Combined mean-field and semiclassical limits of large fermionic systems

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1. Setting and introduction from many particle Schrödinger to Vlasov
2. Reformulation of the many body Schrödinger equation by using Husimi measures and the main result
3. Proof strategy and Uniform estimates

Setting and introduction for the combined limits from many particle Schrödinger to Vlasov

Time dependent N-particle Schrödinger equation

$$\begin{cases} \mathrm{i}\hbar\partial_t \Psi_{N,t} = \left[-\frac{\hbar^2}{2} \sum_{j=1}^{N} \Delta_{q_j} + \frac{1}{2N} \sum_{i \neq j}^{N} V(q_i - q_j) \right] \Psi_{N,t}, \\ \Psi_{N,0} = \Psi_N, \end{cases}$$

- the initial data and the time-dependent states $\Psi_{N,t}$ are both in $L^2_a(\mathbb{R}^{3N})$ functions (fermionic case)
- V is the interacting potential
- $\hbar = N^{-\frac{1}{3}}$ (Combined semiclassical and mean field limit)

Vlasov equation

$$\partial_t m_t(q,p) + p \cdot \nabla_q m_t(q,p) = \nabla\big(V * \rho_t\big)(q) \cdot \nabla_p m_t(q,p),$$

Goal: $N \to$ *infinity*. **Schrödinger → Vlasov**

Mean field limit and semiclassical limit

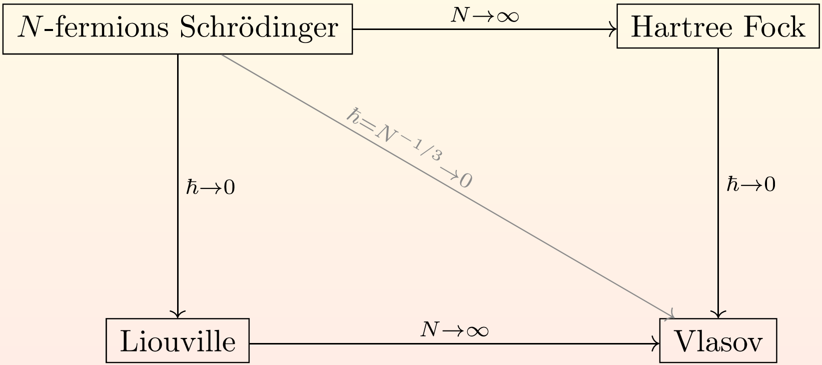

Table-1: Picture from Golse, Mouhot, and Paul

State of arts (not complete!)

- 1980's Narnhofer, Neunzert, Sewell, Spohn ......(for Bosons and Fermions)
- Mean filed limit
  - Bosons (to Hartree) Bardos, Erdös, Golse, Mauser, Yau, Rodnianski, Schlein, Chen, Lee, Pickl, Petrat......
  - Fermions (to Hartree Fock) Benedikter, Porta, ,Schlein, Bach, Breteaux, Petrat, Pickl, Tzaneteas......
- Semiclassical limit (from HF to Vlasov) Benedikter, Porta, Saffirio, Dietler, Rademacher, Schlein...... Wigner transform
- Combined mean field and semiclassical limit (to Vlasov), Golse, Mouhot, Paul Husimi measure (Wigner measure convoluted with Gaussian) (for ∇V Lip)

We hope to contribute in the combined limits with less regular V.
To propose new approach.

Reformulation of the many body Schrödinger equation by using Husimi measures and the main result

Motivated by **Fournais, Lewin and Solovej's work** (stationary case)

Main tool: The Husimi measure.

For any real-valued $f \in L^2(\mathbb{R}^3)$ with $\|f\|_{L^2} = 1$, the **coherent state** is

$$f^{\hbar}_{q,p}(y) := \hbar^{-\frac{3}{4}} f\left(\frac{y-q}{\sqrt{\hbar}}\right) e^{\frac{i}{\hbar} p \cdot y}$$

The k-particle **Husimi measure** is defined as, for any $1 \leqslant k \leqslant N$

$$m^{(k)}_N(q_1, p_1, \ldots, q_k, p_k) := \left\langle \psi_N, a^*(f^{\hbar}_{q_1,p_1}) \cdots a^*(f^{\hbar}_{q_k,p_k}) a(f^{\hbar}_{q_k,p_k}) \cdots a(f^{\hbar}_{q_1,p_1}) \psi_N \right\rangle$$

where $\psi_N \in \mathcal{F}^{(N)}_a$ is the N-fermionic states.

Remark: Husimi measure measures how many particles are in the k semiclassical boxes with length scaled of $\sqrt{\hbar}$ centered in its respectively phase-space pair, $(q_1, p_1), \ldots, (q_k, p_k)$.

Reformulation of the many body Schrödinger equation

Let $\psi_{N,t} \in L^2_a(\mathbb{R}^{3N})$ be anti-symmetric N-particle state satisfying the Schrödinger equation. Then

$$\begin{aligned}&\partial_t m^{(1)}_{N,t}(q_1,p_1) + p_1 \cdot \nabla_{q_1} m^{(1)}_{N,t}(q_1,p_1)\\ =&\frac{1}{(2\pi)^3}\nabla_{p_1}\cdot \iint \mathrm{d}q_2\mathrm{d}p_2 \nabla V(q_1-q_2) m^{(2)}_{N,t}(q_1,p_1,q_2,p_2) + \nabla_{q_1}\cdot R_1 + \nabla_{p_1}\cdot \tilde{R}_1,\end{aligned}$$

where the remainder terms R_1 and $\tilde{R}_1$, are given by

$$\begin{aligned}R_1 :=&\hbar \Im \left\langle \nabla_{q_1} a(f^{\hbar}_{q_1,p_1})\psi_{N,t}, a(f^{\hbar}_{q_1,p_1})\psi_{N,t}\right\rangle,\\ \tilde{R}_1 :=&\frac{1}{(2\pi)^3}\cdot \Re \iint \mathrm{d}w\mathrm{d}u \iint \mathrm{d}y\mathrm{d}v \iint \mathrm{d}q_2\mathrm{d}p_2 \int_0^1 \mathrm{d}s \nabla V\big(su+(1-s)w-y\big)\\ &\quad f^{\hbar}_{q_1,p_1}(w)\overline{f^{\hbar}_{q_1,p_1}(u)} f^{\hbar}_{q_2,p_2}(y)\overline{f^{\hbar}_{q_2,p_2}(v)} \left\langle a_w a_y \psi_{N,t}, a_u a_v \psi_{N,t}\right\rangle\\ &\quad -\frac{1}{(2\pi)^3}\iint \mathrm{d}q_2\mathrm{d}p_2 \nabla V(q_1-q_2) m^{(2)}_{N,t}(q_1,p_1,q_2,p_2).\end{aligned}$$

For $1 < k \leqslant N$, we have the following hierarchy

$$\partial_t m_{N,t}^{(k)}(q_1,p_1,\dots,q_k,p_k) + \vec{p}_k \cdot \nabla_{\vec{q}_k} m_{N,t}^{(k)}(q_1,p_1,\dots,q_k,p_k)$$
$$= \frac{1}{(2\pi)^3}\nabla_{\vec{p}_k} \cdot \iint \mathrm{d}q_{k+1}\mathrm{d}p_{k+1}\nabla V(q_j - q_{k+1}) m_{N,t}^{(k+1)}(q_1,p_1,\dots,q_{k+1},p_{k+1})$$
$$+ \nabla_{\vec{q}_k} \cdot R_k + \nabla_{\vec{p}_k} \cdot \tilde{R}_k + \widehat{R}_k,$$

where the remainder terms are denoted as

$$R_k := \hbar \Im \left\langle \nabla_{\vec{q}_k} \left(a(f^{\hbar}_{q_k,p_k}) \cdots a(f^{\hbar}_{q_1,p_1}) \right) \psi_{N,t}, a(f^{\hbar}_{q_k,p_k}) \cdots a(f^{\hbar}_{q_1,p_1}) \psi_{N,t} \right\rangle,$$

$$(\tilde{R}_k)_j := \frac{1}{(2\pi)^3} \Re \int (\mathrm{d}w\mathrm{d}u)^{\otimes k} \int \mathrm{d}y \left[\int_0^1 \mathrm{d}s \nabla V(su_j + (1-s)w_j - y) \right] \left(f^{\hbar}_{q,p}(w) \overline{f^{\hbar}_{q,p}(u)} \right)^{\otimes k}$$
$$\iint \mathrm{d}\tilde{q} d\tilde{p}\, f^{\hbar}_{\tilde{q},\tilde{p}}(y) \int \mathrm{d}v\, \overline{f^{\hbar}_{\tilde{q},\tilde{p}}(v)} \left\langle a_{w_k} \cdots a_{w_1} a_y \psi_{N,t}, a_{u_k} \cdots a_{u_1} a_v \psi_{N,t} \right\rangle$$
$$- \frac{1}{(2\pi)^3} \iint \mathrm{d}q_{k+1}\mathrm{d}p_{k+1} \nabla V(q_j - q_{k+1}) m_{N,t}^{(k+1)}(q_1,p_1,\dots,q_{k+1},p_{k+1}),$$

$$\widehat{R}_k := \frac{\hbar^2}{2} \Im \int (\mathrm{d}w\mathrm{d}u)^{\otimes k} \sum_{j \neq i}^{k} \left[V(u_j - u_i) - V(w_j - w_i) \right] \left(f^{\hbar}_{q,p}(w) \overline{f^{\hbar}_{q,p}(u)} \right)^{\otimes k}$$
$$\left\langle a_{w_k} \cdots a_{w_1} \psi_{N,t}, a_{u_k} \cdots a_{u_1} \psi_{N,t} \right\rangle$$

What's next?

- $\forall k \geq 1$, the sequence of k particle Husimi measures, $m_{N,t}^{(k)}$, is weakly compact in $L^1(\mathbb{R}^{6k})$
- All the remainder terms, R_k, $\tilde{R}_k$ and $\widehat{R}_k$, converge to 0 in weak sense.

Main result

Theorem $f \in H^1(\mathbb{R}^3)$ and has compact support, $V(-x) = V(x)$ and $V \in W^{2,\infty}(\mathbb{R}^3)$. Let $\psi_{N,t}$ be the solution of Schrödinger equation, $m_{N,t}^{(k)}$ be its k Husimi measures. If $m_N^{(1)}$ satisfies

$$\int \mathrm{d}q_1 \mathrm{d}p_1 (|p_1|^2 + |q_1|) m_N^{(1)}(q_1, p_1) \le C.$$

Then for all $t \ge 0$, $m_{N,t}^{(k)}$ has a subsequence which converges weakly to $m_t^{(k)}$ in $L^1(\mathbb{R}^{6k})$, where $m_t^{(k)}$ is a solution of the following infinite Vlasov hierarchy in the sense of distribution, i.e. it satisfies $\forall k \ge 1$ that

$$\begin{aligned}
&\partial_t m_t^{(k)}(q_1, p_1, \ldots, q_k, p_k) + \vec{p}_k \cdot \nabla_{\vec{q}_k} m_t^{(k)}(q_1, p_1, \ldots, q_k, p_k) \\
&\quad = \frac{1}{(2\pi)^3} \nabla_{\vec{p}_k} \cdot \iint \mathrm{d}q_{k+1} \mathrm{d}p_{k+1} \nabla V(q_j - q_{k+1}) m_t^{(k+1)}(q_1, p_1, \ldots, q_{k+1}, p_{k+1}).
\end{aligned}$$

Remark: This a first stage result, which is rather weak. The main goal of this talk is to introduce our new approach.

Corollary Assume further that the initial data can be factorized, i.e. for all $k \geq 1$,

$$\|m_N^{(k)} - m_0^{\otimes k}\|_{L^1} \to 0, \quad \text{as } N \to \infty.$$

If the infinite hierarchy has a unique solution and m_t be the solution to the classical Vlasov equation. Then

$$W_1\left(m_{N,t}^{(1)},\ m_t\right) \longrightarrow 0, \quad \text{as } N \to \infty.$$

Further remarks

- The assumptions for initial data can be realized by choosing ψ_N to be the Slater-determinant.
- The assumption that f has compact support are expected to be weakened to the situation that $f \in H^1(\mathbb{R}^3)$, $|x|f(x) \in L^2(\mathbb{R}^3)$.
- Within this framework, it is hopeful to weaken the regularity assumption of V, (Goal: Coulomb potential). These will be our future projects.
- In this context, we have applied the BBGKY hierarchy, the intermediate mean field approximation Hartree Fock system has not been benefited. With Hartree Fock approximation, one can do direct factorization in the equation for $m_{N,t}^{(1)}$. In this direction, we expect to derive the convergence rate.

Proof strategy and Uniform estimates

Properties of k-particle Husimi measure
(cited from [Fournais, Lewin, and Solovej])

For $\psi_N \in L^2_a(\mathbb{R}^{3N})$ with $\|\psi_N\| = 1$, the following properties hold true for its Husimi measure $m_N^{(k)}$: $1 \leqslant k \leqslant N$.

- $m_N^{(k)}(q_1, p_1, \ldots, q_k, p_k)$ is symmetric,
- $\frac{1}{(2\pi)^{3k}} \int (\mathrm{d}q\mathrm{d}p)^{\otimes k} m_N^{(k)}(q_1, \ldots, p_k) = \dfrac{N(N-1)\cdots(N-k+1)}{N^k}$,
- $\frac{1}{(2\pi\hbar)^3} \int \mathrm{d}q_k \mathrm{d}p_k \; m_N^{(k)}(q_1, \ldots, p_k) = (N-k+1) m_N^{(k-1)}(q_1, \ldots, p_{k-1})$
- $0 \leqslant m_N^{(k)} \leqslant 1$ a.e.

Remark: These properties hold also for the Husimi measure $m_{N,t}^{(k)}$, which is defined from the solution of the Schrödinger equation $\psi_{N,t}$.

Number operator $\mathcal{N}$ and the total mass

Boundedness of number operator The expectation of number operator $\mathcal{N}$ defined by $\mathcal{N} = \int a_x^* a_x dx$ is conserved in time. More precisely, for finite $1 \leqslant k \leqslant N$, we have

$$\left\langle \psi_{N,t}, \frac{\mathcal{N}^k}{N^k} \psi_{N,t} \right\rangle = \frac{N(N-1)\cdots(N-k+1)}{N^k} \leqslant 1.$$

Remark

$$\begin{aligned}
\langle \psi_{N,t}, \mathcal{N} \psi_{N,t} \rangle &= \int \mathrm{d}x \, \langle \psi_{N,t}, a_x^* a_x \psi_{N,t} \rangle \\
&= \frac{1}{(2\pi\hbar)^3} \iint \mathrm{d}q \mathrm{d}p \int \mathrm{d}x \left\langle \psi_{N,t}, a_x^* f_{q,p}^{\hbar}(x) \left(\int \mathrm{d}y \; a_y \overline{f_{q,p}^{\hbar}(y)} \right) \psi_{N,t} \right\rangle \\
&= \frac{1}{(2\pi\hbar)^3} \iint \mathrm{d}q \mathrm{d}p \, \langle \psi_{N,t}, a^*(f_{q,p}^{\hbar}) a(f_{q,p}^{\hbar}) \psi_{N,t} \rangle \\
&= \frac{1}{(2\pi\hbar)^3} \iint \mathrm{d}q \mathrm{d}p \; m_{N,t}^{(1)}(q,p) = N.
\end{aligned}$$

Bound on localized number operator

Let R be the radius of a ball such that the volume is 1. Then, for all $1 \leqslant k \leqslant N$, we have

$$\int (\mathrm{d}q\mathrm{d}x)^{\otimes k} \left\langle \psi_N, \left(\prod_{n=1}^{k} \chi_{|x_n - q_n| \leqslant \sqrt{\hbar} R} \right) a^*_{x_1} \cdots a^*_{x_k} a_{x_k} \cdots a_{x_1} \psi_N \right\rangle \leqslant \hbar^{-\frac{3}{2}k}.$$

Idea.

$$\int \mathrm{d}x_j \left(\int \mathrm{d}q_j \; \chi_{|x_j - q_j| \leqslant \sqrt{\hbar} R} \right) \left\langle \psi_N, a^*_{x_j} a_{x_j} \psi_N \right\rangle = \hbar^{\frac{3}{2}} \langle \psi_N, \mathcal{N} \psi_N \rangle \leqslant \hbar^{-\frac{3}{2}}.$$

Estimate of oscillation (from harmonic analysis)

For $\phi(p) \in C_0^\infty(\mathbb{R}^3)$ and $\Omega_\hbar := \{x \in \mathbb{R}^3; \; \max_{1 \leq j \leq 3} |x_j| \leq \hbar^\alpha\}$, it holds that for every $\alpha \in (0,1)$, $s \in \mathbb{N}$, and $x \in \mathbb{R}^3 \backslash \Omega_\hbar$,

$$\left| \int_{\mathbb{R}^3} \mathrm{d}p \; e^{\frac{i}{\hbar} p \cdot x} \varphi(p) \right| \leqslant C \hbar^{(1-\alpha)s},$$

where C depends on the compact support and the C^s norm of ϕ.

Kinetic energy and p second moment of the Husimi measure

The kinetic energy operator $\mathcal{K}$ is given by

$$\mathcal{K} = \frac{\hbar^2}{2} \int \mathrm{d}x \; \nabla_x a_x^* \nabla_x a_x,$$

Direct computation shows that

$$\left\langle \psi_{N,t}, \frac{\mathcal{K}}{N} \psi_{N,t} \right\rangle = \frac{1}{(2\pi)^3} \iint \mathrm{d}q_1 \mathrm{d}p_1 \; |p_1|^2 m_{N,t}^{(1)}(q_1, p_1) + \hbar \int \mathrm{d}q \, |\nabla f\,(q)|^2 .$$

Estimate for kinetic energy Assume $V \in W^{1,\infty}$, then the kinetic energy is bounded in the following

$$\left\langle \psi_{N,t}, \frac{\mathcal{K}}{N} \psi_{N,t} \right\rangle \leqslant 2 \left\langle \psi_N, \frac{\mathcal{K}}{N} \psi_N \right\rangle + Ct^2,$$

where C depends on $\|\nabla V\|_\infty$.

q moment estimate for Husimi measure Assume $V \in W^{1,\infty}$, then the first q moment of the Husimi measure is bounded in the following

$$\iint \mathrm{d}q_1 \mathrm{d}p_1 \; |q_1| m_{N,t}^{(1)}(q_1, p_1) \leq C(1 + t^3).$$

Idea. Use the reformulated equation for $m_{N,t}^{(1)}$.

$$\begin{aligned} &\partial_t \iint \mathrm{d}q_1 \mathrm{d}p_1 \; |q_1| m_{N,t}^{(1)}(q_1, p_1) = \iint |q_1| \partial_t m_{N,t}^{(1)}(q_1, p_1) \\ &\leqslant \iint \mathrm{d}q_1 \mathrm{d}p_1 \left(|p_1| m_{N,t}^{(1)}(q_1, p_1) + |R_1| \right), \end{aligned}$$

where R_1 is the remainder term, it can be bounded by the estimates for number operator.

$$\iint \mathrm{d}q_1 \mathrm{d}p_1 \; |R_1| \leqslant (2\pi)^3 \sqrt{\hbar} \left[\int \mathrm{d}\tilde{q} \left| \nabla f(\tilde{q}) \right|^2 \right]^{\frac{1}{2}},$$

moment estimates for k partical Husimi measure

By symmetric property of the Husimi measure, we have the following

$$\int (\mathrm{d}q\mathrm{d}p)^{\otimes k} \ (|\vec{q}_k| + |\vec{p}_k|^2) m_{N,t}^{(k)}(q_1, \dots, p_k) \leqslant C(1+t^3)$$

where C is a constant dependent on k, $\iint \mathrm{d}q\mathrm{d}p(|q| + |p|^2) m_N^{(1)}(q,p)$, and $\|\nabla V\|_\infty$.

Weak compactness of Husimi measures

The above moment estimates together with the L^∞ estimates imply that for any k, the sequence $m_{N,t}^{(k)}$ is uniformly integrable.

Since $\|m_{N,t}^{(k)}\|_{L^1} \leqslant (2\pi)^{3k}$, Dunford-Pettis theorem says that this sequence is weakly compact in L^1.

Uniform estimates for the remainder terms

With the help of all the estimates we have obtained before, especially the number operator, the kinetic energy, the localized number operator, and the oscillation estimate, we are able to obtain the following estimates for remainder terms in the reformulation of the Schrödinger equation.

Let $\Phi \in C_0^\infty(\mathbb{R}^{6k})$ be any test function, then it holds

$$\left| \int (\mathrm{d}q\mathrm{d}p)^{\otimes k} \Phi(q_1, p_1, \ldots, q_k, p_k) \nabla_{\vec{q}_k} \cdot R_k \right| \leqslant C\hbar^{\frac{1}{2}-},$$

$$\left| \int (\mathrm{d}q\mathrm{d}p)^{\otimes k} \Phi(q_1, p_1, \ldots, q_k, p_k) \widehat{R}_k \right| \leqslant C\hbar^{3-},$$

$$\left| \int (\mathrm{d}q\mathrm{d}p)^{\otimes k} \Phi(q_1, p_1, \ldots, q_k, p_k) \nabla_{\vec{p}_k} \cdot \tilde{R}_k \right| \leqslant C\hbar^{\frac{1}{2}-},$$

where C is a constant does not depend on $\hbar$.

The estimate of $\tilde{R}_1$ as an example. For arbitrary test function Φ, we have

$$
\begin{aligned}
&\left| \iint \mathrm{d}q_1 \mathrm{d}p_1 \nabla_{p_1}\Phi(q_1,p_1)\cdot \tilde{R}_1 \right| \\
=&\frac{1}{\hbar^3}\Bigg| \iint \mathrm{d}q_1 \mathrm{d}p_1 \nabla_{p_1}\Phi(q_1,p_1)\cdot \iint \mathrm{d}w\mathrm{d}u \iint \mathrm{d}y\mathrm{d}v \iint \mathrm{d}q_2\mathrm{d}p_2 \\
&\left[\int_0^1 \mathrm{d}s \nabla V(su+(1-s)w-y) - \nabla V(q_1-q_2)\right] e^{\frac{i}{\hbar}p_1\cdot(w-u)} e^{\frac{i}{\hbar}p_2\cdot(y-v)} \\
&f\left(\frac{w-q_1}{\sqrt{\hbar}}\right)\overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} f\left(\frac{y-q_2}{\sqrt{\hbar}}\right)\overline{f\left(\frac{v-q_2}{\sqrt{\hbar}}\right)} \langle a_w a_y \psi_{N,t}, a_u a_v \psi_{N,t}\rangle \Bigg| \\
=&(2\pi)^3\hbar^{\frac{3}{2}}\Bigg| \iint \mathrm{d}q_1 \mathrm{d}p_1 \nabla_{p_1}\Phi(q_1,p_1)\cdot \iint \mathrm{d}w\mathrm{d}u \iint \mathrm{d}y\mathrm{d}\tilde{q}_2 \\
&\left[\int_0^1 \mathrm{d}s \nabla V(su+(1-s)w-y) - \nabla V(q_1-y+\sqrt{\hbar}\tilde{q}_2)\right] \\
&f\left(\frac{w-q_1}{\sqrt{\hbar}}\right)\overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} e^{\frac{i}{\hbar}p_1\cdot(w-u)} \left|f\left(\tilde{q}_2\right)\right|^2 \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t}\rangle \Bigg|.
\end{aligned}
$$

Insert a term $\nabla V(q_1 - y)$ and use the triangle inequality, we will have to estimates two terms I and II. Now we again use II as an example to show the main idea.

$$II = (2\pi)^3 \hbar^{\frac{3}{2}} \Bigg| \iint \mathrm{d}q_1 \mathrm{d}p_1 \nabla_{p_1} \Phi(q_1, p_1) \cdot \iint \mathrm{d}w \mathrm{d}u \iint \mathrm{d}y \mathrm{d}\tilde{q}_2 \left[\nabla V(q_1 - y) - \nabla V(q_1 - y + \sqrt{\hbar}\tilde{q}_2)\right] \left(\chi_{(w-u)\in\Omega_\hbar^c} + \chi_{(w-u)\in\Omega_\hbar}\right) f\left(\frac{w - q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u - q_1}{\sqrt{\hbar}}\right)} e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \left|f(\tilde{q}_2)\right|^2 \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle \Bigg|$$

Two integrals are important with $\Omega_\hbar = \{x : |x| \leq \hbar^\alpha\}$.

$$\left| \int \mathrm{d}p_1 \; e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \chi_{(w-u)\in\Omega_\hbar^c} \nabla_{p_1} \Phi(q_1, p_1) \right| \quad \text{gives an } \hbar^{(1-\alpha)s}$$

and $\left| \int \mathrm{d}p_1 \; e^{\frac{i}{\hbar} p_1 \cdot (w-u)} \chi_{(w-u)\in\Omega_\hbar} \nabla_{p_1} \Phi(q_1, p_1) \right|$

By the oscillation estimate, Lip of ∇V, and the estimate for localized number operator, we have

$$II_1 \leqslant C\hbar^{\frac{3}{2}+\frac{1}{2}+(1-\alpha)s} \int \mathrm{d}q_1 \iint \mathrm{d}w\mathrm{d}u \int \mathrm{d}y \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right|$$
$$\chi_{|w-q_1|\leqslant\sqrt{\hbar}R}\chi_{|u-q_1|\leqslant\sqrt{\hbar}R}\|a_w a_y \psi_{N,t}\|\|a_u a_y \psi_{N,t}\| \leqslant C\hbar^{(1-\alpha)s-1}.$$

For the case where $w-u\Omega$, the oscillation is useless we have

$$II_2 \leqslant C\hbar^{\frac{3}{2}+\frac{1}{2}} \int \mathrm{d}q_1 \iint \mathrm{d}w\mathrm{d}u \int \mathrm{d}y \; \chi_{(w-u)\in\Omega_\hbar} \cdot \left| f\left(\frac{w-q_1}{\sqrt{\hbar}}\right) \overline{f\left(\frac{u-q_1}{\sqrt{\hbar}}\right)} \right|$$
$$\chi_{|w-q_1|\leqslant\sqrt{\hbar}R}\chi_{|u-q_1|\leqslant\sqrt{\hbar}R} \left| \langle a_w a_y \psi_{N,t}, a_u a_y \psi_{N,t} \rangle \right|$$
$$\leqslant C\hbar^{-1} \left(\int \mathrm{d}\tilde{w} \, |f(\tilde{w})|^2 \int \mathrm{d}\tilde{u} \; \chi_{|\tilde{w}-\tilde{u}|\leq\hbar^{\alpha+\frac{1}{2}}} |f(\tilde{u})|^2 \right)^{\frac{1}{2}} \leq C\hbar^{\alpha-\frac{1}{2}}.$$

Choose $s = \frac{1+2\alpha}{2(1-\alpha)}$ such that $II_1 + II_2$ is of the order $\hbar^{\alpha-\frac{1}{2}}$, $\forall \frac{1}{2} < \alpha < 1$.

Summary and future projects

- We showed an alternative strategy to prove the combined mean field and semiclassical limit from many body Fermionic Schrödinger to Vlasov equation.
- The main contribution in this work is the uniform estimates for the remainder terms in the reformulated Schrödinger equation.

Future projects

- Weaken the assumption of f to $f \in H^1(\mathbb{R}^3)$, $|x|f(x) \in L^2(\mathbb{R}^3)$.
- Weaken the regularity assumption of V, (Goal: Coulomb potential).
- Convergence rate estimate in an appropriate distance. (In this context, we have applied the BBGKY hierarchy, the intermediate mean field approximation Hartree Fock system has not been benefited. With Hartree Fock approximation, one can do direct factorization in the equation for $m^{(1)}_{N,t}$.)

THANK YOU!