Generalised FEs: Domain Decomposition, Optimal Local Approximation & Model Order Reduction

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Parallel Solution Methods for Systems Arising from PDEs

CIRM – Luminy, Marseille, September 16th, 2019

Overview

- Problem Formulation & Motivation
- Robust Subspace Correction vs. Multiscale Discretisation
- **Key message:** use weighted norms for contrast independence

Three Theoretical Tools

(Three) Practical 'Knobs'

- Beyond scalar elliptic problem: anisotropic linear elasticity
- High performance implementation of GenEO
- Some Numerical Results
- Outlook – GenEO as a surrogate in Multilevel MCMC

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Model Problem

- Elliptic PDE in bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$

$$-\nabla \cdot (\boldsymbol{\alpha} \nabla u) = f \qquad + \text{ suitable BCs on } \partial\Omega$$

Issues adressed even more pronounced in other eqns., e.g. linear elasticity.

- **Strongly varying** coefficient $\boldsymbol{\alpha}(x) \geq 1$ (otherwise rescale eqn.) (scalar $\boldsymbol{\alpha}$, or quasi-isotropic tensor $\boldsymbol{\alpha}$ with $\lambda_{\max}(\boldsymbol{\alpha}) \sim \lambda_{\min}(\boldsymbol{\alpha})$)

- FE discretisation (p.w. lin. V^h): $a(u_h, v_h) = (f, v_h)\ \forall v_h \in V_h$

- **Two possible aims:**
  - h-optimal, α-robust parallel **solver** (fine mesh $\mathcal{T}^h$, α **resolved**)
  - H-optimal, α-robust **approximation** in coarse space V^H (α **not resolved: "Upscaling"** – no scale separation!)

- **Key Question** (for both): **Robust coarse space**

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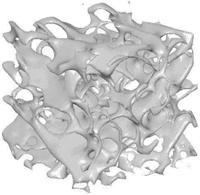
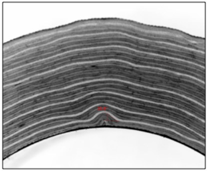
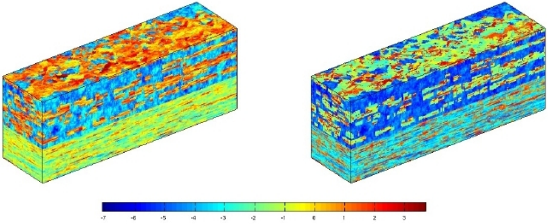
- **Elasticity**, e.g. in bone or carbon fibre composites





- **Subsurface flow**, e.g. in an oil reservoir
  (SPE10 benchmark)



- ... many more ...

Initial Remarks

- Complicated variation of $\alpha(x)$ on many scales ($h \ll \mathrm{diam}(\Omega)$) Hard to **resolve** by "geometric" **coarse** mesh!
- **Goal A:** Efficient & scalable multilevel parallel solver
  - **robust** w.r.t. mesh size h ($\Leftrightarrow$ w.r.t. problem size n: $O(n)$ cost)
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  - Discretisation in **"special"** coarse space V^H $\rightarrow$ **Upscaling**
  - Approximation depends on (subgrid) variation & contrast in α **!**
- Robust multiscale space is **expensive** for **general** coefficients
- Unless we have periodicity, scale separation, multiple RHSs, parameter dependence, not clear why Goal B over Goal A
- Coefficient-robust theory for **Goal B** much less well developed **!**

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Types of Multiscale Methods & Theory (incomplete list)

- Adaptive FEs ..., [Babuska, Rheinboldt, 1978]
- Generalised FEs [Babuska, Osborn, 1983]
- Numerical Upscaling ..., [Durlofsky, 1991]
- Multiscale Finite Elements [Hou, Wu, 1997], ...
- Variational Multiscale Method [Hughes et al, 1998]
- Multigrid Based Upscaling [Moulton, Dendy, Hyman, 1998]
- Multiscale Finite Volume Methods [Jenny, Lee, Tchelepi, 2003]
- Heterogeneous Multiscale Method [E, Engquist, 2003]
- Multiscale Mortar Spaces [Arbogast, Wheeler et al, 2007]
  (& other DD based methods)
- Adaptive Multiscale FVMs/FEs [Durlovsky, Efendiev, Ginting, 2007]
- Energy minimising bases [Dubois, Mishev, Zikatanov, 2009]
- Locally spectral (Generalised MsFEM) [Efendiev, Galvis, Wu, 2010]
- ... etc ...

Simplifying Assumptions & Theory (incomplete list of refs)

1. Periodicity $\Rightarrow$ Homogenisation theory ..., [Hou, Wu, 1997], ...
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   (high contrast, no periodicity, no scale separation)
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Combine **5** and **6** to cover **more general high contrast coeffs.**

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Classical theory and more recent ideas

Classical theory

- in H^1 and $H^{1/2}$-norm
- based on standard **Poincaré inequalities**
- and robustness of **weighted L_2-projections**

  [Bramble, Xu, Math Comp 91], ... **(for resolving coarse grids!)**

More recent ideas

- directly in the **energy norm**
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  [Galvis, Efendiev, 2010], [Pechstein, RS, 2011 & 2012]
- and an **abstract Bramble-Hilbert Lemma** $\longleftarrow$ $\boxed{\text{This Talk!}}$

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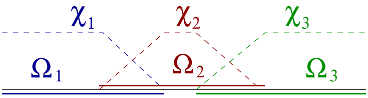
Precondition by solving (exactly or approximately) in subspaces $V_0, V_1, \ldots V_L \subset V_h$ in parallel **(additive)** or successively **(multiplicative)**

Two-level overlapping Schwarz

$V_\ell = \{v_h \in V_h : \mathrm{supp}(v_h) \subset \Omega_\ell\}$ with overlapping partitioning $\{\Omega_\ell\}_{\ell=1}^L$ of Ω



and $V_0 = \mathrm{span}\{\Phi_j \in V_h : j = 1, \ldots, N\}$ (abstract)

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Geometric Multigrid & BPX

Subspace Correction Methods (e.g. two-level Schwarz or multigrid)

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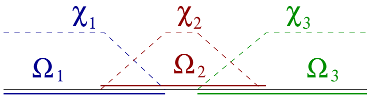

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Geometric Multigrid & BPX

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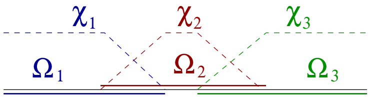

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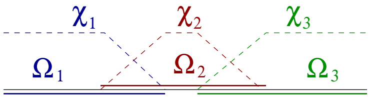

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Two-level Overlapping Schwarz – Abstract Theory

Let
$$\|v\|_{0,\alpha}^2 = \int_\Omega \alpha v^2 \,\mathrm{d}x \quad \text{(weighted } L_2\text{-norm)}$$

Analogously to classical theory (in H^1-seminorm and L^2-norm) we have:

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If there exists an operator $\Pi : V_h \to V_0$ such that, for all $v \in V_h$,

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"Nodal" Coarse Spaces, e.g. Piecewise Linears

- $V_0 = V_H$ cts. p.w. linears on a **shape-regular** grid $\mathcal{T}_H$
  No assumption that **coefficient is resolved** on $\mathcal{T}_H$!

Theorem [RS, Vassilevski, Zikatanov, SINUM 2012]

For all $K \in \mathcal{T}_H$, let $C_K^P > 0$ be the best constant s.t. for all $v \in V_h$

$$\inf_{\xi \in \mathbb{R}} \|v - \xi\|_{0,\alpha,\omega_K}^2 \le C_K^P \, H^2 \, \|v\|_a^2 \qquad \textbf{(WPI)}$$

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(WPI) linked **directly** to **local quasi-monotonicity** **[Pechstein, RS, 2012]**

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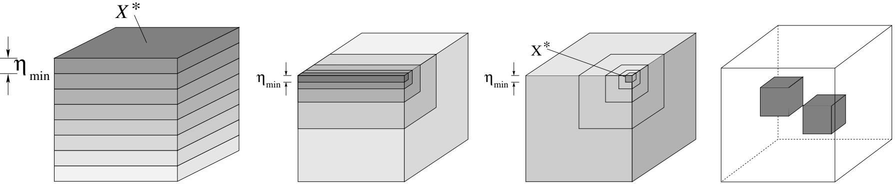
- **Bounds** for the effective Poincaré constant C_T^P :
  Darker colour means higher permeability.



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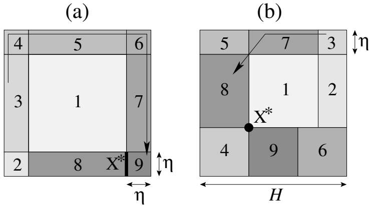

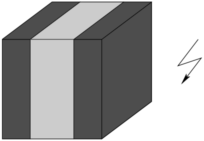

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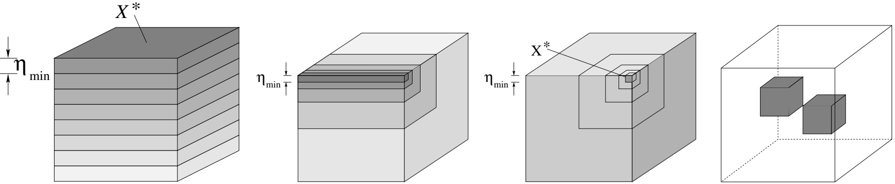

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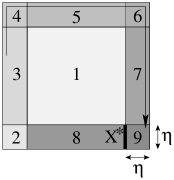

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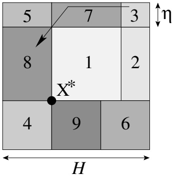

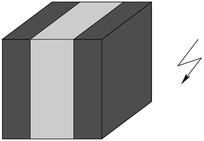

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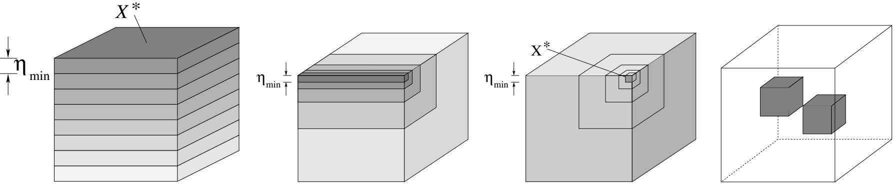

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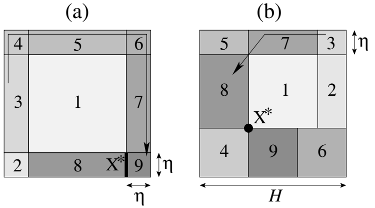

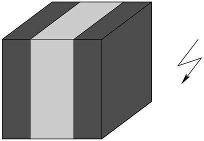

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	C_K^P bounded for all ω_K		C_K^P **not** bdd. on some ω_K	
$\widehat{\alpha}$	λ_1^{-1}	#MG Its (tol $= 10^{-8}$)	λ_1^{-1}	#MG Its (tol $= 10^{-8}$)
10^1	1.69	10	1.72	10
10^2	2.75	14	3.87	19
10^3	3.32	12	14.5	23
10^4	3.42	10	115.5	70
10^5	3.42	10	1125	76

In right table islands closer to each other!

Guiding principle for choice of "nodal" coarse spaces

$\mathcal{T}_H$ sufficiently fine (**locally**) s.t. $\alpha(x)$ **quasi-monotone** on all ω_K

When it is difficult to ensure **quasi-monotonicity** on all ω_K

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Suppose $\{\Omega_\ell\}_{\ell=1}^L$ is an overlapping partition of Ω and $\{\chi_\ell\}_{\ell=1}^L$ an associate partition of unity w. $\|\chi_\ell\|_\infty \lesssim 1$ & $\|\nabla\chi_\ell\|_\infty \lesssim \delta_\ell^{-1} \lesssim H_\ell^{-1}$

(This could be a set of FE basis functions and their supports.)

Local Energy Minimization subject to Functional Constraints

For each subdomain Ω_ℓ, assume that we have a collection of **linear functionals** $\{f_{\ell,j}\}_{j=1}^{m_\ell} \subset V_h(\Omega_\ell)'$ and let

$$\Psi_{\ell,j} = \arg\min_{v \in V_h(\Omega_\ell)} \|v\|_{a,\Omega_\ell}^2 \quad \text{subject to} \quad f_{\ell,k}(\Psi_{\ell,j}) = \delta_{jk}\,. \qquad (2)$$

Now, with I_h the standard nodal interpolant onto V_h, let

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A1. a is positive semi-definite and defines a semi-norm $|\cdot|_a$ on V and $\sqrt{\|v\|^2 + |v|_a^2}$ defines a norm on V.

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An abstract Bramble–Hilbert Lemma – Tool 1

Theorem (RS, Vassilevski, Zikatanov, MMS 2011)

Let Assumptions **A1-3** *hold. Then* $\pi u = \sum_k f_k(u)\psi_k$ *satisfies*

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(Note that this is **independent** *of the constants* c_q *and* c_f *in* **A2** *and* **A3***.)*

Proof.

- First one notes that given $u \in V$, πu minimizes energy subject to $f_k(v) = f_k(u)$. Thus $|\pi u|_a \le |u|_a$ by construction.
- Secondly, from **A3**, the fact that $f_k(v - \Pi v) = 0 \ \forall k$ and the stability estimate, we get

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- First one notes that given $u \in V$, πu minimizes energy subject to $f_k(v) = f_k(u)$. Thus $|\pi u|_a \le |u|_a$ by construction.
- Secondly, from **A3**, the fact that $f_k(v - \Pi v) = 0\ \forall k$ and the stability estimate, we get

$$\begin{aligned} \|v - \Pi v\|^2 &\le c_a|v - \Pi v|_a^2 + c_f \sum_{l=1}^{m} |f(v - \Pi v)|^2 \\ &= c_a|v - \Pi v|_a^2 \le 2c_a(|v|_a^2 + |\Pi v|_a^2) \le 4c_a|v|_a^2\,. \end{aligned}$$

(can lose factor 2 by more careful bound)

In our **specific model problem** considered above

- Assumption **A1** is naturally satisfied on any subdomain Ω_ℓ with $\mathcal{H} = L_2(\Omega_\ell)$ and $\|v\| = \int_{\Omega_\ell} \alpha v^2 \, dx$ (**weighted** L^2-norm **!**)

- Assumption **A2** simply means the functionals $\{f_k\}$ should be linearly independent.

- Coarse space robustness reduced to verifying Assumption **A3**
  - For **one** functional reduces to **(WPI)** and quasi-monotonicity.
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More importantly: can be applied also to other problems, e.g. **elasticity, Stokes, ...**

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Choice of functionals – 'Knob' 1

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But also in multiscale literature:

[Babuska, Lipton '11]:

[Peterseim, RS '16] (LOD):

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How can we use this **abstract Bramble-Hilbert Lemma** to obtain a contrast-robust approximation theory for LOD or GFEM?

Localizable Orthogonal Decomposition (LOD)

- FE space $V_H := \text{span}\{\Phi_j\}$ associated with (coarse) FE mesh $\mathcal{T}_H$
- Quasi-interpolation operator $\Pi : V_h \to V_H$ with

$$\Pi v := \sum_j \frac{(v, \Phi_j)_{L^2(\Omega)}}{(1, \Phi_j)_{L^2(\Omega)}} \Phi_j$$

(Π invertible on V_H!)

Decomposition

$$V_h = V_H \oplus V_h^{\text{f}} \quad \text{with} \quad V_h^{\text{f}} := \text{kernel}\,\Pi = \{v \in V_h \mid \Pi v = 0\}$$

- For each $v \in V_h$ define the fine scale projection $P^{\text{f}} v \in V_h^{\text{f}}$ by

$$a(P^{\text{f}} v, w) = a(v, w) \quad \text{for all } w \in V_h^{\text{f}} \qquad \text{(global!)}$$

a–Orthogonal Decomposition [Malqvist, Peterseim, '11]

$$V_h = V_H^{\text{ms}} \oplus V_h^{\text{f}} \quad \text{and} \quad a(V_H^{\text{ms}}, V_h^{\text{f}}) = 0 \quad \text{with } V_H^{\text{ms}} := (1 - P^{\text{f}})V_H$$

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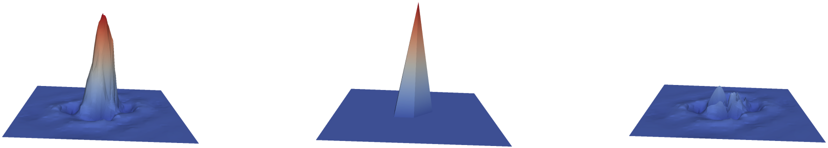
Modified (multiscale) nodal basis

- $\{\Phi_j \mid j = 1, \ldots, N\} \subset V_H$ denotes classical nodal basis
- $\varphi_j^f := P^{\mathrm{f}} \Phi_j \in V_h^{\mathrm{f}}$ denotes the fine scale correction of Φ_j

Ideal multiscale FE space

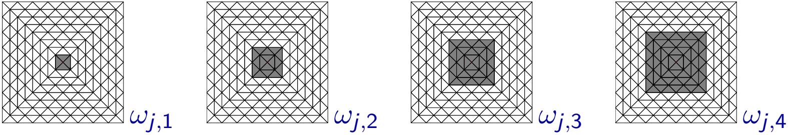
$$V_H^{\mathrm{ms}} = \mathrm{span}\left\{\Phi_j - \varphi_j^f \mid j = 1, \ldots, N\right\}$$

Example



$$\underbrace{\Phi_j - \varphi_j^f}_{\in V_H^{\mathrm{ms}}} = \underbrace{\Phi_j}_{\in V_H} - \underbrace{\varphi_j^f}_{\in V_h^{\mathrm{f}}}$$

- Define nodal patches $\omega_{j,k}$ of k-th order around vertex x_j of $\mathcal{T}_H$



- Can show that $|\varphi_j^f|_{H^1(\Omega\setminus\omega_{j,k})} \lesssim \gamma^k |\varphi_j^f|_{H^1(\Omega)}$ (with $\gamma < 1$).

- Define $\varphi_{j,k}^f \in V_h^{\mathrm{f}}(\omega_{j,k}) := \{v \in V_h^{\mathrm{f}} \mid \operatorname{supp} v \subset \omega_{j,k}\}$ (the localised correction) s.t.

$$a(\varphi_{j,k}^f, w) = a(\Phi_j, w) \quad \text{for all} \quad w \in V_h^{\mathrm{f}}(\omega_{j,k})$$

Localized multiscale FE spaces

$$V_{H,k}^{\mathrm{ms}} := \operatorname{span}\{\Phi_j^H - \varphi_{j,k}^f \mid j = 1, \ldots, N\}$$

Multiscale Coarse Problem & Approximation Result

Multiscale approximation

Seek $u^{\text{ms}}_{H,k} \in V^{\text{ms}}_{H,k}$ such that

$$a(u^{\text{ms}}_{H,k}, v) = (f, v) \quad \text{for all} \quad v \in V^{\text{ms}}_{H,k}$$

- $\dim V^{\text{ms}}_{H,k} = \dim V_H = N$ & basis functions have local support
- Overlap of the supports is proportional to the parameter k

Theorem (Malqvist & Peterseim, 2011)

$$|u - u^{\text{ms}}_{H,k}|_{H^1(\Omega)} \lesssim k^d \gamma^k \|f\|_{H^{-1}(\Omega)} + H\|f\|_{L_2(\Omega)} + |u - u_h|_{H^1(\Omega)}$$

Thus, provided $k \gtrsim \log_\gamma(\frac{1}{H})$ and h is suff'ly small we have **optimal** $\mathcal{O}(H)$ convergence without any assumptions on scales or regularity.

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But unfortunately $\gamma \to 1$ as the contrast $\dfrac{\alpha_{\max}}{\alpha_{\min}} \to \infty$ and the hidden constant depends also on $\dfrac{\alpha_{\max}}{\alpha_{\min}}$

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A contrast-robust theory (work again in weighted norms)

Theorem (Peterseim & RS, 2016)

If $\exists$ linear, cont. quasi-interpolation operator $\Pi : V_h \to V_H$ s.t.

(QI1) $(\Pi|_{V_H})^{-1}\Pi v_H = v_H$, for all $v_H \in V_H$

(QI2) $H_T^{-2}\|v - \Pi v\|^2_{0,\alpha,T} + \|v - \Pi v\|^2_{a,T} \le C_2\|v\|^2_{a,\omega_T}$,
for all $v \in V_h$ and $T \in \mathcal{T}_H$

(QI3) for all $v_H \in V_H$ there exists a $v \in V_h$, s.t. $\Pi v = v_H$,
$\operatorname{supp} v \subset \operatorname{supp} v_H$ and $\|v\|_a \le C_3\|v_H\|_a$.

then (with some universal constant $m \lesssim 1$)

$$\|u - u^{\mathrm{ms}}_{H,k}\|_a \lesssim \left(\frac{\alpha_{\max}}{\alpha_{\min}}\right)^m \frac{e^{-k}}{H}\|f\|_{H^{-1}(\Omega)} + \frac{H}{\alpha_{\min}^{-1/2}}\|f\|_{L_2(\Omega)} + \|u - u_h\|_a$$

Thus, provided $k \gtrsim \ln\left(\frac{\alpha_{\max}}{\alpha_{\min}}\frac{1}{H}\right)$ and h suff'ly small we have **optimal** $\mathcal{O}(H)$ convergence without assumptions on regularity or contrast.

A suitable quasi-interpolation operator

- For simplicity assume α p.w. constant w.r.t. some grid $\mathcal{T}_\eta$, with $h < \eta < H$, but not by $\mathcal{T}_H$ ($\mathcal{T}_H \subset \mathcal{T}_\eta \subset \mathcal{T}_H$ nested)
- Choose $\Pi v := \sum_{j=1}^{N} \frac{(\alpha v, \Phi_j)_{L^2(\Omega)}}{(\alpha, \Phi_j)_{L^2(\Omega)}} \Phi_j$ **(again weighted!)**

Theorem [Peterseim, RS '16]

For all $T \in \mathcal{T}_H$, let $C_T^P > 0$ be the best constant s.t.

$$\inf_{\xi \in \mathbb{R}} \|v - \xi\|_{0,\alpha,\omega_T}^2 \leq C_T^P H_T^2 \|v\|_{a,\omega_T}^2 \quad \forall v \in V_h . \quad \textbf{(WPI)}$$

Then

$$H_T^{-2} \|v - \Pi v\|_{0,\alpha,T}^2 + \|v - \Pi v\|_{a,T}^2 \lesssim C_2 \|v\|_a^2$$

where $C_2 \eqsim \frac{H}{\eta} \max_{T \in \mathcal{T}_H} C_T^P$, i.e. Assumption (QI2).

Moreover, (QI1) and (QI3) hold with $C_3 \eqsim \left(\frac{H}{\eta}\right)^2$.

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$$\inf_{\xi\in\mathbb{R}} \|v - \xi\|^2_{0,\boldsymbol{\alpha},\omega_T} \le C_T^P\, H_T^2 \|v\|^2_{a,\omega_T} \quad \forall v \in V_h\,. \quad \textbf{(WPI)}$$

Then

$$H_T^{-2}\|v - \Pi v\|^2_{0,\alpha,T} + \|v - \Pi v\|^2_{a,T} \;\lesssim\; C_2\,\|v\|^2_a$$

where $C_2 \eqsim \frac{H}{\eta} \max_{T\in\mathcal{T}_H} C_T^P$, i.e. Assumption (QI2).

Moreover, (QI1) and (QI3) hold with $C_3 \eqsim \left(\frac{H}{\eta}\right)^2$.

A suitable quasi-interpolation operator

- For simplicity assume α p.w. constant w.r.t. some grid $\mathcal{T}_\eta$, with $h < \eta < H$, but not by $\mathcal{T}_H$ ($\mathcal{T}_H \subset \mathcal{T}_\eta \subset \mathcal{T}_H$ nested)
- Choose $\Pi v \;:=\; \sum_{j=1}^{N} \frac{(\alpha v, \Phi_j)_{L^2(\Omega)}}{(\alpha, \Phi_j)_{L^2(\Omega)}} \Phi_j$ **(again weighted!)**

Theorem [Peterseim, RS '16]

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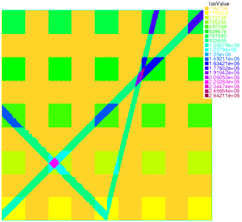
Discussion & Tool 2

In summary, we do get **contrast independent** convergence rates, but so far only under **fairly stringent** assumptions on the type of coefficient variation
(i.e. locally quasi-monotone & p.w. constant w.r.t. $\mathcal{T}_\eta$ for moderate H/η)

Key tool: Weighted Caccioppoli-type Inequality

Let $\omega \subset \Omega$ s.t. $\mathrm{dist}(\partial\omega, \partial\Omega) > \delta > 0$. Then

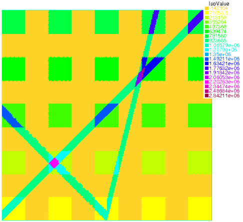
$$\|u\|_{a,\omega} \leq 2\delta^{-1} \|u\|_{0,\alpha,\Omega} \qquad \text{for all } a\text{-harmonic } u \text{ on } \Omega.$$



Adapt grid or **enrich local space** or **change functionals !**

- **LOD**: Refine base grid $\mathcal{T}_H$ locally where C_I^P depends on contrast (similar to multiresolution idea in gamblets)

- **GFEM**: Use eigenproblem with $\Omega_\ell \subset \Omega_\ell^*$ and combine (abstract) **Bramble-Hilbert** (Tool 1) with (weighted) **Caccioppoli** (Tool 2) [Babuska, Lipton '11], [Smetana, Patera '16], [Buhr, Smetana '18]



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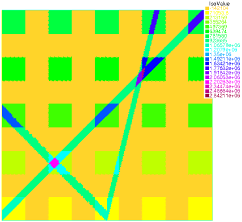

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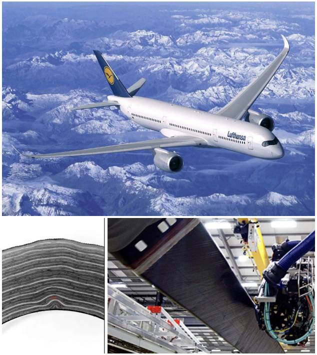
Beyond scalar elliptic problems

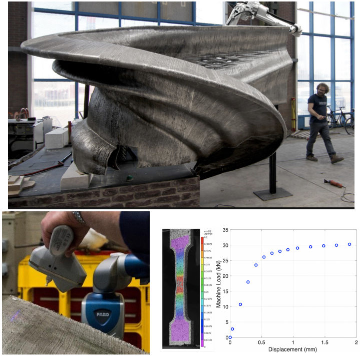
Linear elasticity equations:

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega C(\mathbf{x})\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Gamma (\sigma\cdot\mathbf{n})\cdot\mathbf{v}\;d\mathbf{x} \quad \forall \mathbf{v}\in V$$

small length scales (<mm), **high contrast** and **strongly anisotropic**



CERTEST (EPSRC Project)
Bristol, Bath, Exeter, Heidelberg,...



STEAM (Turing/Royce Project)
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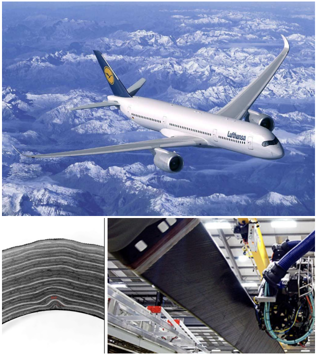

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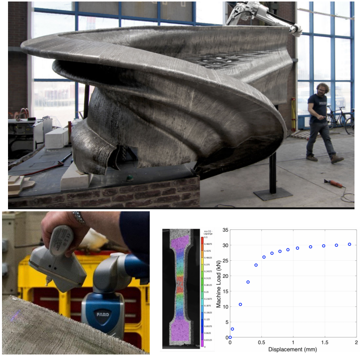

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Key lemma in subspace correction theory to bound $\kappa(M_{\text{add}}^{-1}A)$:

Lions' Lemma – Stable splitting

$$\exists C_0 > 0 : \ \forall v \in V_h : \ \exists v_\ell \in V_\ell : \quad v = \sum_{\ell=0}^{L} v_\ell \ \text{ and } \ \sum_{\ell=0}^{L} \|v_\ell\|_a^2 \leq C_0^2 \|v\|_a^2$$

Key observation in [Spillane, Dolean, Hauret, Nataf, Pechstein, RS '14]:

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Suppose that $\exists C_1 > 0 : \ \forall \ell = 1, \ldots, L : \quad \|v_\ell\|_{a,\Omega_\ell}^2 \leq C_1^2 \|v\|_{a,\Omega_\ell}^2$. Then the splitting above is stable with $C_0^2 = 2 + k_0 C_1^2 + 2k_0^2 C_1^2$

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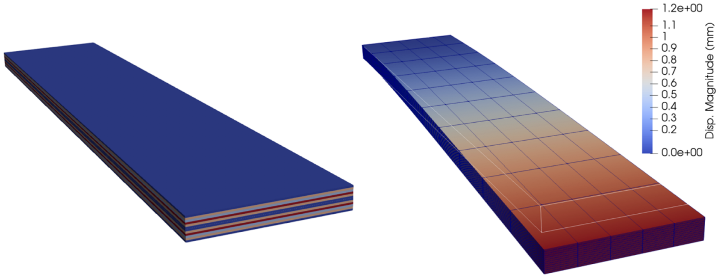
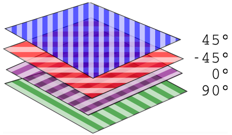
Toy Composite Example for Demonstration - Cantilever

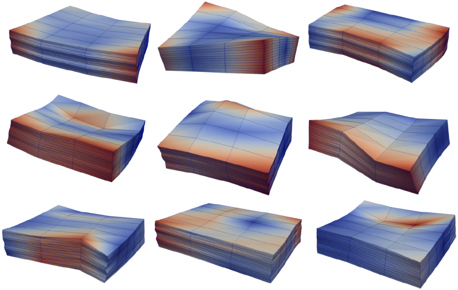


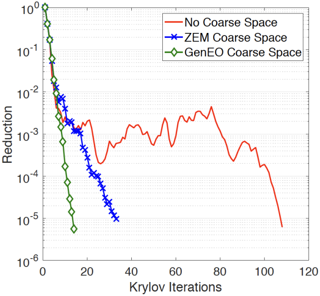
- Flat composite plate $[0, 100\text{mm}] \times [0, 20\text{mm}]$
- Cantilever under uniform pressure (top surface)
- 12 Layers - 11 weak interfaces

$$[\mp 45^\circ/0^\circ/90^\circ/\pm 45^\circ/\mp 45^\circ/90^\circ/0^\circ/\pm 45^\circ].$$



- 20-node serendipity elements



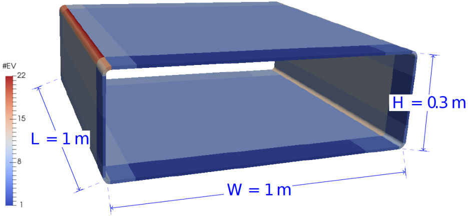


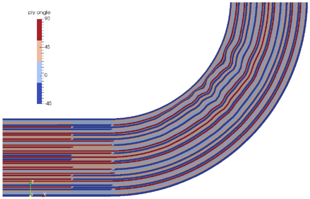
N	AS		ZEM			GenEO			BoomerAMG	
	it	cond κ	it	cond κ	$\dim(V_H)$	it	cond κ	$\dim(V_H)$	it	Num. levels
4	89	79,735	26	394	12	16	10	78	258	10
8	97	84,023	30	245	42	15	9	126	258	11
$16^\star$	107	98,579	36	177	84	16	10	182	257	12
32	158	226,871	42	230	168	16	9	526	263	12

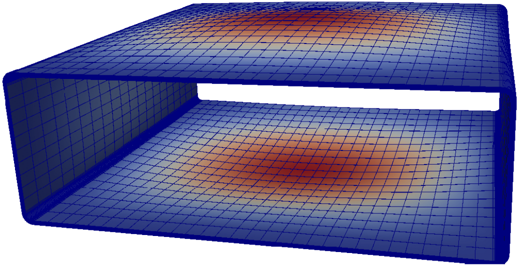
[Butler, Dodwell, Reinarz, Sandhu, RS, Seelinger '19]

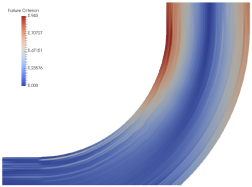
Wingbox section with defect under internal fuel pressure

(ply-scale stress resolution!!)



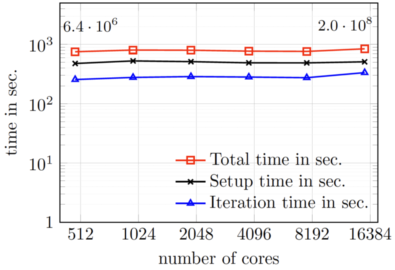


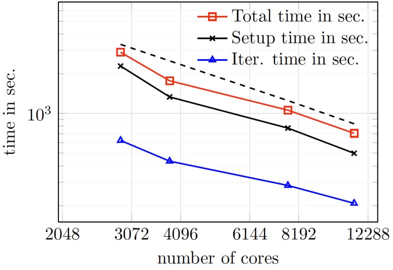




Parallel Efficiency of HPC Implementation up to 15,360 cores

- HPC implementation of GenEO within Dune
- Parallel performance on UK National HPC Cluster archer





[Butler, Dodwell, Reinarz, Sandhu, RS, Seelinger '19]

This scale of computations brings composites problems that would otherwise be unthinkable into the feasible range.

Current Work & Final Remarks

- Extend to **nonlinear** elasticity & composite failure
- More complicated geometries & **bigger overlap**
- GenEO as a **GFEM**: first results in [Dodwell, Sandhu, RS '17]
  (different functional as [Babuska, Lipton], [Buhr, Smetana]: **'Knob' 1**)
- Bayesian inference: **surrogate** in **multilevel MCMC**
- **'Knob' 2**: Choice of **partition of unity** (seems to have big effect)
- **'Knob' 3**: ARPACK eigensolver vs. randomised eigensolver

Some initial experiments below!

- **Theoretical Aim:** Prove **contrast-independent** approximation results for (versions of) **LOD** and **GFEM** !

THANK YOU!

[If anybody is up for rock climbing on Wed afternoon, please let me know!]

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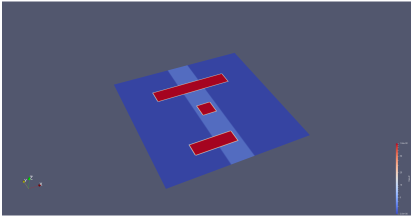
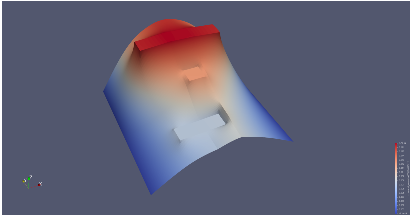
References

1. **RS**, PS Vassilevski & LT Zikatanov, Weak Approximation Properties of Elliptic Projections with Functional Constraints, *Multiscale Model Sim (SIAM)* **9**, 2011.
2. N Spillane, V Dolean, P Hauret, F Nataf, C Pechstein & **RS**, Abstract Robust Coarse Spaces for Systems of PDEs via Generalized Eigenproblems in the Overlaps, *Numer Math* **126**, 2014.
3. D Peterseim & **RS**, Robust Numerical Upscaling of Elliptic Multiscale Problems at High Contrast, *Comput Meth Appl Math* **16**, 2016.
4. TJ Dodwell, A Sandhu & **RS**, Customized Coarse Models for Highly Heterogeneous Materials, in "Bifurcation and Degradation of Geomaterials with Engineering Applications" (Papamichos et al Eds.), Springer Series in Geomechanics and Geoengineering, 2017.
5. R Butler, TJ Dodwell, A Reinarz, A Sandhu, **RS** & L Seelinger, High-performance `dune` modules for solving large-scale, strongly anisotropic elliptic problems with applications to aerospace composites arXiv:1901.05188, 2019.

Coefficient function



Harmonic POU - 1st Mode

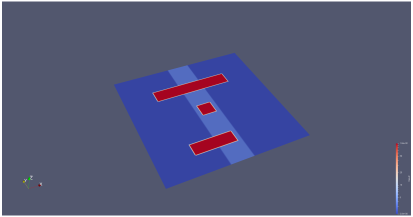


First 6 eigenmodes in each domain: [Arne Strehlow]

P.O.U.		λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
p.w. const.	Ω_1	0.00272	0.00536	0.19743	0.22077	0.28599	0.34094
	Ω_2	0.00315	0.00749	0.20680	0.22085	0.30189	0.34094
Sarkis	Ω_1	0.01963	0.05366	0.09788	1.05319	1.05517	1.05974
	Ω_2	0.01599	0.04153	0.09416	0.99473	1.05318	1.05327
Harmonic	Ω_1	0.03357	0.21091	0.78878	1.05086	1.05326	1.05974
	Ω_2	0.03444	0.28577	0.83536	1.00547	1.00852	1.00915

Eigenfunctions for Different Partitions of Unity (scalar elliptic)

Coefficient function



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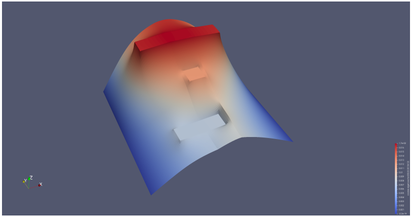

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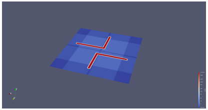
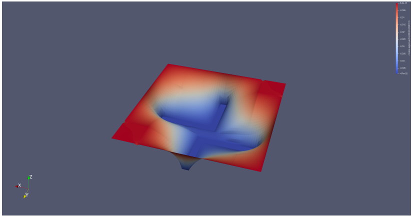
'Disconnected' cross
(4 subdomains)

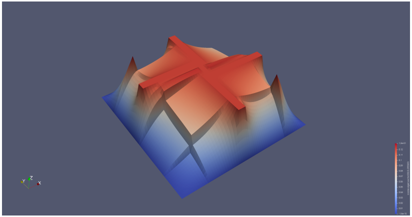


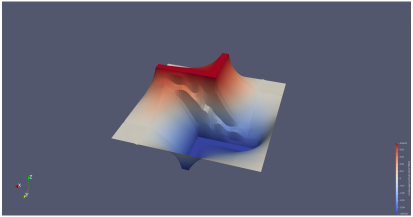
[Arne Strehlow]

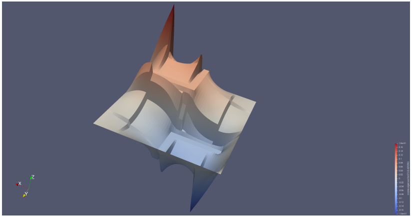
Harmonic POU

Piecewise constant POU







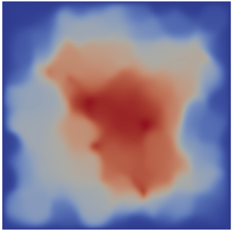


GenEO as GFEM for scalar elliptic problem

[Tim Dodwell]

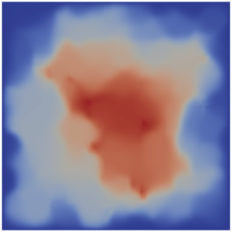
First 5 eigenfunctions on Ω_6 (16 subdomains; $a(\mathbf{x})$ is log-normal sample):

$\lambda_6^{(1)} = 0.0$ $\quad \lambda_6^{(2)} = 0.0193$ $\quad \lambda_6^{(3)} = 0.0511$ $\quad \lambda_6^{(4)} = 0.0937$ $\quad \lambda_6^{(5)} = 0.2056$



Fine model

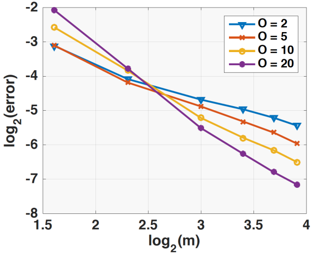
dim $V_h = 4 \times 10^4$



Coarse Model

dim $V_H = 320$

$(m = 20, O = 5)$



$$\text{err} = \frac{\|\mathbf{u}_h - \mathbf{R}_H^T \mathbf{U}_H\|_2}{\|\mathbf{u}_h\|_2}$$

Coarse Approximation Error (channels & islands)

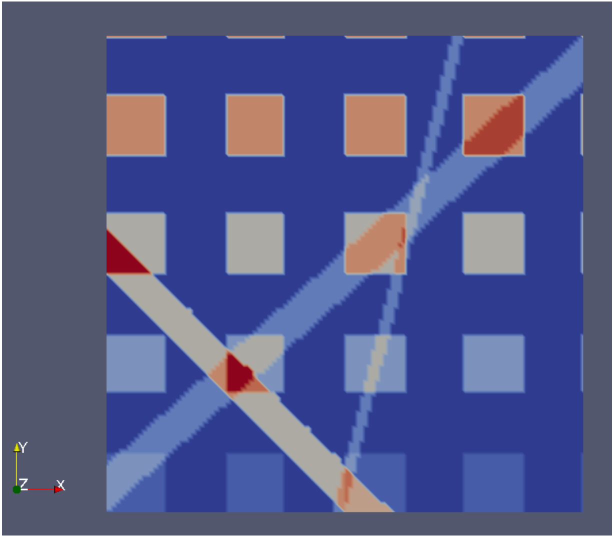


Figure: Parameter Distribution

Coarse Approximation Error (channels & islands)

[Linus Seelinger]

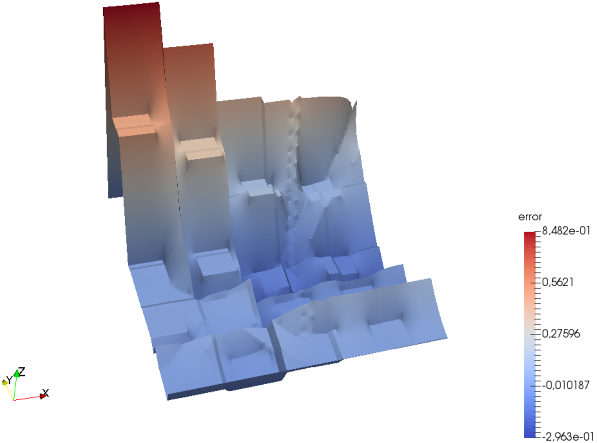


Figure: Coarse Error – 1 EV/subdomain

Coarse Approximation Error (channels & islands)

[Linus Seelinger]

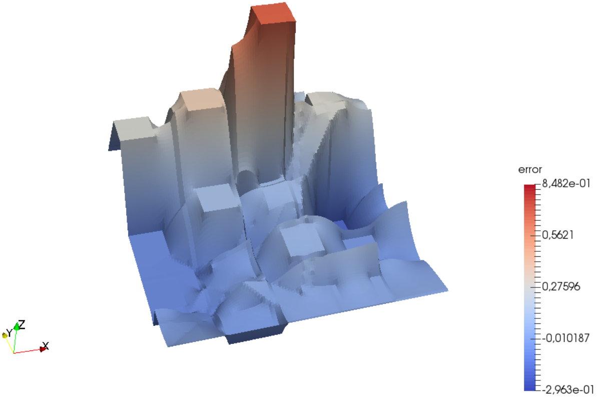


Figure: Coarse Error – 2 EV/subdomain

Coarse Approximation Error (channels & islands)

[Linus Seelinger]

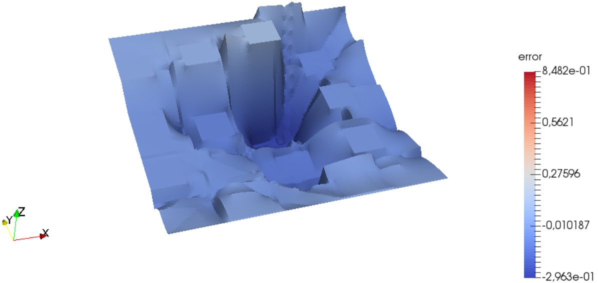


Figure: Coarse Error – 3 EV/subdomain

Coarse Approximation Error (channels & islands)

[Linus Seelinger]

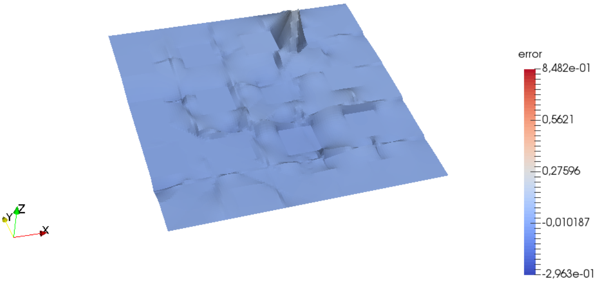


Figure: Coarse Error – 4 EV/subdomain

Coarse Approximation Error (channels & islands)

[Linus Seelinger]

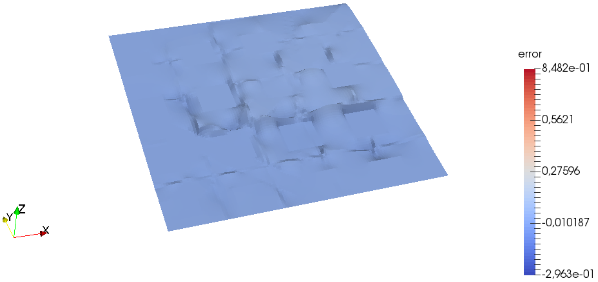


Figure: Coarse Error – 5 EV/subdomain