Remark

All the smooth quartics $V(f)$ have the same Hilbert function

$$H(M(f); t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + t^6,$$

but the other invariants may change.

$V(f)$ smooth quartic

The Fermat type polynomial of degree 4, $f_F = x_0^4 + x_1^4 + x_2^4$ has the following Hilbert function:

$$H(A(f_F); t) = 1 + 3t + 3t^2 + 3t^3 + t^4,$$

and the minimal resolution of $A(f_F)$ is given by

$$0 \to Q(-7) \to Q(-3)^2 \oplus Q(-5)^3 \to Q(-2)^3 \oplus Q(-4)^2 \to Q,$$

in particular $A(f_F)$ is not a complete intersection. The second order Milnor algebra is $M^2(f_F) = R/(x_0^2, x_1^2, x_2^2)$, hence a complete intersection, with Hilbert function

$$H(M^2(f_F); t) = 1 + 3t + 3t^2 + t^3.$$

The smooth Caporali quartic is given by $f_C = x_0^4 + x_1^4 + x_2^4 + (x_0 + x_1 + x_2)^4$. We get

$$H(A(f_C); t) = 1 + 3t + 4t^2 + 3t^3 + t^4,$$

and the minimal resolution of $A(f_C)$ is given by

$$0 \to Q(-7) \to Q(-4) \oplus Q(-5)^2 \to Q(-2)^2 \oplus Q(-3) \to Q.$$

Hence $A(f_C)$ is a complete intersection of multi-degree $(2, 2, 3)$. The second order Milnor algebra $M^2(f_C)$ has a Hilbert function given by

$$H(M^2(f_C); t) = 1 + 3t + 2t^2,$$

in particular this algebra is not Gorenstein.

$V(f)$ singular quartic

The cuspidal rational quartic defined by $f_{\mathbf{C}} = x_0^3 x_1 + x_2^4$ satisfies $H(A(f_{\mathbf{C}}); t) = H(A(f_F); t)$ and the minimal resolution for $A_(f_{\mathbf{C}})$ is

$$0 \to Q(-7) \to Q(-3)^2 \oplus Q(-5)^3 \to Q(-2)^3 \oplus Q(-4)^2 \to Q.$$

Hence the algebra $A(f_{\mathbf{C}})$ has the same resolution as a graded R-module as the algebra $A(f_F)$. An isomorphism $A(f_F) \simeq A(f_{\mathbf{C}})$ of $\mathbb{K}$-algebras would imply that the two nets of conics

$$N_F : aX_0X_1 + bX_0X_2 + cX_1X_2 \text{ and } N_{\mathbf{C}} : aX_1^2 + bX_0X_2 + cX_1X_2$$

are equivalent. But

$$N_F \text{ singular } \Leftrightarrow N_F \in \cup_{i=1}^3 \ell_i \quad (\ell_1 : a = 0, \ell_2 : b = 0, \ell_3 : c = 0)$$

while

$$N_{\mathbf{C}} \text{ singular } \Leftrightarrow N_{\mathbf{C}} \in \cup_{i=1}^2 \ell_i \quad (\ell_1 : a = 0, \ell_2 : b = 0).$$

For the associated Milnor algebras, one has

$$H(M(f_{\mathbf{C}}); t) = 1 + 3t + 6t^2 + 7t^3 + 7t^4 + 6\frac{t^5}{1-t},$$

and

$$H(M^2(f_{\mathbf{C}}); t) = 1 + 3t + 3t^2 + 2\frac{t^3}{1-t}.$$

Hence $M^2(f_{\mathbf{C}})$ is not Artinian.

The line arrangement, $f=(x_0^3+x_1^3)x_2$, has an algebra $A(f)$ with exactly the same resolution as a graded R-module as the algebra $A(f_C)$, hence a complete intersection of multi-degree $(2,2,3)$. An isomorphism $A(f_C)\simeq A(f)$ of $\mathbb{K}$-algebras would imply that the two pencils of conics

$$P_C : a(y_0y_1-y_1y_2)+b(y_0y_2-y_1y_2) \text{ and } P_f : ay_0y_1+by_2^2$$

are equivalent. But

$$P_C \text{ singular } \Leftrightarrow P_C\in\cup_{i=1}^3\ell_i \quad (\ell_1 : a=0, \ell_2 : b=0, \ell_3 : a+b=0)$$

while

$$P_f \text{ singular } \Leftrightarrow P_f\in\cup_{i=1}^2\ell_i \quad (\ell_1 : a=0, \ell_2 : b=0).$$

Higher order Jacobians and polar maps

Let

$$0 \to I_k \to Q_k \to J^k_{d-k} \to 0$$

be a exact sequence where $I = \mathrm{Ann}(f)$ and the map $Q_k \to J^k_{d-k}$ is given by evaluation $\alpha \to \alpha(f)$.

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We have also the natural exact sequence

$$0 \to I_k \to Q_k \to A_k \to 0.$$

Remark

For J^k_{d-k}, $\mathbb{K}$-vector space, we have

$$\dim J^k_{d-k} = \dim Q_k - \dim I_k = \binom{n+k}{k} - \dim I_k = \dim A_k.$$

Definition

Let $X = V(f) \subset \mathbb{P}^n$ be an hypersurface, the k-th polar mapping is the rational map $\Phi_X^k : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+k}{k}-1}$ given by the k-th partial derivatives of f.

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Definition

The k-th polar image of X is the closure of the image of the k-th polar map: $\widetilde{Z}_k = \overline{\Phi_X^k(\mathbb{P}^n)}$.

Definition

Let $\{\alpha_1, \ldots, \alpha_{a_k}\}$ be a $\mathbb{K}-$linear basis of A_k, the relative k-th polar map is the rational map $\varphi_X^k : \mathbb{P}^n \dashrightarrow \mathbb{P}^{a_k-1}$ given by the linear system J_{d-k}^k:

$$\varphi_X^k = (\alpha_1(f)(p) : \ldots : \alpha_{a_k}(f)(p)).$$

Definition

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$$\varphi_X^k = (\alpha_1(f)(p) : \ldots : \alpha_{a_k}(f)(p)).$$

Definition

The k-th relative polar image of X is the closure of the image of the relative k-th polar map: $Z_k = \overline{\varphi_X^k(\mathbb{P}^n)} \subseteq \mathbb{P}^{a_k-1}$.

It is possible to prove ([W]) that for $k \leq \frac{d}{2}$ and f generic, $a_k = \binom{n+k}{k}$ and hence $\Phi_X^k = \varphi_X^k$. In general, the exact sequence

$$0 \to I_k \to Q_k \to J_{d-k}^k \to 0$$

gives a linear projection $\mathbb{P}^{\binom{n+k}{k}-1} \to \mathbb{P}^{a_k-1}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{P}^n & \to & \mathbb{P}^{\binom{n+k}{k}-1} \\ & \searrow & \downarrow \\ & & \mathbb{P}^{a_k-1}. \end{array}$$

Moreover since $\widetilde{Z}_k \subset \mathbb{P}(J^k_{d-k})$ the secant variety of $\widetilde{Z}_k$ does not meet the center of projection, hence $Z_k \simeq \widetilde{Z}_k$.

Moreover since $\widetilde{Z}_k \subset \mathbb{P}(J^k_{d-k})$ the secant variety of $\widetilde{Z}_k$ does not meet the center of projection, hence $Z_k \simeq \widetilde{Z}_k$.
The next result is the formalization of the intuitive idea that the mixed hessian $\text{Hess}_f^{(1,k)}$ is the Jacobian matrix of the polar map φ^k of order k.

Theorem 10

With above hypothesis, we have

$$\dim Z_k = \dim \widetilde{Z}_k = \mathrm{rk}(\mathrm{Hess}_X^{(1,k)}) - 1.$$

In particular the following conditions are equivalent:

1. φ^k *is a degenerate map;*
2. $\mathrm{rk}(\mathrm{Hess}_X^{(1,k)}) < n+1$;
3. *the map* $\bullet L^{d-k-1}\colon A_1 \to A_{d-k}$ *has not maximum rank for all* $L \in A_1$.

It follows a partial generalization of Gordan - Nöether Hessian criterion to the case of higher order Hessian:

Corollary 11

Let $k \leq \lfloor \frac{d}{2} \rfloor$ *be the greatest integer such that* $\bullet L\colon A_{k-1} \to A_k$ *is injective for some* $L \in A_1$. *For any* $j \leq k$ *if* φ^j *is degenerate then* $\mathrm{hess}_X^j = 0$.

The converse is not true, as one can see in the following example:

Example 3

Let $f = xu^3 + yu^2v + zuv^2 + v^4 \in \mathbb{K}[x,y,z,u,v]_4$ and $A = Q/\operatorname{Ann}(f)$. The map of multiplication $\bullet L\colon A_1 \to A_2$ is injective for $L = U + V$. For $j = 2$, we have:

$$\operatorname{Hess}_f^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 6 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 24 \end{pmatrix}.$$

We have $\mathrm{hess}_f^2 = \det \mathrm{Hess}_f^2 = 0$. Calculating the $\mathrm{Hess}_f^{(1,2)}$, we get:

$$\mathrm{Hess}_f^{(1,2)} = \begin{pmatrix} 0 & 0 & 0 & 6u & 0 \\ 0 & 0 & 0 & 2v & 2u \\ 0 & 0 & 0 & 0 & 2v \\ 0 & 0 & 0 & 2u & 0 \\ 0 & 0 & 0 & 2v & 2u \\ 0 & 2u & 2v & 2y & 2z \\ 6u & 2v & 0 & 6x & 2y \\ 0 & 0 & 2u & 2z & 24v \end{pmatrix}$$

The rank $\mathrm{rk}(\mathrm{Hess}_f^{(1,2)}) = 5$, hence φ^2 is not degenerate by Theorem (4).

Corollary 12

Let f be an homogeneous polynomial of degree d and let $1 < k < d-1$. Let φ_f^k be the k-th polar map of f. If $\mathrm{hess}_f \neq 0$ then φ_f^k is not degenerate. In particular, if $X = V(f) \subset \mathbb{P}^n$, with $n \leq 3$, is not a cone, then φ_X^k is not degenerate.

Example 4

Let $X = V(f) \subset \mathbb{P}^3$ be Ikeda's surface given by $f = xuv^3 + yu^3v + x^2y^3$ as in [MW]. Since X is not a cone, by Gordan - Nöether criterion, $\text{hess}_f \neq 0$. Furthermore, $\text{hess}_f^2 = 0$. By Corollary (12), we know that the second polar map is not degenerate. Moreover, $\varphi_X^2 = \Phi_X^2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^9$ since $\dim A_2 = 10$.

- Let $\Phi_X^k\colon \mathbb{P}^n \to \mathbb{P}^N$ be the k-th polar map associated to a smooth hypersurface $X = V(f)$, where $N = \binom{n+k}{k} - 1$. The Corollary (12) implies that the image of this map has dimension n for any $1 < k < d - 1$;
- let $\psi_X^k = \Phi_X^k|X\colon X \to \mathbb{P}^N$ be the restriction.

In [D] it is shown that the map ψ_X^k is finite for $1 \leq k < d$. The case $k = 2$ and the general case are stated in [D].

- Let $\Phi_X^k \colon \mathbb{P}^n \to \mathbb{P}^N$ be the k-th polar map associated to a smooth hypersurface $X = V(f)$, where $N = \binom{n+k}{k} - 1$. The Corollary (12) implies that the image of this map has dimension n for any $1 < k < d - 1$;
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In [D] it is shown that the map ψ_X^k is finite for $1 \leq k < d$. The case $k = 2$ and the general case are stated in [D].
We prove that the map ϕ_X^k is finite, which implies $\dim Z_k = n$. We note that this map Φ_X^k is well defined as soon as $M^k(f)$ is Artinian.

Theorem 13

Let $\Phi_X^k : \mathbb{P}^n \to \mathbb{P}^N$ be the k-th polar map associated to an hypersurface X such that $M^k(f)$ is Artinian, where $N = \binom{n+k}{k} - 1$ and $0 < k < d$. Then Φ_X^k is finite. In particular

$$\dim Z_k = \dim \Phi_X^k(\mathbb{P}^n) = n$$

and one has $\deg \Phi_X^k \cdot \deg \Phi_X^k(\mathbb{P}^n) = (d-k)^n$.

Thank you!

Bibliography

D. Bernstein and A. Iarrobino,
A non-unimodal graded Gorenstein Artin algebra in codimension five,
Comm. Algebra 20 (1992), 2323–2336.

A.Dimca,
On the dual and hessian mappings of projective hypersurfaces,
Math. Proc. Cambridge Phil. Soc. 101 (1987) 461–468.

A. Dimca, R. Gondim and G. Ilardi,
Higher order Jacobians, Hessians and Milnor algebras,
in press on Collectanea Mathematica.

A. Dimca and G. Sticlaru,
On the jumping lines of logarithmic vector fields along plane curves,
arXiv: 1804.06349.

Bibliography

A. Dimca and G. Sticlaru,
Plane curves with three syzygies, minimal Tjurina curves andnearly free cuspidal curves,
arXiv:1810.11766.

R. Gondim and G. Zappalà,
On mixed Hessians and the Lefschetz properties,
J. of Pure and Appl. Algebra 223(10), 4268–4282.

S. Hassanzadeh and A. Simis.
Plane Cremona maps: saturation and regularity of the base ideal,
J. Algebra 371 (2012), 620–652.

K. Hulek,
Stable rank 2 vector bundles on $\mathbb{P}^2$ with c_1 odd,
Math. Ann. 242 (1979), 241–266.

Bibliography

G. Ilardi,
Jacobian ideals, arrangements and the Lefschetz properties,
J. Algebra 508 (2018), 418–430.

T. Maeno and J. Watanabe,
Lefschetz elements of artinian Gorenstein algebras and Hessians of homogeneous polynomials,
Illinois J. Math. 53 (2009), 593–603.

Z. Wang,
On homogeneous polynomials determined by their higher jacobians,
arXiv: 1809.02422.