Microscopic description of Coulomb gases

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Setup

- Energy

$$H_N(x_1,\dots,x_N)=\frac{1}{2}\sum_{i\neq j} w(x_i-x_j)+N\sum_{i=1}^N V(x_i) \qquad x_i\in\mathbb{R}^d,\, d\geq 1$$

- interaction potential : Coulomb

$$w(x)=-\log|x| \quad \text{if } d=2$$

$$\text{or} \quad w(x)=\frac{1}{|x|^{d-2}} \qquad d\geq 3$$

- V confining potential, sufficiently smooth and growing at ∞

  Gibbs measure

$$d\mathbb{P}_{N,\beta}(x_1,\cdots,x_N)=\frac{1}{Z_{N,\beta}}e^{-\beta N^{\frac{2}{d}-1}H_N(x_1,\dots,x_N)}dx_1\dots dx_N$$

  $Z_{N,\beta}$ partition function

Motivation

- Random matrices and β-ensembles in the logarithmic cases Dyson, Mehta, Wigner
  quantum mechanics models, Laughlin wave-function in the fractional quantum Hall effect, self-avoiding paths in probability, see [Forrester '10]
- $d \geq 2$ classical **Coulomb gas**
  [Lieb-Lebowitz '72,Lieb-Narnhofer '75, Penrose-Smith '72, Sari-Merlini '76, Kiessling-Spohn '99, Alastuey-Jancovici '81, Jancovici-Lebowitz-Manificat' 93...]

Mean Field limit: the equilibrium measure

- μ_V is the unique minimizer of

$$\mathcal{E}(\mu) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} w(x-y)\, d\mu(x)\, d\mu(y) + \int_{\mathbb{R}^d} V(x)\, d\mu(x).$$

  among probability measures.
- Examples: $V(x) = |x|^2$ (**Ginibre ensemble** in RMT) then $\mu_V = \frac{1}{c_d} \mathbb{1}_{B_1}$ (circle law).
- For fixed β,

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \sim \mu_V \quad \text{except with exponentially small probability}$$

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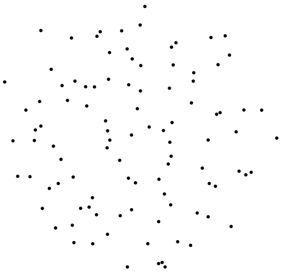
A 2D log gas for $V(x) = |x|^2$



$w = -\log$, $V = |x|^2$, 100 points, $\beta \in [0.7, 400]$ (Thomas Leblé)

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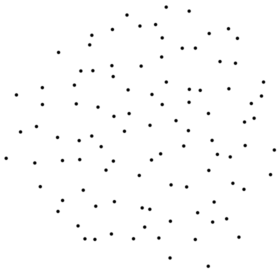

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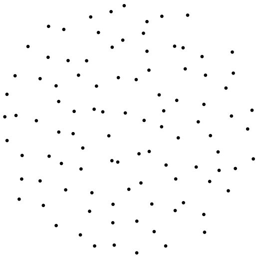

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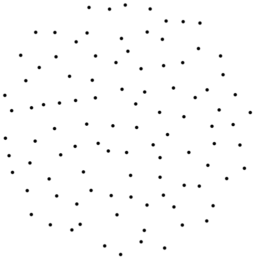

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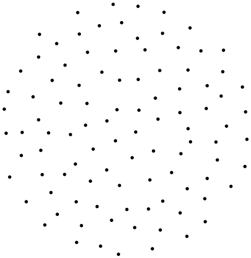

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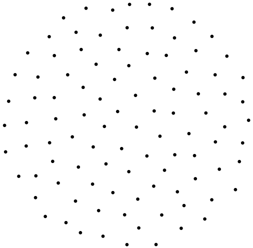

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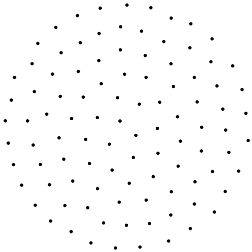

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Questions

- Rigidity of the points?

$$\int_{B_R} \left(\sum_{i=1}^N \delta_{x_i} - N\mu_V \right) \ll NR^d?$$

For which R? Down to microscale $N^{-1/d}$??

- Behavior of the point configurations at the microscale? Limit point processes?
- Fluctuations of linear statistics

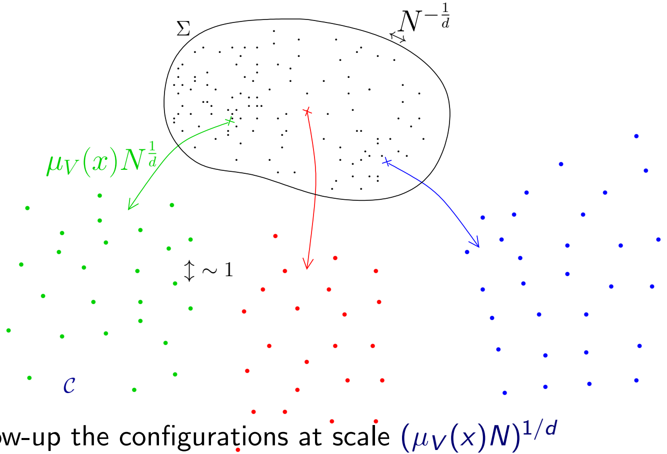
$$\int \xi(x) d \left(\sum_{i=1}^N \delta_{x_i} - N\mu_V \right)(x)$$

Are they Gaussian? For which ξ? Supported at which scale?

- Dependence in β?
- Free energy expansions

$$-\frac{1}{\beta} \log Z_{N,\beta} = N^2 \mathcal{E}(\mu_V) - \frac{1}{4} N \log N + A_\beta N + B_\beta N^{\frac{1}{2}} + C_\beta \log N + \dots$$

The blow-up procedure



- blow-up the configurations at scale $(\mu_V(x)N)^{1/d}$
- define interaction energy $\mathbb{W}$ for infinite configurations of points in $\mathbb{R}^d$ with uniform negative background -1 (*jellium*)
- the total energy will be the average $\overline{\mathbb{W}}$ of $\mathbb{W}$ over all blow-up centers in supp μ_V.

Properties of the jellium energy $\mathbb{W}$

- defined in [Sandier-S '12, Rougerie-S '16, Petrache-S '17]
- Rigidity and equidistribution results for minimizers by a bootstrap on scales [Rota Nodari-S '17, Petrache - Rota Nodari '17, Armstrong-S'19]
- In dimension $d = 1$, the minimum of $\mathbb{W}$ over all possible configurations is achieved for the **lattice** $\mathbb{Z}$.
- In dimension $d = 8$ the minimum of $\mathbb{W}$ is achieved by the E_8 lattice and in dimension $d = 24$ by the Leech lattice: consequence (see [Petrache-S '19] of the Cohn-Kumar conjecture proven in [Cohn-Kumar-Miller-Radchenko-Viazovska '19]
- the Cohn-Kumar conjecture remains open in dimension 2. If true, the minimum of $\mathbb{W}$ is achieved at the *triangular lattice* (cf equivalent conjectures of [Sandier-S, Brauchart-Hardin-Saff, Bétermin-Sandier]).

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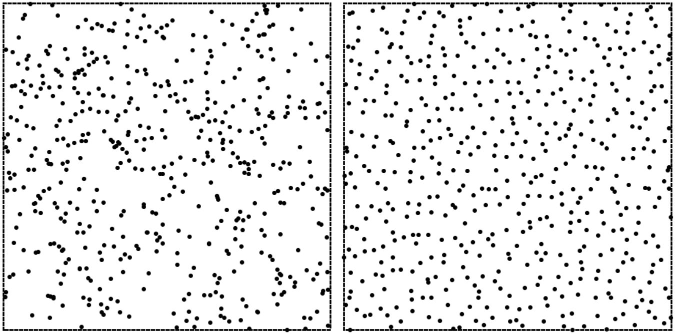
Pictures of limiting point processes



The **Poisson** point process and the **Ginibre** point process
(limit as $N \to \infty$ for $\mathbb{P}_{N,\beta}$ when $w = -\log$, V quadratic, $\beta = 2$)
(pic. Alon Nishry)

A "Large Deviations Principle" for limiting point processes

Theorem (Leblé-S, '17)

For Coulomb (or log or Riesz interactions with $d-2 \le s < d$), the Gibbs measure concentrates on configurations whose limiting point processes P^x (after zoom around x) minimize

$$\mathcal{F}_\beta(P) := \int_{\text{supp } \mu_V} (\mathbb{W} dP^x + \frac{1}{\beta} \text{ent}[P^x|\Pi])\, dx, \qquad \Pi = \textit{Poisson intensity 1}$$

- $\beta \gg 1$ rigid behavior expected (complete crystallization proven in 1D)
- $\beta \ll 1$ entropy dominates $\rightsquigarrow$ Poisson point process
- $\beta \propto 1$ intermediate, **phase-transition for cristallization?**

Generalization to the 2D "two component plasma" Leblé-S-Zeitouni

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Next order free energy expansion

Corollary (Leblé-S '17)

$$\log Z_{N,\beta} = -\beta N^{1+\frac{2}{d}} \mathcal{E}_\theta(\mu_\theta) + \left(\frac{\beta}{4} N \log N\right) \mathbb{1}_{d=2} - \frac{N\beta}{4} \mathbb{1}_{d=2} \int \mu_\theta \log \mu_\theta - N\beta \int f(\beta \mu_\theta^{1-\frac{2}{d}}) d\mu_\theta + o(N)$$

where we denote

$$f(\beta) = \min_P \left(\frac{1}{2c_d} \mathbb{W}(P) + \frac{1}{\beta} \mathrm{ent}[P|\Pi] \right)$$

P=*stationary point processes of intensity* 1.

to be compared with [Borot-Guionnet '13, Shcherbina '13] ($d = 1$, log), [Wiegmann-Zabrodin '09] ($d = 2$, log) (formal)

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Treating general β : the thermal equilibrium measure

Instead of μ_V minimizing

$$\mathcal{E}(\mu) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} w(x-y)\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^d} V(x)\,d\mu(x).$$

use μ_θ minimizing

$$\mathcal{E}_\theta(\mu) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} w(x-y)\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^d} V d\mu + \frac{1}{\theta}\int_{\mathbb{R}^d} \mu\log\mu$$

with

$$\theta := \beta N^{\frac{2}{d}}.$$

If β fixed or $\beta \gg N^{-\frac{2}{d}}$, then $\theta \to \infty$ and $\mu_\theta \to \mu_V$ (precise asymptotics obtained in [Armstrong-S '19]).
Introduces new lengthscale $\frac{1}{\sqrt{\theta}} = \beta^{-\frac{1}{2}} N^{-\frac{1}{d}}$ for *macroscopic rigidity*

Splitting with respect to the thermal equilibrium measure

$$H_N(X_N) = N^2\mathcal{E}_\theta(\mu_\theta) - \frac{N}{\theta}\sum_{i=1}^{N}\log\mu_\theta(x_i)$$

$$\underbrace{+\frac{1}{2}\iint_{\triangle^c} w(x-y)d\left(\sum_{i=1}^{N}\delta_{x_i} - N\mu_\theta\right)(x)d\left(\sum_{i=1}^{N}\delta_{x_i} - N\mu_\theta\right)(y)}_{F(X_N,\mu_\theta)}$$

This way

$$Z_{N,\beta} = \exp\left(-\beta N^{1+\frac{2}{d}}\mathcal{E}_\theta(\mu_\theta)\right)$$

$$\times\int_{(\mathbb{R}^d)^N} e^{-\beta N^{\frac{2}{d}-1}F(X_N,\mu_\theta)}d\mu_\theta(x_1)\ldots d\mu_\theta(x_N)$$

The electric formulation

Define the potential generated by the distribution $\sum_i \delta_{x_i} - N\mu_\theta$

$$h = w * \left(\sum_i \delta_{x_i} - N\mu_\theta \right)$$

$$-\Delta h = c_d \left(\sum_i \delta_{x_i} - N\mu_\theta \right)$$

and rewrite the energy as

$$F(X_N, \mu_\theta) \simeq \int |\nabla h|^2$$

(renormalized with truncations)
Formally

$$\mathbb{W} = \lim_{R\to\infty} \fint_{\square_R} |\nabla h|^2$$

for the h computed after blow-up at scale $N^{1/d}$.

Local laws

$$\chi(\beta) = \begin{cases} 1 & \text{if } d \geq 3 \ \text{ or } d = 2 \text{ and } \beta \geq 1 \\ |\log \beta| + 1 & \text{if } d = 2 \text{ and } \beta \leq 1. \end{cases}$$

Theorem (Armstrong-S. '19)

Let Σ *be a set where* $\mu'_\theta \geq m > 0$ *(blown-up by* $N^{1/d}$ *of* μ_θ*),* $x'_i = N^{1/d} x_i$. *There exists a minimal scale* $\rho_\beta \simeq \max(\beta^{-1/2}\chi(\beta)^{1/2}, 1)$ *and* $C(d,m,M)$ *such that if* $R \geq C\rho_\beta$ *and* $\mathrm{dist}(\square_R, \partial\Sigma) \geq N^{\frac{1}{d+2}}$

- *(Local energy control)*

$$\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}} \left(\exp \left(\frac{1}{2} \beta F^{\square_R}(X'_N, \mu'_\theta) \right) \right) \right| \leq C\beta\chi(\beta) R^d$$

- *(Rigidity of number of points) Set* $\omega_N = \sum_{i=1}^N \delta_{x'_i} - d\mu'_\theta$,

$$\left|\log \mathbb{E}_{\mathbb{P}_{N,\beta}} \left(\exp \left(\frac{\beta}{C} \frac{(\omega_N(\square_R))^2}{R^{d-2}} \min(1, \frac{|\omega_N(\square_R)|}{R^d}) \right) \right) \right| \leq C\beta\chi(\beta) R^d$$

previous results:
[Leblé, Bauerschmidt-Bourgade-Nikula-Yau] $d = 2$, β fixed, mesoscales $R \geq N^{\varepsilon}$, $\varepsilon > 0$.

Corollary

Up to a subsequence, and after blow-up by $N^{1/d}$, there exists a limiting point process.

CLT for fluctuations in $d = 2$

Theorem (Leblé-S. '16)

Assume $d = 2$, $\beta > 0$ *arbitrary fixed,* $V \in C^{3,1}$. *Assume* $\Sigma = \mathrm{supp}\,\mu_V$ *has one connected component. Let* $\xi \in C_c^{3,1}(\mathbb{R}^2)$ *or* $C_c^{2,1}(\Sigma)$ *and* $\xi^\Sigma =$ *harmonic extension of* ξ *outside* Σ. *Then*

$$\sum_{i=1}^N \xi(x_i) - N \int_\Sigma \xi \, d\mu_V$$

converges in law as $N \to \infty$ *to a Gaussian distribution with*

$$mean = \frac{1}{2\pi}\left(\frac{1}{\beta} - \frac{1}{4}\right) \int \Delta\xi \, (\mathbb{1}_\Sigma + \log \Delta V)^\Sigma \qquad var = \frac{1}{2\pi\beta} \int_{\mathbb{R}^2} |\nabla \xi^\Sigma|^2.$$

$\rightsquigarrow \Delta^{-1}\left(\sum_{i=1}^N \delta_{x_i} - N\mu_V\right)$ *converges to the Gaussian Free Field.*
The result can be localized with ξ *supported on mesoscales* $N^{-\alpha}$, $\alpha < \frac{1}{2}$.

Simultaneous result by [Bauerschmidt-Bourgade-Nikula-Yau] for $\xi \in C_c^4(\mathrm{supp}\,\mu_V)$

CLT for fluctuations in $d = 2, 3$, all temperatures

Theorem (S. '19)

Assume $d = 2, 3$. Assume $V, \xi_0 \in C^p$ for some p large enough.
If $d = 3$ assume in addition that $f \in C^p$ ("no phase transitions" near that β) and $|f^{(k)}(\beta)| \le C\beta^{-k}$ for all $k \le p$.
Assume $\ell \gg \rho_\beta N^{-\frac{1}{d}}$ (in $d = 3$, $\ell \gg \rho_\beta N^{-\frac{1}{d}} N^{\alpha/p}$).
Assume $\beta \ll (N\ell^d)^{1-\frac{2}{d}-\frac{4}{3d}}$.
Assume $\xi := \xi_0(\frac{x-x_0}{\ell})$ is supported in $\{\mathrm{dist}(x, \partial\{\mu_\theta > m\}) \ge N^{\frac{1}{d+2}-\frac{1}{d}}\}$. Then

$$\beta^{\frac{1}{2}} N^{\frac{1}{d}-\frac{1}{2}} \ell^{1-\frac{d}{2}} \left(\sum_{i=1}^N \xi(x_i) - N \int \xi \, d\mu_\theta \right) - C_{N,\beta,\ell,\xi}$$

converges in law to a Gaussian with mean 0 and variance $\frac{1}{2c_d} \int |\nabla \xi_0|^2$ (with cv rate).

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Assume $\ell \gg \rho_\beta N^{-\frac{1}{d}}$ *(in* $d = 3$, $\ell \gg \rho_\beta N^{-\frac{1}{d}} N^{\alpha/p}$*).*
Assume $\beta \ll (N\ell^d)^{1-\frac{2}{d}-\frac{4}{3d}}$.
Assume $\xi := \xi_0(\frac{x-x_0}{\ell})$ *is supported in*
$\{\mathrm{dist}(x, \partial\{\mu_\theta > m\}) \ge N^{\frac{1}{d+2}-\frac{1}{d}}\}$. *Then*

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converges in law to a Gaussian with mean 0 *and variance* $\frac{1}{2c_d} \int |\nabla \xi_0|^2$ *(with cv rate).*

CLT for fluctuations in $d = 2, 3$, all temperatures

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Comparison with the literature

- 2D log case
  - [Rider-Virag] same result for $\beta = 2$, $V(x) = |x|^2$
  - [Ameur-Hedenmalm-Makarov] same result for $\beta = 2$, $V \in C^\infty$ and analyticity in case the support of ξ intersects $\partial\Sigma$
  - Concentration bounds (in N^ε, but with quantified error in probability), including at mesoscale, on $\|\sum_{i=1}^N \delta_{x_i} - N\mu_V\|$ [Sandier-S, Leblé], [Chafai-Hardy-Maida], [Bauerschmidt-Bourgade-Nikula-Yau]
  - Number fluctuations for *hierarchical* Coulomb gas [Chatterjee] (d=2,3), [Ganguly-Sarkar] (all d).
- 1D log case
  - [Johansson] 1-cut, V polynomial
  - [Borot-Guionnet], [Shcherbina] 1-cut and V, ξ locally analytic, multi-cut and V analytic
  - new proof by [Lambert-Ledoux-Webb] 1-cut, Stein method, [Bekerman-Leblé-S]

Method of proof for local laws

Use idea of sub/superadditive quantities of [Armstrong-Smart] (in homogenization theory), like Dirichlet-Neumann bracketing: in any cube $\square_R$ define the partition functions $K_N(\square_R)$ and $L_N(\square_R)$ for the energies $\int_{\square_R} |\nabla u|^2$, resp. $\int_{\square_R} |\nabla v|^2$ where u solves

$$\begin{cases} -\Delta u = c_d\left(\sum_{i=1}^N \delta_{x_i} - 1\right) & \text{in } \square_R \\ u = 0 & \text{on } \partial\square_R. \end{cases}$$

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The first one works well by restriction $\rightsquigarrow$ subadditive, while the second one works well by patching $\rightsquigarrow$ superadditive.
$\frac{\log K_N(\square_R)}{|\square_R|}$ and $\frac{\log L_N(\square_R)}{|\square_R|}$ both converge monotonically to the same limit $f(\beta)$.
Moreover by "screening procedure", they differ only by $O(R^{d-1})$.
Hence *almost additivity* on cubes and expansion of the true partition function up to R^{d-1}.

Method of proof for the CLT

- Compute the Laplace transform of the fluctuations

$$\mathbb{E}_{\mathbb{P}_{N,\beta}}\left[-e^{\beta t N^{\frac{2}{d}}\left(\sum_{i=1}^N \xi(x_i) - N\int \xi \mu_\theta\right)}\right],$$

  with $t = \frac{\tau}{N}$, and show it converges to that of a Gaussian.

- it amounts to computing

$$\frac{Z(V_t)}{Z(V)}$$

  where $V_t := V + t\xi$, thermal equilibrium measure μ_θ^t.

- use map Φ_t that transports μ to μ^t, $\Phi_t \simeq I + t\psi$. By using change of variables $y_i = \Phi_t(x_i)$, we find

$$\frac{K_N(\mu^t)}{K_N(\mu)} = \mathbb{E}_{\mathbb{P}_{N,\beta}}\left(F_N(\Phi_t(X_N), \Phi_t\#\mu) - F_N(X_N, \mu)\right)$$

- use expansion in t small for the rhs + expansion of $\log Z_{N,\beta}$ with a rate to evaluate this with $o(1)$ error when $t = \tau/N$.

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Free energy expansions

$\log K$ is known for constant densities on cubes. By transport, we can evaluate it for nonconstant densities that are close to their average, on cubes. Then use almost additivity (with surface errors) on cubes to obtain

Theorem (S '19+)

$$\log Z_{N,\beta} = -\beta N^{1+\frac{2}{d}} \mathcal{E}_\theta(\mu_\theta) + \left(\frac{\beta}{4} N \log N\right) \mathbb{1}_{d=2} - \frac{N\beta}{4} \mathbb{1}_{d=2} \int \mu_\theta \log \mu_\theta - N\beta \int f(\beta \mu_\theta^{1-\frac{2}{d}}) d\mu_\theta + \mathrm{Rem}$$

where f is as above.

[Leblé-S '15] any $d \geq 2$: $\mathrm{Rem} = o_\beta(N)$ (also for 1D log gas)
[Bauerschmidt-Bourgade-Nikula-Yau '16] $d = 2$: $\mathrm{Rem} = O_\beta(N^{1-\varepsilon})$
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