Jordan types in height two graded Artinian algebras

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Lefschetz Properties in Algebra, Geometry and Combinatorics II
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- The diagonal lengths of $P_{\ell,A}$ are given by the Hilbert function of A. [IY]

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- For $P \in \mathcal{P}(T)$, the cell $\mathbb{V}(E_P)$ of G_T consists of algebras in which multiplication by a generic linear form has Jordan type P. [Iarrobino-Yaméogo]

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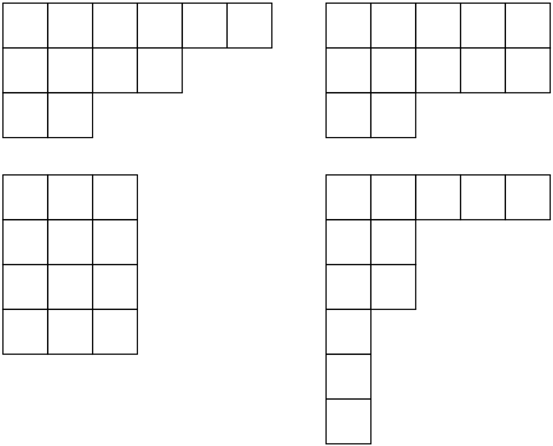

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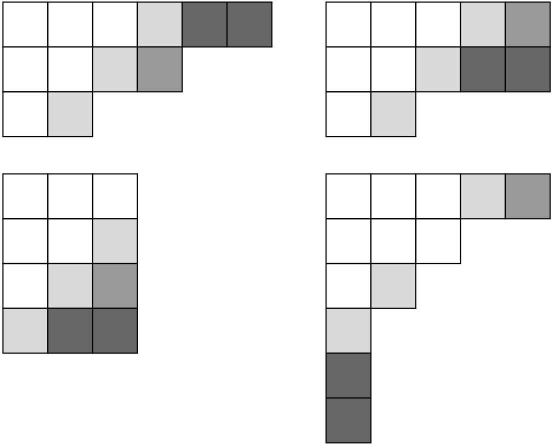

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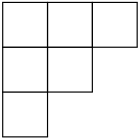

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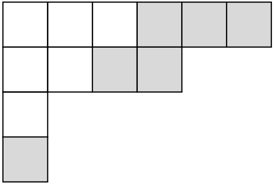

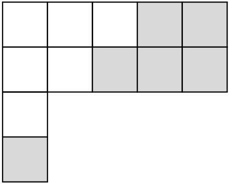

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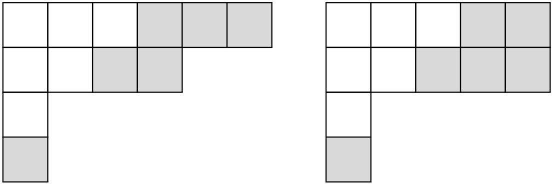

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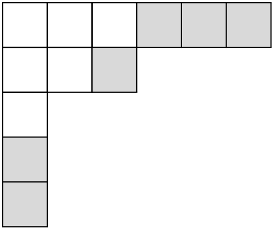

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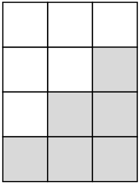

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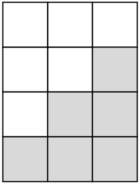

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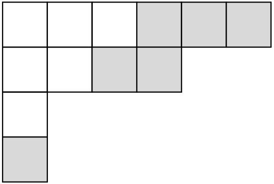

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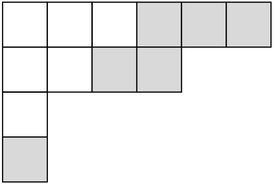

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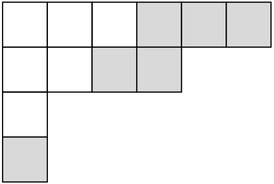

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Theorem [Altafi, Iarrobino,K]. Let $T = (1, 2, \ldots, d-1, d^k, d-1, \ldots, 2, 1)$ such that $d \geq 2$. A partition P with diagonal lengths T can occur as the Jordan type of multiplication by a linear form in a CI Artinian algebra of Hilbert function T if and only if the vertical part of the "branch label" of P is either empty or the single interval $(1, 2, \ldots, e)$, for some $1 \leq e \leq d$.

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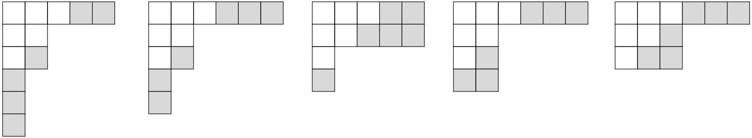

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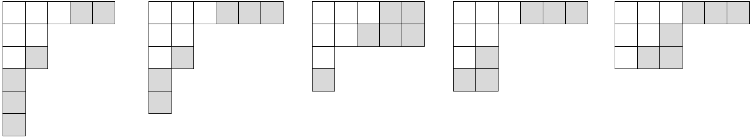

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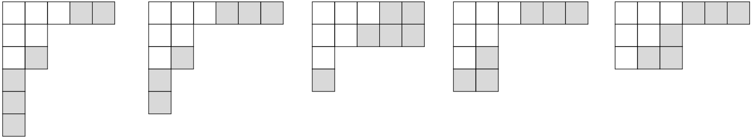

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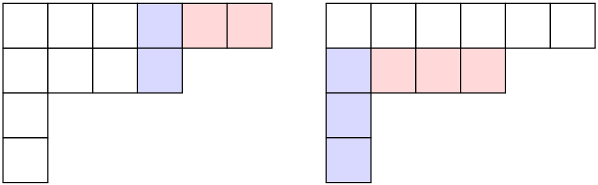
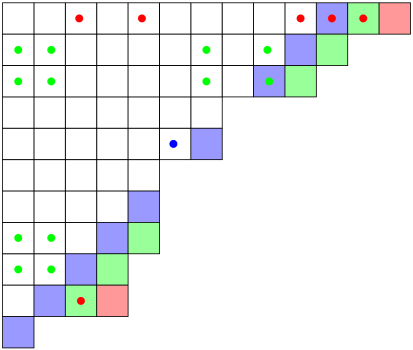


Consider the Ferrer's diagram of a partition P. Let $c = x^a y^b$ be a monomial in F_P. The hook with corner c consists of monomials

$$\begin{array}{cccc} c & x^{a+1}y^b & \dots & x^{a+h}y^b \\ x^a y^{b+1} & & & \\ \vdots & & & \\ x^a y^{b+f} & & & \end{array}$$

in F_P where $x^{a+h+1}y^b, x^a y^{b+f+1} \in I$.
The hook has hand length h and foot length f and is called a difference-one hook if $h = f + 1$.

Let $T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0)$ with $d \geq t_d \geq \cdots \geq t_j > 0$. Each partition P with diagonal lengths T is uniquely determined by its difference-one hook code.



$$T = (1, 2, \ldots, 9, 10, {\color{blue}9}, {\color{green}6}, {\color{red}2}, 0)$$

$$\mathfrak{Q}(P) = \Big({\color{blue}(1, 0^2)_{10}}, \ {\color{green}(4^2, 2^2)_{11}}, \ {\color{red}(5, 1)_{12}} \Big)$$

Let $T = (1, 2, \ldots, d, t_d, \ldots, t_j, 0)$ with $d \geq t_d \geq \cdots \geq t_j > 0$. Assume that P is a partition with diagonal lengths T. Find $\kappa(P)$, the minimal number of generators for an ideal in $\mathbb{V}(E_P)$.

Theorem. [Altafi, Iarrobino, Yaméogo, K]
Assume that $T = (1, \ldots, d, t, 0)$ and let P be a partition with diagonal lengths T and difference-one hook code $\mathfrak{Q}(P) = (h_1^{\ell_1}, \ldots, h_n^{\ell_n})$. For $k = 1, \ldots, n$, let $\tau_k = \sum_{i=k}^{n} \ell_i - h_k$. Also let $s = d + 1 - t$. Then

$$\kappa(P) = s + \max\{t + 1 - s, 0, \tau_k\}_{k=1,\ldots n}.$$

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Theorem. [Altafi, Iarrobino, Yaméogo, K] For a positive integer k, we define $\mu(T, k)$ to be the number of partitions P with diagonal lengths T and $\kappa(P) = k$. Then

$$\mu(T,k) = \begin{cases} \binom{s+t}{s} - \binom{s+t}{s+\delta+1}, & \text{if } k = s + \delta \ (\kappa(P) \text{ minimal}) \\ \\ \binom{s+t}{k} - \binom{s+t}{k+1}, & \text{if } s + \delta < k \leq s + t \\ \\ 0, & \text{otherwise .} \end{cases}$$

Let P be a partition with Hilbert function
$T = (1, \ldots, d, t_d, \ldots, t_j, 0)$ and hook code $\mathfrak{Q}(P) = \big(\mathfrak{h}_d, \ldots, \mathfrak{h}_j\big)$.
Set $t_{d-1} := d$ and $t_{j+1} := 0$ and for $i = d, \ldots, j$, let

$$T_i = (1, \ldots, t_{i-1} - t_{i+1}, t_i - t_{i+1}, 0)\,.$$

Then we define the i-th block of P, denoted by P_i, to be the partition with diagonal lengths T_i and hook code $\mathcal{Q}(P_i) = \mathfrak{h}_i$.

Theorem. [Altafi, Iarrobino, Yaméogo, K]

$$\kappa(P) = \kappa(P_d) + \cdots + \kappa(P_j) - (t_d - t_j) - (j - d).$$

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$$\kappa(P)=\kappa(P_d)+\cdots+\kappa(P_j)-(t_d-t_j)-(j-d).$$

Corollary. P is "special" if and only if at least one of its components is special.

Thank you!