Circulant matrices and Galois-Togliatti systems

Pietro De Poi

Dipartimento di Scienze Matematiche, Informatiche e Fisiche
Università degli Studi di Udine

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Outline

1. Circulant Matrices
   - Basic Definitions
   - Some results
2. Our Results
   - Galois-Togliatti systems
   - Our results

Collaboration with Emilia Mezzetti, Rosa Maria Miró-Roig, Mateusz Michałek and Eran Nevo.

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Notations and assumptions

- $R := \mathbb{C}[x_0, \ldots, x_n] = \oplus_i R_i$
- $P \in R$,
$$P = \sum_{i_0+\cdots+i_n} [P]_{i_0,\ldots,i_n} x_0^{i_0} \cdots x_n^{i_n}$$
- $d \in \mathbb{N}$, $d \geq 3$
- ϵ a primitive d-th root of 1
- S_d the symmetric group

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Circulant matrix: definition

Definition

A $d \times d$ *circulant matrix* is a square matrix of the form

$$\mathrm{Circ}(x_0, \dots, x_{d-1}) := \begin{pmatrix} x_0 & x_1 & \cdots & x_{d-2} & x_{d-1} \\ x_{d-1} & x_0 & \cdots & x_{d-1} & x_{d-2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_1 & x_2 & \cdots & x_{d-1} & x_0 \end{pmatrix} \quad (1)$$

where $x_0, \dots, x_{d-1} \in \mathbb{C}$, or more generally elements of a ring.

Successive rows are circular permutations of the first row $x_0, \dots, x_{d-1}$.

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Why to study Circ$(x_0, \ldots, x_{d-1})$?
First part

Remark

Circulant matrices are a particular form of *Toeplitz matrices*, i.e. matrices whose elements are constant along the diagonals.

As observed by I. Kra and S. Simanca in [*On circulant matrices*, Notices AMS, 2012]:

- Circulant matrices are prevalent in many parts of mathematics.

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First part: applied mathematics

For instance

- in applications involving the discrete Fourier transform
- the study of cyclic codes for error correction.

More generally, in the last decades circulant matrices have been related to various fields of applied mathematics such as

- cryptography
- coding theory
- digital signal processing
- image compression
- physics
- engineering simulations
- theory of statistical designs...

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First part: (pure) mathematics

But there are interesting connections of circulant matrices also to (pure) mathematics such as

- complex analysis
- number theory
- commutative algebra
- algebraic geometry

Reason:

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We will see that these matrices appear naturally in areas of mathematics where the roots of unity play a role.

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Other basic definitions

We recall that, given a square $d \times d$ matrix

$$M = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,d-1} & x_{1,d} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,d-1} & x_{2,d} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{d,1} & x_{d,2} & \cdots & x_{d,d-1} & x_{d,d} \end{pmatrix} \quad (2)$$

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Other basic definitions

Definitions

- The *determinant* of M is

$$\det(M) := \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \prod_{i=1}^{d} x_{i,\sigma i}.$$

- The *permanent* of M is

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Some basic questions on $\mathrm{Circ}(x_0, \ldots, x_{d-1})$

Questions

- What is $\det(\mathrm{Circ}(x_0, \ldots, x_{d-1}))$?
- What is $\mathrm{perm}(\mathrm{Circ}(x_0, \ldots, x_{d-1}))$?

Answer

$\mathrm{Circ}(x_0, \ldots, x_{d-1})$ has d eigenvalues, namely

$$x_0 + \epsilon^j x_1 + \epsilon^{2j} x_2 + \cdots + \epsilon^{j(d-1)} x_{d-1} \qquad j = 0, \ldots, d-1.$$

therefore

$$\det(\mathrm{Circ}(x_0, \ldots, x_{d-1})) = \prod_{j=0}^{d-1} \left(\sum_{k=0}^{d-1} \epsilon^{jk} x_k\right).$$

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$\det(\mathrm{Circ}(x_0, \ldots, x_{d-1})), \mathrm{perm}(\mathrm{Circ}(x_0, \ldots, x_{d-1})) \in R_d$ and if $i_0 + \cdots + i_{d-1} = d$

- $[\det(\mathrm{Circ}(x_0, \ldots, x_{d-1}))]_{i_0, \ldots, i_{d-1}} \in \mathbb{Z}$
- $[\mathrm{perm}(\mathrm{Circ}(x_0, \ldots, x_{d-1}))]_{i_0, \ldots, i_{d-1}} \in \mathbb{Z}^+$

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- For which indices $[\det(\mathrm{Circ}(x_0, \ldots, x_{d-1}))]_{i_0, \ldots, i_{d-1}} \neq 0$?
- For which indices $[\mathrm{perm}(\mathrm{Circ}(x_0, \ldots, x_{d-1}))]_{i_0, \ldots, i_{d-1}} \neq 0$?
- If so, find explicit expressions for these coefficients.

Obviously, $[\det(\mathrm{Circ}(x_0, \ldots, x_{d-1}))]_{i_0, \ldots, i_{d-1}} \neq 0 \Rightarrow [\mathrm{perm}(\mathrm{Circ}(x_0, \ldots, x_{d-1}))]_{i_0, \ldots, i_{d-1}} \neq 0$.

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Some answers

A necessary condition for being $\neq 0$ is

$$0i_0 + i_1 + 2i_2 + \cdots + (d-1)i_{d-1} \equiv 0 \pmod{d}.$$

Indeed, the monomials explicitly appearing are invariant under the action on the polynomial ring $\mathbb{C}[x_0, \ldots, x_{d-1}]$ of the cyclic group of order d generated by the matrix

$$M_d = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \epsilon & 0 & \cdots & 0 \\ 0 & 0 & \epsilon^2 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & \epsilon^{d-1} \end{pmatrix}.$$

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also sufficient?

Answer

Yes, for the permanent:

Theorem (Brualdi-Newman)

$[\mathrm{perm}(\mathrm{Circ}(x_0, \ldots, x_{d-1}))]_{i_0,\ldots,i_{d-1}} \neq 0$ *if and only if* $i_0 + \cdots + i_{d-1} = d$ *and* $i_1 + 2i_2 + \cdots + (d-1)i_{d-1} \equiv 0 \pmod{d}$.

What about $[\det(\mathrm{Circ}(x_0, \ldots, x_{d-1}))]_{i_0,\ldots,i_{d-1}} \neq 0$?

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Examples for d small

Examples

- $d = 3$: $\det(\mathrm{Circ}(x, y, z)) = x^3 + y^3 + z^3 - 3xyz$;
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- $d = 5$: $\det(\mathrm{Circ}(x, y, z, t, u)) = x^5 + y^5 + z^5 + t^5 + u^5 - 5x^3yu - 5x^3zt - 5xy^3z - 5y^3tu - 5xz^3u - 5yz^3t - 5xyt^3 - 5zy^3u - 5xtu^3 - 5yzu^3 + 5x^2y^2t + 5x^2yz^2 + 5x^2zu^2 + 5x^2t^2u + 5xy^2u^2 + 5xz^2t^2 + 5y^2z^2u + 5y^2tu^2 + 5yt^2u^2 + 5z^2tu^2 - 5xyztu$.

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Some answers

Theorem (Hugh Thomas, 2004; Colarte–Mezzetti–Miró-Roig–Salat, 2019)

The condition $i_1 + 2i_2 + \cdots + (d-1)i_{d-1} \equiv 0 \pmod{d}$ *is also sufficient if and only if* d *is a prime or a power of a prime.*

In these cases the monomials appearing explicitly in the development of the determinant are precisely all monomials invariant under the action of M_d, and their number is equal to the number of terms in the development of the permanent.

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Varieties satisfying Laplace equations

Why are we interested in circulant matrices? Recall

Definition

A k-dimensional projective variety $X \subset \mathbb{P}^n$ is said to *satisfy r Laplace equations of order d* if for any parametrization $F = F(t_1, \ldots, t_k)$ of X around a general (smooth) point, F satisfies a system of r (linearly independent) PDE's with constant coefficients of order d.

In other words, the osculating spaces of order d of X have all dimension $e - r$, where e is the expected dimension for a osculating space of order d.

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1. *I fails the Weak Lefschetz Property in degree $d-1$;*
2. *$F_1, \dots, F_r$ become linearly dependent in $R/(L)$, for any linear form L;*
3. *X satisfies at least one Laplace equation of order $d-1$, where X is parametrized by generators of the apolar module $I_d^{\perp}$ in degree d, i.e. the osculating spaces of order $d-1$ of X have all dimension less than expected.*

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Togliatti systems

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- The ideals I as in the Tea Theorem are called *Togliatti systems*.
- A Togliatti system which can be generated by monomials is callous a *monomial Togliatti system*.
- A monomial Togliatti system is *minimal* if there is no proper subset of the set of generators defining a monomial Togliatti system.

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Examples

Case $n = 2$: surfaces

- $I_1 = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$ is apolar to the parametrization of the famous Togliatti surface in $\mathbb{P}^5$;
- $I_2 = (x_0^7, x_1^7, x_2^7, x_0^4x_1x_2^2, x_0^2x_1^4x_2, x_0x_1^2x_2^4)$.

- I_1 is generated by the monomials invariant under the action of $M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix}$, where ϵ is a primitive third root of 1.
- I_2 is generated by the monomials invariant under the action of $A_{1,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^3 \end{pmatrix}$, where ϵ is a primitive 7th root of 1.

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Theorem (Mezzetti–Miró-Roig, 2018)

Fix an integer $d \geq 3$ and let $I_{a,b} \subset K[x_0, x_1, x_2]$ be the ideal generated by all monomials of degree d invariant under the action of the cyclic group of order d, generated by

$$A_{a,b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^a & 0 \\ 0 & 0 & \epsilon^b \end{pmatrix}, \text{ where } 1 \leq a < b \leq d-1,$$

$\mathrm{GCD}(a,b,d) = 1$ *and* ϵ *is a root of unity of order* d.
Then, $I_{a,b}$ is a monomial Togliatti system, called Galois-Togliatti system (GT-system for short).

Problem (Minimality of the GT-systems)

Is any Togliatti system strictly contained in $I_{a,b}$?

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Second part: Minimal GT-Systems

Proving the minimality of the GT-system $I_{a,b}$ is equivalent to proving that the monomials of degree d invariant under the action of $A_{a,b}$ all appear with non-zero coefficient in the development of the product of linear forms

$$(x_0 + x_1 + x_2)(x_0 + \epsilon^a x_1 + \epsilon^b x_2)(x_0 + \epsilon^{2a} x_1 + \epsilon^{2b} x_2) \cdots \\ \cdots (x_0 + \epsilon^{(d-1)a} x_1 + \epsilon^{(d-1)b} x_2).$$

This is the determinant of the $d \times d$ circulant matrix

$$\mathrm{Circ}(x_0, 0, \dots, 0, x_1, 0, \dots, 0, x_2, 0, \dots, 0)$$

where x_1 is in the position of index a (counting from zero) and x_2 in the position of index b: it is a three-lines circulant matrix.

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r-lines circulant matrices

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An r-lines circulant matrix is a circulant matrix $\mathrm{Circ}(x_0, x_1 \dots, x_{d-1})$ of order $d > r$, where $d - r$ among $x_0, \dots, x_{d-1}$ are specialized to 0.

Question

We ask if for some pairs (r, d), $r < d$, conditions

$$\begin{cases} i_0 + \cdots + i_{d-1} = d \\ i_1 + 2i_2 + \cdots + (d-1)i_{d-1} \equiv 0 \pmod{d} \end{cases} \quad (3)$$

are sufficient for the non-vanishing of the corresponding coefficient in the determinant.

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Our results

Theorem (PDP–Mezzetti–Miró-Roig–Michałek–Nevo, 2018)

Conditions (3) *are sufficient in the case of* 3*-lines circulant matrices of order d, of the form*

$$\mathrm{Circ}(x_0, 0, \ldots, 0, x_1, 0, \ldots, 0, x_2, 0, \ldots, 0),$$

where x_1 appears in position a, x_2 appears in position b, and $\mathrm{GCD}(a, b, d) = 1$.

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Theorem (PDP–Mezzetti–Miró-Roig–Michałek–Nevo, 2018)

- *There are examples where the analogous property fails for*
  - *3-lines circulant matrices with* $\mathrm{GCD}(a, b, d) \neq 1$,
  - *r-lines circulant matrices with* $r \geq 4$ *and similar* GCD *equal to one.*
- *The coefficient of any specific monomial in a 3-lines circulant determinant is always equal, up to the sign, to the analogous coefficient in the permanent of the same matrix under the assumption* $\mathrm{GCD}(a, b, d) = 1$.
- *We give an explicit formula for calculating it.*

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Final remarks

Loehr, Warrington and Wilf (2004) proved sufficiency in case $a = 1$.

The more general question of finding a formula for the coefficients of the monomials in the expansion of the the determinant of a circulant matrix had already been considered in 1951 by Oystein Ore, who gave an explicit expression. Other expressions were given more recently by Malenfant and Wyn-Jones.

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For Further Reading I

- I. Kra and S. R. Simanca, *On circulant matrices*, Notices AMS **59** (3) (2012), 368–377.
- H. Thomas, *The number of terms in the permanent and the determinant of a generic circulant matrix*, J. Algebraic Combin. **20** (2004), no. 1, 55–60.
- E. Mezzetti, R.M. Miró-Roig and G. Ottaviani, *Laplace Equations and the Weak Lefschetz Property*, Canad. J. Math. **65** (2013), 634–654.
- E. Mezzetti and R.M. Miró-Roig, *Togliatti systems and Galois coverings*, J. Algebra **509** (2018), 263–291.

- L. Colarte, E. Mezzetti, R.M. Miró-Roig, M. Salat, *On the coefficients of the permanent and the determinant of a circulant matrix. Applications*, Proc. AMS **147** (2019), 547–558.
- P. De Poi, E. Mezzetti, M. Michałek, R.M. Miró-Roig, E. Nevo, *Circulant matrices and Galois-Togliatti systems*, arXiv:1808.08387
- N. A. Loehr, G. S. Warrington and H. S. Wilf, *The combinatorics of a three-line circulant determinant*, Israel J. Math. **143** No. 1 (2004), 141–156.