Problems on Jordan type of Multiplication Maps

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Abstract

The Jordan type P_ℓ of the multiplication m_ℓ on an Artinian algebra A is the partition giving the sizes in the Jordan block decomposition for m_ℓ, which is nilpotent. It in general gives more information than whether ℓ is weak or strong Lefschetz. I will discuss the set of Jordan types possible for a given Artin algebra A, the relation of Jordan type to the Hilbert function of A, and the loci of elements ℓ for which P_ℓ is fixed, as invariants of A. Some problems concern the behavior of these invariants for constructions such as tensor product, free extensions, and, for local Artinian Gorenstein algebras. Others concern which pairs of Jordan types are compatible.

Reference: arXiv:math.AC/1802.07383 and AC/1806.02767 (2018) with Pedro Marques and Chris McDaniel.

Plan

1. Introduction: Multiplication maps on an Artinian algebra A, Jordan type.
2. Kinds of Artin algebras to consider.
3. Questions on Jordan type of $P_{\ell,A}$ for $\ell \in \mathfrak{m}_A$.
4. Free extensions C of A with fibre B (T. Harima and J. Watanabe).
5. Strong Lefschetz and Tensor Product.
6. Cells $\mathbb{V}(E_P)$.
7. What sets of partitions $\{P_a, a \in \mathfrak{m}_A\}$ occur for a fixed A?
A. Appendices.

1. Multiplication maps on an Artinian algebra A

Let ℓ be an element in the maximal ideal of an Artinian algebra A. Recall that the Jordan type P_ℓ of the multiplication map $m_\ell : A \to A$ is the partition giving the sizes in the Jordan block decomposition for m_ℓ, which is nilpotent. The Jordan type in general gives more information than whether ℓ is weak or strong Lefschetz. We first give an overview of some directions of research about Jordan type that have been pursued in the past year, by several groups. We then propose some open problems.

Strong Lefschetz Jordan type.

Assume that A is graded Artinian, not necessarily standard graded. We say A is *strong Lefschetz* if there is a linear form $\ell \in A_1$ such that P_ℓ is $H(A)^\vee$, the conjugate of the Hilbert function.

For $\mathcal{A}$ local Artinian, we say that $\ell \in \mathfrak{m}_{\mathcal{A}}$ has *strong Lefschetz Jordan type* if $P_\ell = H(\mathcal{A})^\vee$.

The pair (A, ℓ) is *weak Lefschetz* if $\#$ parts of $P_\ell =$Sp(A); Sperner number Sp$(A) = \max_i H(A)_i$.
Prop.[H-W Prop. 3.5] If A is graded, unimodal, then A is weak Lefschetz iff $m_\ell : A_i \to A_{i+1}$ has maximal rank for each $i \geq 0$.

Example ($P_\ell = H(A)^\vee$ and strong Lefschetz)

Let $A = \mathsf{k}[x, y]/(x^2, xy, y^n)$, with Hilbert funct. $H(A) = (1, 2, 1, \ldots, 1_{n-1})$. Then $m_y : 1 \to y \to y^2 \to \cdots \to y^{n-1}$ and $m_y : x \to 0$. So $P_y = (n, 1)$. Likewise, $P_{y+ax} = (n, 1)$. (A is SL).

But $P_x = (2, 1, 1, \ldots 1)$ since the chains are $1 \to x \to 0, y \to 0, \ldots, y^{n-1} \to 0$.

Note: Viewed as a partition, $H = \{1, 2, 1, \ldots 1_{n-1}\}$ has conjugate $H^\vee = (n, 1) = P_{y+ax}$, the *generic Jordan type* of multiplication by a linear form. Also, $y + ax$ is a strong Lefschetz element, but x is not even a weak Lefschetz element ($H_{\max} = 2$).

2. Kinds of Artin algebras to consider for JT:

A. Standard Graded Artinian Gorenstein:

$H(A)$ is symmetric:

2 variables: (Macaulay) $H = (1, 2, \dots d, d \dots, 2, 1)$,
Variety GGor(H), fibration to $\mathbb{P}^d$ understood.

(N. Altafi, L. Khatami,I.: JT possible for CI, any ℓ.)

3 variables (Buchsbaum-Eisenbud) H understood,
Variety GGor(H) irreducible. Catalecticants.

4 variables (Srinivasan-I, Boij): Some H understood,
GGor(H) has several irred. components, related to space curves, when $H = (1, 4, 7, \dots, 7, 4, 1)$.

5 variables: non-unimodal Hilbert functions.

Kinds of Artin algebras . . .

B. Graded Artinian, G_T
2 variables: G_T smooth, projective:
(J. Briançon;J. Yameogo-I. Fibration $Z_T \to G_T$.
$r \geq 3$ variables: poorly understood.
B′. Non-standard graded Artinian: T. Harima, J. Watanabe- study Lefschetz Properties.
C. Gorenstein local: use symmetric decomposition (P. Macias Marques-I.).
C′. Local Artinian: understood in 2 variables.

3. Questions on Jordan type of P_ℓ for $\ell \in \mathfrak{m}_A$.

a. Compare P_ℓ with $H(A)^\vee$. Is $P_\ell \leq H(A)^\vee$ in dominance partial order? Open in general!
b. For which A is $P_\ell = H(A)^\vee$ (strong Lefschetz), for generic ℓ? (Many authors, see Harima, Maeno, Morita, Numata, Wachi,Watanabe: SLN 2080; and survey by Migliore-Nagel, also recent work of participants.)
c. Fix H, consider $GGor_H$, graded Gorenstein: which P_ℓ occur? (2 vars. N. Altafi, L. Khatami, I.). Open in $r \geq 3$ variables! despite Pfaffian structure theorem of D. Buchsbaum-D. Eisenbud for $r = 3$!

Questions on Jordan type of P_ℓ . . .

d. Behavior under tensor product $A \otimes_{\mathsf{k}} B$, or free extensions (a generalization of TP), or for connected sums $A \#_T B$ (C. McDaniel, A. Secealanu, I -).

e. Which pairs of Jordan types can occur in the same algebra A? Thorny area of commuting nilpotent Jordan types, very basic questions open:
e′. P. Oblak conjecture on the maximum commuting Jordan type $Q(P)$ for P. [known when $Q(P)$ has ≤ 3 parts – P. Oblak, L. Khatami.]
Fact: Parts of $Q(P)$ differ pairwise by at least two (P. Oblak, T. Košir): CI property of $\mathsf{k}\{A, B\}, B$ generic $\in \mathcal{N}_A$.

4. Free extensions (T. Harima, J. Watanabe)

C is a *free extension* of $\mathcal{A}$ with fiber B if there is an inclusion of algebras $\iota : A \to C$ making C into a free A-module, and $B \cong C/\mathfrak{m}_A C$. Thus C is an A-algebra such that the associated map $\gamma : \mathrm{Spec}(C) \to \mathrm{Spec}(\mathcal{A})$ is finite and flat, and with a closed embedding $\pi : \mathrm{Spec}(B) \to \mathrm{Spec}(C)$ inducing a cartesian diagram:

$$\begin{array}{ccc} \mathrm{Spec}(B) & \xhookrightarrow{\pi} & \mathrm{Spec}(C) \\ \downarrow & & \downarrow \gamma \\ \mathrm{Spec}(\mathsf{k}) & \hookrightarrow & \mathrm{Spec}(\mathcal{A}). \end{array}$$

Free extensions as deformation

Assume k is infinite, A, B, C graded Artinian algebras.

Theorem (C. McDaniel, P. Marques,I-,referee)

A free extension C of A with fiber B is a flat deformation of $A \otimes_{\mathsf{k}} B$ as an A algebra.

Set $P_A = P_\ell$ for ℓ generic in A_1.

Theorem (C. McDaniel, P. Marques,I-)

Let C be a free extension of A with fiber B. Then $P_C \geq P_{A \otimes_{\mathsf{k}} B}$ in the dominance partial order.

Here A, B are graded algebras. The following is Theorem 6.1 of T. Harima-J.Watanabe (2007).

Corollary

Suppose that C is a free extension of A with fiber B. Assume that $\mathrm{char}\ \mathsf{k} = 0$ *or* $\mathrm{char}\ \mathsf{k} > j_A + j_B$*, that the Hilbert functions of both A and B are symmetric, and that both A and B are strong Lefschetz. Then C is also strong Lefschetz.*

Proof idea. By Clebsch-Gordan $A \otimes_{\mathsf{k}} B$ is SL; but the HF of C equals that of $A \otimes_{\mathsf{k}} B$, so P_C is at least $P_{A \otimes_{\mathsf{k}} B} = H(A \otimes_{\mathsf{k}} B)^{\vee} = H(C)^{\vee}$, but this is the maximum possible.

Counterexample: Free Extension C SL $\not\Rightarrow$ B SL.

Even if A, B, C are standard graded with symmetric Hilbert functions, the free extension C being strong Lefschetz does not imply that B is strong Lefschetz.

Example

Take dual generator $F_B = (XU^{[2]} + YUV + ZV^{[2]})$, $A = \mathsf{k}[t]/(t^2)$ and $F_C = TF_B + G$, $G = X^{[2]}UV + XYV^{[2]}$. The Hilbert functions are $H(A) = (1, 1), H(B) = (1, 5, 5, 1)$ and $H(C) = (1, 6, 10, 6, 1)$.

(This answered a question of J. Watanabe).

6. Strong Lefschetz and tensor product

Clebsch-Gordan for $A = \mathsf{k}[x,y]/(x^m, y^n)$.
Use Clebsch-Gordan theorem (SLN 2080, Thm. 3.66) to determine the Jordan type in a tensor product $A \otimes B$ of $\ell = a \otimes 1 + 1 \otimes b$, from P_a, P_b.

Key idea. The Jordan type of $[m] \otimes [n]$ [1] is the same as the generic Jordan type of $\ell = x + y$ in $\mathsf{k}[x,y]/(x^m, y^n)$, which is (for $\mathrm{char}\ k$ not small) $P_\ell = (m+n-1, m+n-3, \cdots)$ ending in 2 or 1. For example $[3] \times [3] = (5, 3, 1)$.

[1] $[m]$ is the partition having a single part "m".

Thm. (See LNM 2080, Props. 3.64,3.66) If A, B are standard graded with symmetric Hilb. functions, and strong Lefschetz then $A \otimes B$ is strong Lefschetz.

Converse ($C = A \otimes B$ SL $\Rightarrow A, B$ SL) is open if $H(A), H(B)$ are not assumed symmetric!

The assumption of $H(A), H(B)$ symmetric is necessary because of counterexample:

Example (J. Migliore, U. Nagel, H. Schenck)

$A = B \cong \mathsf{k}[x, y]/(x^2, xy, y^2)$, then A, B are SL (codimension two) but $A \otimes B$ is not SL.
$H(A) = (1, 2, 0), H(A \otimes B) = (1, 4, 4)$.

Inequality between P_ℓ and $H(A)^\vee$.

Dominance partial order: Given
$P = (p_1, \ldots, p_s), p_1 \geq p_2 \geq \cdots \geq p_s$: then
$P \geq P'$ if $\sum_1^k p_i \geq \sum_1^k p_i'$ for all k.

Thm. If A is graded and $\ell \in A_i, i > 0$ is homogeneous, then $P_\ell \leq H(A)^\vee$.(Dominance partial order)

Open whether this is true if we take $\ell \in \mathfrak{m}_A$;

Thm.[P. Marques, C. McDaniel, I.] Let $\mathcal{A}$ be a local Artinian algebra (maybe graded, maybe not). Then for every $\ell \in \mathfrak{m}_A$ we have

$$P_\ell \leq H(\mathcal{A})^\vee.$$

What other Jordan types are possible for $\ell \in \mathfrak{m}_A$?

Answer: for $m = n = 3$, JT workgroup at Mittag-Leffler 2017: N. Altafi, H. L. Dao, I., L. Khatami, A. Seceleanu, and Y-S. Shin:

$y + x^2 : P_\ell = (4, 3, 2)$
$y + x^3 : P_\ell = (3, 3, 3)$ (same as m_y)
$y^2 + x^2 : P_\ell = (3, 2, 2, 1, 1)$
$y^2 + x^3 : P_\ell = (2, 2, 2, 1, 1, 1)$ (same as m_{y^2}.
There are two more, one inaccessible from $\otimes$:
$xy : P = (3, 2, 2, 1, 1)$ accessible.
$x^2y; P = (2, 2, 1^5)$ inaccessible.

J.T. workgroup 2017: began a list of modular case generic Jordan types $\lambda(m,n,p)$ for algebras $A(m,n) = \mathsf{k}[x,y]/(x^m,y^n)$ ($p = \mathrm{char}\ \mathsf{k} \leq j_A$).

There has been extensive work on this in RepTh: S.P.Glasby, C.E. Praeger, and B. Xia; Kei-Ichiro Iima and Ryo Iwamatsu (Schur functions), J.-C. Renaud. See §3.2 of "Artinian Algebras and J.T" for references.

Thm. [GPX] i. $A(m,n)$ in $\mathrm{char}\ \mathsf{k} = p$ is always WL; ii. it is SL if $n \neq \pm 1, \pm 2, \ldots, \pm m \mod p$, $(m \leq n)$. iii. $\exists$ Finite number $\leq 2^{m-1}$ of deviation vectors $\epsilon_{m,n,p} = \lambda(m,n,p) - (n,n,\ldots,n) \forall n \geq m, \forall p$.

Theorem (L. Nicklasson in 2018) Completely answers when $A(m,n)$ is SL $\mod p$ for each triple (m,n,p), using the base p- expansions of m and n.

6. Cells $\mathbb{V}(E_P)$ for $A = R/I, R = \mathsf{k}[x,y], H(A) = H$.

Point: fix a C.I. Hilbert function graded height two, $H = (1, 2, \dots, d-1, d, \dots, d, d-1, \dots, 2, 1)$. Choose a partition P of diagonal lengths T (so $P \in \mathcal{P}(H)$). We define an affine space cell $\mathbb{V}(E_P)$.

Theorem

i. $GGor(H) = \bigcup_{P \in \mathcal{P}(H)} \mathbb{V}(E_P)$.

ii. For $A \in \mathbb{V}(E_P)$, $\Rightarrow P_{x,A} = P$.

This cellular decomposition is an element in our study of which Jordan types are possible for a C.I. quotient of R (N.Altafi, L. Khatami-I, arXiv 1810.00716)

Definition (Initial ideal of I, and the Cell $\mathbb{V}(E_P)$)

The *initial monomial* $\mu(f) = \mathrm{in}(f)$ of a form $f = \sum_k a_k y^k x^{i-k}$, $a_k \in \mathsf{k}$ in the y-direction is the monomial $\mu(f) = y^s x^{i-s}$ of highest y-degree s among those with non-zero coefficients a_k. Given an ideal $I \subset R = \mathsf{k}[x, y]$, we denote by $\mathrm{in}(I)$ the ideal

$$\mathrm{in}(I) = (\{\mathrm{in} f, f \in I\})$$

generated by the initial monomials of all elements of I. We may identify $\mathrm{in} I$ with an initial ideal E_P for a partition $P \in \mathcal{P}(T)$. See next slide
We denote by $\mathbb{V}(E_P)$ the affine variety parametrizing all ideals $I \subset R$ having initial ideal E_P.

$P \in \mathcal{P}(T)$, the partitions of diagonal lengths T

Given a partition $P = (p_1 \geq p_2 \geq \cdots \geq p_s)$ we denote by C_P the set of monomials:

$$\begin{array}{llll} 1, & x, & x^2, & \ldots \quad , x^{p_1 - 1}; \\ y, & yx, & yx^2, & \ldots \quad , yx^{p_2 - 1} \\ \ldots & & & \\ \multicolumn{4}{l}{y^{s-1}, y^{s-1}x, \ldots, y^{s-1}x^{p_s - 1}.} \end{array}$$

Let $E_P = (x^{p_1}, yx^{p_2}, \cdots, y^{s-1}x^{p_s}, y^s)$.
The Hilbert function $H = H(R/E_P)$ is the *diagonal lengths* of P: we write $P \in \mathcal{P}(T)$.

Example: Cell $\mathbb{V}(E_P)$, $P = (5,2,2)$, $H = (1,2,3,2,1)$.

$$C_P : \begin{array}{ccccc} 1 & x & x^2 & x^3 & x^4 \\ y & yx & & & \\ y^2 & y^2x & & & \end{array} \qquad E_P = (x^5, yx^2, y^3).$$

$A = R/I, I = I_{a,b,c}$: (Actually, $I = (f_3, f_1)$ so is C.I.)

$$\begin{aligned} f_0 &= x^5; \\ f_1 &= yx^2 + cx^3 = x^2 g_1, \text{ where } g_1 = y + cx; \\ f_2 &= yf_1 = yx^2 g_1; \\ f_3 &= y^3 + ay^2x + bx^3 = g_3. \end{aligned}$$

Key: The multiplication m_x has strings on A

$1, x, x^2, x^3, x^4; g_1, xg_1; yg_1, yxg_1$. So $P_{x,A} = (5,2,2)$.

7. What sets of partitions $\{P_a, a \in \mathcal{A}\}$ are possible?

Question 1 (Compatibility) Given partitions P, P' of n, can we find $\mathcal{A}; a, a' \in \mathcal{A}$ with $P_a = P, P_{a'} = P'$?

Answer 1. Not all pairs can occur: for example $P_a = 4, P_{a'} = (3, 1)$ is impossible (see Maximum commuting orbit, below). See also P. Oblak [Ob]).

Question 2 (Intermediate P) For which $\mathcal{A}$ does a generic $\ell \in \mathfrak{m}_{\mathcal{A}}$ give P_ℓ that are WL but not SL?

Answer 2. Examples have been provided by A.I., R. Gondim, and by E. Mezzetti, R.M Miro, J. Vallés. R. Gondim's "On higher Hessians and the Lefschetz properties" [Gon] gives many examples and results.

Questions, continued

Problem. Characterize Artin algebras where the generic $P_\ell = P$ given. OR also fix $H(A)$.

Question 3 (Locus in $\mathbb{P}(\mathcal{A})$.) Find equations of the locus $\{a \mid P_a = P\}$ in the projective space $\mathbb{P}(\mathcal{A})$.

Answer 3. Very open. Some examples by J. Migliore et al in process. See also "Loci" in Appendix 1,2 "Maximum commuting orbit" for related problems.

Question 4 (Invariants) Use Questions 1,2,3 to give invariants of $\mathcal{A}$. Use as obstructions to deformation.

Answer 4 This has been furthest explored by groups motivated by modular representation theory: D. Benson, L. Carson, E. Friedlander, J. Pevtsova, A. Suslin and others.. See the talk notes of J. Pevtsova from Porto 2015.

Question 5 What are the Jordan type relationships for C that are A-free extensions with fibre B?
Answer 5: See Section 4 above for partial answers. For discussion, questions see arXiv 1806.02767, 1807.02881 (to appear, J.Commut. Alg.) - McDaniel-Marques-I. Also arXiv 1807.05869 (online in Lin and Mult. Alg) by McDaniel, Chen, Marques,-I: it has a connection to open combinatorial problems of G. Almkvist.

Appendices

A1. Maximum commuting orbit of a partition P.
A2. Determining the loci of $P_\ell = P$ fixed.
Loci questions arising from the Examples.
A3. Central simple modules and Jordan type (T. Harima and J. Watanabe).
A4. Partial idealization - U. Perazzo's cubic, Togliatti system (Mezzetti, Miro-Roig et al).
Hesssians (T. Maeno, J.Watanabe, 2009),
Mixed Hessians (B. Costa, R. Gondim, 2019).
A5. Overview, conclusion.

App. 1: Maximum commuting orbit of a partition P.

Given a partition P of n, denote by $\mathfrak{Q}(P)$ the maximum partition Q such that there is an Artin $\mathcal{A}$, and $a, a' \in \mathfrak{m}_{\mathcal{A}}$ with $P_a = P, P_{a'} = Q$.

Problem: determine the map $P \to \mathfrak{Q}(P)$.
P. Oblak's conjecture about this is half shown (L.Khatami, I.). See also [IKhVZ] and references.

Example

Let $P = (5, 4, 3, 3, 2, 1)$. Then $\mathfrak{Q}(P) = (12, 5, 1)$. So any larger partition in the Bruhat order, such as $(13, 6)$ cannot "commute" with P.

Theorem

(T. Košir, P. Oblak, 2009). $\mathfrak{Q}(P) = H(\mathcal{A})^\vee$ *for* $\mathcal{A} = \mathsf{k}\{A, B\}$ *where* A *is a general enough element of the commutator of* B, $P_B = P$. *Then* $\mathcal{A}$ *is a complete intersection, so* $\mathfrak{Q}(P)$ *has parts differing by at least two (F.H.S. Macaulay, 1904).*

Problem. Given a stable Q (whose parts differ pairwise by at least two $\mathfrak{Q}(Q) = Q$) let B be a Jordan matrix of type Q. Let $\mathfrak{Q}(P) = Q$. Determine the locus $\mathfrak{Z}_P = \{A \in C_B \mid P_A = P\}$ in the commutator $C_B = \{A \in \mathrm{Mat}_\mathsf{k}(n) \mid [A, B] = 0\}$. See [IKhVZ] and work in progress with M. Boij et al.

A2. Determining the loci of $P_\ell = P$ fixed.

We denote by $[n]^k$ the unique partition of n into k parts differing pairwise by at most 1.[2]

Example (Affine space deomposition of $\mathbb{P}^n$)

Let $\mathcal{A} = \mathsf{k}\{y/(y^n)\}$. Let u be a unit in $\mathcal{A}$ and let $\ell = \mathrm{u}y^k, 1 \le k \le n$. Then $P_\ell = [n]^k$. The locus $\mathfrak{Z}_P, P = ([n]^k)$ in $\mathbb{P}^n$ is an affine space $\mathbb{A}^{n-1-k}$ parametrizing $\ell = y^k + \sum_{i=k+1}^{n-1} a_i y^i$.

Reason: The y strings of $\mathcal{A}/(y^k)$ are $(y^s \to y^{k+s} \to y^{2k+s}, \ldots), 0 \le s < k$.

[2] If $n = kq + r, 0 \le r \le k$ then $[n]^k = (q^{k-r}, (q+1)^r)$. It is called the *almost rectangular* partition of n with k parts.

Example (One projective line in $\mathbb{P}^{n+1}$)

[3] Let $\mathcal{A} = \mathsf{k}[x, y]/(x^2, xy, y^n)$, for which $H(\mathcal{A}) = (1, 2, 1, \ldots, 1_{n-1})$ but $P_{y+ax} = (n, 1) = H^\vee$ and $H_x = H$. What other partitions P_ℓ and loci are possible for this $\mathcal{A}$?

Answer For $\ell = \mathsf{u}y^k + ax$ we have $P_\ell = P_k = ([n]^k, 1)$, hence for $k \le n-2$ we have $\mathfrak{Z}_{P_k} = \mathbb{A}^{n-k} \subset \mathbb{P}^n$. But for $P_0 = (2, 1^{n-1})$ we have $\mathfrak{Z}_{P_0} = p_x \cup A^1 = \mathbb{P}^1 \subset \mathbb{P}^n$, a projective line. Here $\mathbb{A}^1$ parametrizes $y^{n-1} + ax$.

Task Build a library of simple examples like these.

[3] This was our first example.

Loci questions arising from the examples.

LQ1. Determine all Artin algebras for which the loci stratification by $\{\mathfrak{Z}_{P_\ell}, \ell \in \mathbb{P}(\mathcal{A})$ is that of the first (usual affine spaces), or the second (usual except for $\mathbb{P}^1$.

LQ2. We can stratify $\mathcal{A}$ by its $\mathfrak{m}$-adic or Loewy $(0 : \mathfrak{m}^b)$ ideals and their intersections $\mathfrak{m}^a \cap (0 : \mathfrak{m}^b)$. Fixing dimensions of these strata (whichever we choose) and considering the Jordan type loci for each, we obtain a finer loci-stratification (LS) invariant $\mathfrak{G}$. Parametrize $\mathcal{A}$ of given $\mathfrak{G}$.

LQ3 What are the closures of $\mathfrak{Z}_P$?

A3. Central simple modules and Jordan type.

T. Harima and J. Watanabe introduced "central simple modules" with respect to a linear form L in a (standard or non-standard) graded ring A, in [HW1, HW2, HW3]. These are closely related to Jordan type for such rings, and may be generalized to ideals J in general Artinian algebras (see [BjI]). For Artinian Gorenstein algebras this viewpoint can generalize that used for the symmetric decomposition (use the maximum ideal $\mathfrak{m}$) and there is a similar reflexive structure for subquotients of the associated graded algebra with respect to J.

A4. Partial Idealization-U. Perazzo's cubic [Pe]

Example [R. Gondim, 2016] $F = XU^3 + YUV^2 + ZU^2V$. Then $A = R/\operatorname{Ann} F$ is AG (Artinian Gorenstein) algebra of HF $H(A) = (1,5,6,5,1) = (1,2,3,3) + (0,3,3,2,1)$, whose conjugate is $H^\vee = (5,3,3,3,3,1)$. But the Jordan type for a generic L is $P_L = (5,3,3,3,2,2)$, so A is WL but not SL.That it is WL is from Prop. 3.6 of [Gon].

Not SL: The partial derivatives $x \circ F = U^{[3]}, y \circ F = UV^{[2]}$ and $z \circ F = U^{[2]}V$ are algebraically dependent, so by Gordan-Noether criterion ([GorN, Gon, MW]) the Hessian $\mathrm{Hess}^1_F = 0$. By the Maeno-Watanabe criterion ([MW], [H-W, Theorem 3.76]) this implies that the map $m_{\ell^2} : A_1 \to A_3$ does not have max. rank, so A fails the strong Lefschetz Property - see §3.6 of SLN #2080.)

Togliatti System: **Example** [E.Mezzetti, R.M. Miro-Roig, J. Vallés] Let $I = (x^d, y^d, z^d, x^{d-2}yz, x^{d-4}y^2z^2, \ldots, y^{d/2}z^{d/2})$ (in case d is even). Then I fails WL in degree $d-1$.

Question. What is P_ℓ here?

Use of Hessians and mixed Hessians

Theorem (B. Costa and R. Gondim[4])

The Jordan type of any standard graded Artinian Gorenstein algebra $A = Q/\operatorname{Ann} f$ depends only on the ranks of certain mixed Hessians of f.

They use string diagrams to visualize the Jordan degree type of A (more information than just the Jordan type). And they complete determine the possible Jordan types for some low socle degree examples of Hilbert functions, as $H = (1, r, r, 1)$.

[4] Advances in Applied Math. 111 (Oct 2019) 101 941.

A5 Overview, conclusion.

1. The Jordan type of the multiplication map $m_\ell, \ell \in \mathfrak{m}_A$ is a partition of $n = \dim_k \mathcal{A}$ giving more information than the "yes/no" of WL or SL. The Jordan type behaves in natural ways under deformation of $\mathcal{A}$, and the map $\mathcal{A} \to \mathcal{A}^*$. The generic Jordan type for A is not in general that of $\mathcal{A}^*$.

2. The Jordan type of $(\ell, \mathcal{A})$ is easily calculated by finding the ranks of $m_\ell, m_{\ell^2}, \ldots$ (see "Algorithm").

3. The set $\{P_\ell, \ell \in \mathfrak{m}_\mathcal{A}\}$ is an invariant of $\mathcal{A}$, as well as the loci $\mathfrak{Z}_P = \{\ell \mid P_\ell = P\}$ in $\mathbb{P}(\mathcal{A})$.

4. Some pairs (P_1, P_2) are compatible for Jordan types of (ℓ_1, ℓ_2) in the same Artin algebra $\mathcal{A}$, but many are not. This relates to the study of commuting pairs of nilpotent matrices: P. Oblak, T. Košir; R. Basilli; D.I. Panyushev; M.Boij, L. Khatami, B. Van Steirteghem, R.Zhao, & A.I.

5. Is there a deeper theory of parametrizing Artinian algebras with given invariants determined by the Jordan types of elements? Any universal objects?

6. Analogously to Constant JT modules in representation theory, do we get interesting vector bundles on the loci $\mathfrak{Z}_P$ or on their closures?

END

THANK YOU!

See I-, P. Macias Marques, C. McDaniel, *Artinian Algebras and Jordan Type*, arXiv:math.AC/1802.07383 (2018). (version iii.),

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