Robust coarse spaces for the boundary element method

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ANR project NonlocalDD

Introduction

Boundary Integral Equation

We want to solve a PDE in Ω using

Boundary Integral Equations (BIE)

- Reformulation on $\partial\Omega$ using its fundamental solution
- Non-local integral operators (pseudo-differential operators)
- Dense matrices using Galerkin approximation

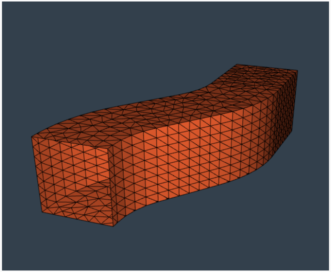

Figure 1: Mesh of a cavity

Some practical difficulties

- Compression ($\mathcal{H}$-matrices, FMM,SCSD,...)
- Parallelism and vectorization (MPI, OpenMP,...)

$\Longrightarrow$ Htool library by P.-H. Tournier and P.M. (available on GitHub)

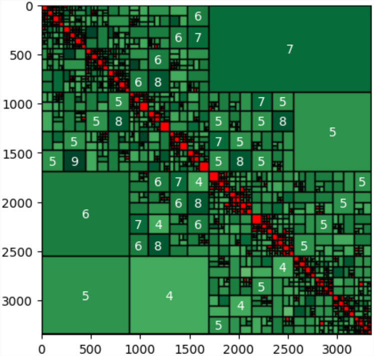

Figure 2: $\mathcal{H}$-matrice for COBRA cavity

- free and open-source
- $\sim$ 460 commits
- $\sim$ 6800 lines of C++

Different points of view for DDM

Volume domain decomposition THEN boundary integral

- PMCHWT formulation
- Boundary Element Tearing and Interconnecting (BETI) method
  $\rightarrow$ boundary element counterpart of the FETI methods
- Multitrace formulation
  $\rightarrow$ the local variant is equivalent to Optimal Schwarz Method for particular parameters[1]

[1] Claeys, Dolean, and M. Gander 2019; Claeys and Marchand 2018.

Different points of view for DDM

Boundary integral formulation THEN surface domain decomposition: Additive Schwarz Method (ASM).

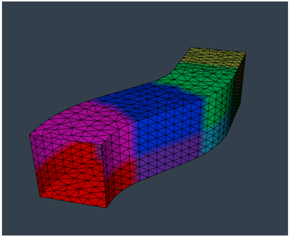

Figure 3: Surface decomposition for COBRA cavity

- Two-level Schwarz preconditioners with coarse mesh[2]
- In our turn, we develop GenEO-type preconditioners

[2]Hahne and Stephan 1996; Heuer 1996; Stephan 1996; Tran and Stephan 1996.

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Boundary Integral Equations

Function spaces

Geometry

- $\Omega \subset \mathbb{R}^d$ for $d = 2$ or $d = 3$, Lipschitz domain
- $\Gamma \subseteq \partial\Omega$

Sobolev spaces

- $H^{1/2}(\Gamma) := \{u|_\Gamma \mid u \in H^{1/2}(\partial\Omega)\}$
- $\widetilde{H}^{1/2}(\Gamma) := \{u \in H^{1/2}(\partial\Omega) \mid \mathrm{supp}(u) \subset \overline{\Gamma}\}$
- By duality: $\widetilde{H}^{-1/2}(\Gamma) := H^{1/2}(\Gamma)^*$ and $H^{-1/2}(\Gamma) := \widetilde{H}^{1/2}(\Gamma)^*$

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Associated norms

- $\|\varphi\|^2_{H^{1/2}(\partial\Omega)} := \|\varphi\|^2_{L^2(\partial\Omega)} + \int_{\partial\Omega\times\partial\Omega} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{d+1}} \, \mathrm{d}\sigma(x, y)$
- $\|\varphi\|^2_{\widetilde{H}^{1/2}(\Gamma)} := \|E_\Gamma(\varphi)\|^2_{H^{1/2}(\partial\Omega)}$

where E_Γ is the extension by zero.

Model problem

$$\begin{cases} L(u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^d \\ + \text{ condition at infinity if } \Omega \text{ is an unbounded domain} \end{cases}$$

L is a general linear, elliptic differential operator with constant coefficient and G the associated fundamental solution

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Fundamental solution

$$L(G) = \delta_0 \text{ in } \mathbb{R}^d$$

Example of a fundamental solution

Laplacian in $\mathbb{R}^3$:

$$G(\mathbf{x}) := \frac{1}{4\pi\|\mathbf{x}\|} \quad \text{for } \mathbf{x} \in \mathbb{R}^3 \setminus \{0\}.$$

Surface potentials

Single and double layer potential

$$\mathsf{SL}(q)(\mathsf{x}) := \int_\Gamma G(\mathsf{x}-\mathsf{y})q(\mathsf{y})\,\mathrm{d}\sigma(\mathsf{y}),$$
$$\mathsf{DL}(v)(\mathsf{x}) := \int_\Gamma \mathsf{n}(\mathsf{y}) \cdot (\nabla G)(\mathsf{x}-\mathsf{y})v(\mathsf{y})d\sigma(\mathsf{y}),$$

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Properties

- $L \circ \mathsf{SL}(q) = 0$ and $L \circ \mathsf{DL}(v) = 0$ in $\mathbb{R}^d \setminus \Gamma$
- $\mathsf{SL}(q)$ and $\mathsf{DL}(v)$ satisfy appropriate conditions at infinity

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Dirichlet (resp. Neumann) problem

- Dirichlet data $g_D \in H^{1/2}(\Gamma) \implies V(q) = g_D$ with $V = \gamma_D \circ \mathsf{SL}$
- Neumann data $g_N \in H^{-1/2}(\Gamma) \implies W(v) = g_N$ with $W = \gamma_N \circ \mathsf{DL}$

Considered problem

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- Discretization using the *Boundary Element Method* (BEM): find $u_h \in \mathcal{V}_h \subset \widetilde{H}^s(\Gamma)$ such that

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  where $\mathcal{V}_h = \mathrm{Span}(\varphi_i, i = 1 \ldots N)$.

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Hypothesis: a is symmetric positive definite

Considered problem

Remarks

- Laplace equation on screens, Laplace equation with Dirichlet conditions on closed surface, Modified Helmholtz...
- Example of analytical expression for Laplacian in 3D:

$$\langle V(q), \varphi \rangle = \int_{\Gamma} \int_{\Gamma} \frac{1}{4\pi \|\mathsf{x} - \mathsf{y}\|} q(\mathsf{y}) \varphi(\mathsf{x}) \, \mathrm{d}s_{\mathsf{y}} \, \mathrm{d}s_{\mathsf{x}}$$

- Condition number for the linear system associated with the preceding bilinear form and obtained with finite element:

$$\kappa(\mathsf{V}) \leq C h^{-1}.$$

Context

Algebraic system

$$\mathsf{A}_h \mathbf{u}_h = \mathbf{f}, \quad \text{with } \mathbf{u}_h \in \mathbb{R}^d$$

A_h a dense matrix usually compressed (Fast Multipole Method, hierarchical matrices, Sparse Cardinal Sine Decomposition,...)

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Solvers

- Direct methods:
  - ⊕ Factorisation can be stored for multi-rhs
  - ⊖ Expensive for dense matrices (complexity in $O(N^3)$)
  - ⊕ Possibility to use $\mathcal{H} - LU$ decomposition
- Iterative methods:
  - ⊕ Less intrusive
  - ⊕ Only matrix-vector products ($O(N^2)$ or quasi linear complexity with compression)
  - ⊖ But ill-conditioned, especially when the mesh is refined

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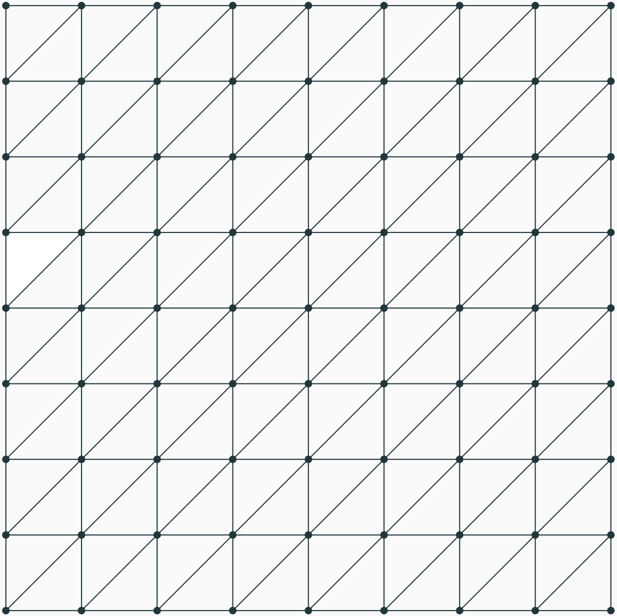
$\Longrightarrow$ preconditioning techniques: $\mathsf{P}\mathsf{A}_h \mathbf{u}_h = \mathsf{P}\mathbf{f}$

Domain Decomposition Methods

Example of decomposition



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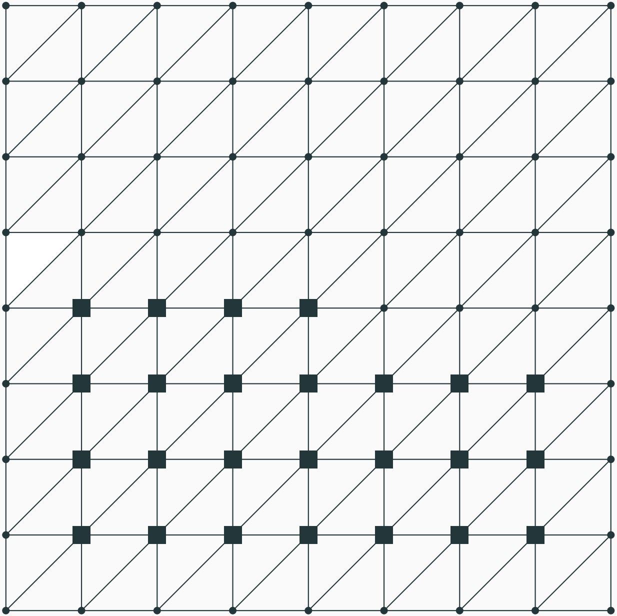

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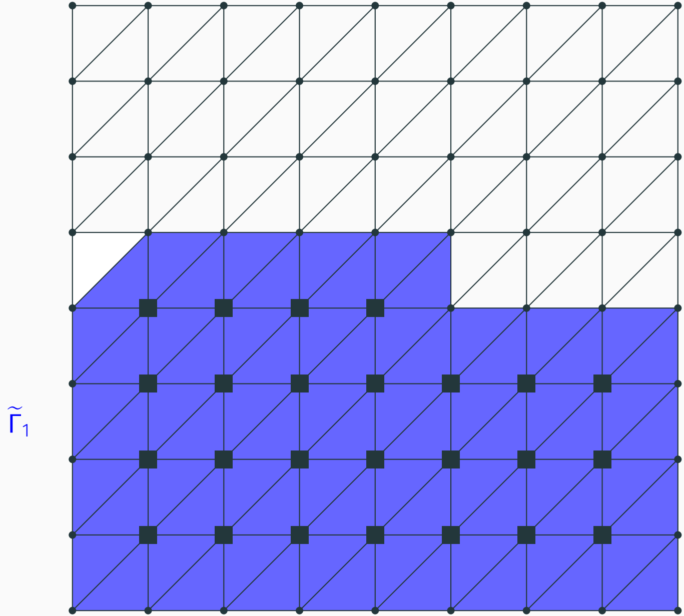
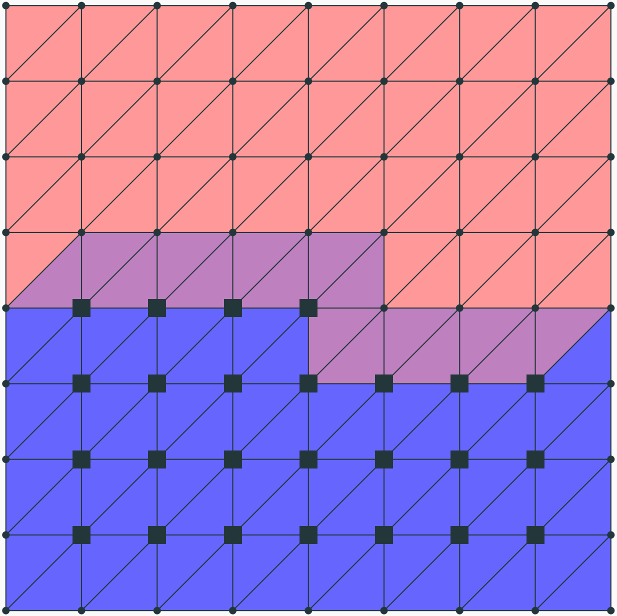


Example of decomposition

$\widetilde{\Gamma}_1$



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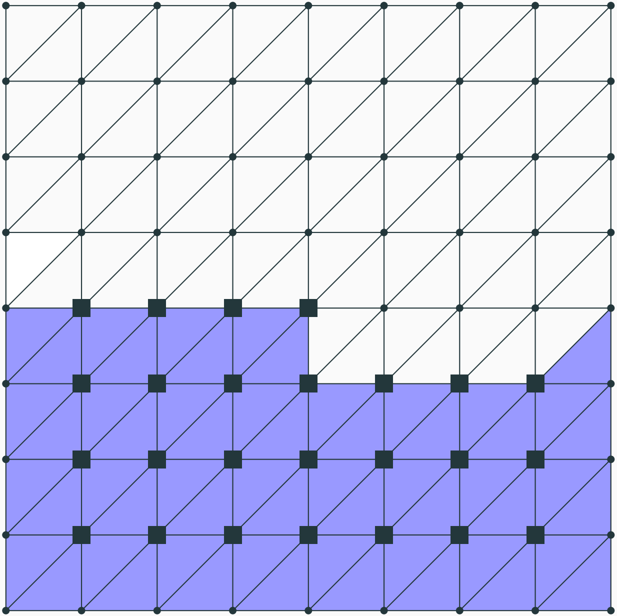

$\Gamma_1 \subset \widetilde{\Gamma}_1$

Notations

Subdomains

- $\mathrm{dof}_{h,p} \subset \{1, \ldots, N\}$
- $\widetilde{\Gamma}_p := \cup_{j \in \mathrm{dof}_{h,p}} \mathrm{supp}(\varphi_j)$
- $\Gamma_p := \widetilde{\Gamma}_p \setminus \cup_{j \notin \mathrm{dof}_{h,p}} \mathrm{supp}(\varphi_j) \subset \widetilde{\Gamma}_p$

Decomposition

- Number of unknowns in the subdomain p: N_p,
- Extension by zero: $\mathsf{R}_p^T \in \mathbb{R}^{N \times N_p}$,
- Restriction matrices: R_p
- Partition of unity: diagonal matrices $\mathsf{D}_p \in \mathbb{R}^{N_p \times N_p}$ s.t.

$$\sum_{p=1}^{n} \mathsf{R}_p^T \mathsf{D}_p \mathsf{R}_p = \mathsf{I}_d.$$

Preconditioners for BEM

Additive Schwarz preconditioner

Additive Schwarz Preconditioner[3]

$$\mathsf{P}_{ASM} = \mathsf{R}_0^T(\mathsf{R}_0\mathsf{A}_h\mathsf{R}_0^T)^{-1}\mathsf{R}_0 + \sum_{p=1}^{n} \mathsf{R}_p^T(\mathsf{R}_p\mathsf{A}_h\mathsf{R}_p^T)^{-1}\mathsf{R}_p$$

- $\mathsf{Z} = \mathsf{R}_0^T \in \mathbb{R}^{N\times N_0}$, an interpolation operator from the *coarse space* to the finite element space
- The coarse space $\mathcal{V}_{h,0}$ is spanned by the columns of Z

[3] Widlund and Dryja 1987.

Additive Schwarz Preconditioner and Fictitious Space Lemma

Hypotheses of the Fictitious Space lemma[4]

(H1) $\|\sum_{p=0}^{n} \mathsf{R}_p^T \mathbf{u}_h^p\|_{\mathsf{A}_h}^2 \leq c_R \sum_{p=0}^{n} \|\mathsf{R}_p^T \mathbf{u}_h^p\|_{\mathsf{A}_h}^2 \quad \forall (\mathbf{u}_h^p)_{p=0}^n \in \prod_{p=0}^n \mathbb{C}^{N_p}$,

(H2) For $\mathbf{u}_h \in V_h$, how can we define $(\mathbf{u}_h^p)_{p=0}^n \in \prod_{p=0}^n \mathbb{C}^{N_p}$ s.t. $\mathbf{u}_h = \sum_{p=0}^n \mathsf{R}_p^T \mathbf{u}_h^p$ and

$$c_T \sum_{p=0}^{n} \|\mathsf{R}_p^T \mathbf{u}_h^p\|_{\mathsf{A}_h}^2 \leq \|\mathbf{u}_h\|_{\mathsf{A}_h}^2,$$

Result

$$\mathrm{cond}_2(\mathsf{P}_{ASM}\mathsf{A}_h) \leq \frac{c_R}{c_T}.$$

[4] Nepomnyaschikh 1992.

Lemmas

Lemma (Sauter and Schwab 2011, Lemma 4.1.49 (b))

For $(u_p)_{1 \le p \le n} \in \prod_{p=1}^n \widetilde{H}^{1/2}(\widetilde{\Gamma}_p)$, *we have the following inequality:*

$$\left\| \sum_{p=1}^{n} E_{\widetilde{\Gamma}_p}(u_p) \right\|^2_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \sum_{p=1}^{n} \|u_p\|^2_{\widetilde{H}^{1/2}(\widetilde{\Gamma}_p)} .$$

Proof for (H1)

$$\left\| \sum_{p=0}^{n} \mathsf{R}_p^T \mathbf{u}_h^p \right\|^2_{\mathsf{A}_h} \lesssim \|\mathsf{R}_0^T \mathbf{u}_h^0\|^2_{\mathsf{A}_h} + \underbrace{\left\| \sum_{p=1}^{n} \mathsf{R}_p^T \mathbf{u}_h^p \right\|^2_{\mathsf{A}_h}}_{\lesssim \sum_{p=1}^{n} \|\mathsf{R}_p^T \mathbf{u}_h^p\|^2_{\mathsf{A}_h}}$$

Spectral Coarse Space: GenEO

- Assume there exists $(\mathsf{B}_p)_{p=1}^{n} \in (\mathbb{C}^{N_p \times N_p})^n$, s.t.

$$\sum_{p=1}^{n} (\mathsf{B}_p \mathsf{R}_p \mathsf{u}_h, \mathsf{R}_p \mathsf{u}_h) \leq \|\mathsf{u}_h\|_{\mathsf{A}_h}^2$$

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- Idea of the GenEO coarse space[5]: a sufficient condition is

$$(\mathsf{A}_h \mathsf{R}_p^T \mathsf{u}_h^p, \mathsf{R}_p^T \mathsf{u}_h^p) \leq \tau (\mathsf{B}_p \mathsf{R}_p \mathsf{u}_h, \mathsf{R}_p \mathsf{u}_h).$$

[5] Spillane et al. 2011, 2014.

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We introduce the following eigenvalue problem: find $(\lambda_k^p, \mathsf{v}_k^p)$ s.t.

$$\mathsf{D}_p \mathsf{R}_p \mathsf{A}_h \mathsf{R}_p^T \mathsf{D}_p \mathsf{v}_k^p = \lambda_k^p \mathsf{B}_p \mathsf{v}_k^p,$$

[5] Spillane et al. 2011, 2014.

Spectral Coarse Space: GenEO

We define $\mathsf{Z}_{p,\tau} = \ker(\mathsf{B}_p) \cup \mathrm{Span}(\mathbf{v}_k^p \,|\, \lambda_k^p > \tau)$, Π_p, the projector on $Z_{p,\tau}$ and,

$$\mathcal{V}_{h,0} = \mathrm{Span}(\mathsf{R}_p^T \mathsf{D}_p \mathbf{v}_h^p \,|\, 1 \leq p \leq N,\ \mathbf{v}_h^p \in \mathsf{Z}_{p,\tau})$$

$\mathsf{R}_0^T = \mathsf{Z}_\tau \in \mathbb{R}^{N \times N_0}$ be a column matrix so that $\mathcal{V}_{h,0}$ is spanned by its columns and $N_0 = \dim(\mathcal{V}_{h,0})$.

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$$\mathbf{u}_h^0 = (\mathsf{R}_0 \mathsf{R}_0^T)^{-1} \mathsf{R}_0 \left(\sum_{p=1}^{n} \mathsf{R}_p^T \mathsf{D}_p \Pi_p \mathsf{R}_p u \right) \text{ and } \mathbf{u}_h^p = \mathsf{D}_p (\mathsf{I}_d - \Pi_p) \mathsf{R}_p \mathbf{u}_h, \quad \forall 1 \leq p \leq n$$

Spectral Coarse Space: GenEO

$$\implies \sum_{p=0}^{n} \mathsf{R}_p^T \mathsf{u}_h^p = \mathsf{u}_h \quad \text{and} \quad \sum_{p=0}^{n} \|\mathsf{R}_p^T \mathsf{u}_h^p\|_{\mathsf{A}_h}^2 \lesssim \tau \|\mathsf{u}_h\|_{\mathsf{A}_h}^2$$

Spectral Coarse Space: GenEO

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Theorem

With the previous coarse space, there exists $C_\Gamma > 0$ independent of the meshsize and the number of subdomains such that

$$\mathrm{cond}_2(\mathsf{P}_{ASM}\mathsf{A}_h) < C_\Gamma \tau.$$

Concrete coarse spaces: three possibilities

For the hypersingular operator W ($s = 1/2$):

(i) Continuous injection

$$\sum_{p=1}^{n} \|u_h|_{\Gamma_p}\|^2_{L^2(\Gamma_p)} \lesssim \|u_h\|^2_{L^2(\Gamma)} \lesssim \|u_h\|^2_{\widetilde{H}^{1/2}(\Gamma)} \simeq \|\mathbf{u}_h\|^2_{\mathsf{A}_h},$$

(ii) Inverse inequality[6]:

$$\sum_{p=1}^{n} \|h_{\mathcal{T}} \nabla u|_{\Gamma_p}\|^2_{L^2(\Gamma_p)} \lesssim \|h_{\mathcal{T}} \nabla u_h\|^2_{L^2(\Gamma)} \lesssim \|u_h\|^2_{\widetilde{H}^{1/2}(\Gamma)} \simeq \|\mathbf{u}_h\|^2_{\mathsf{A}_h},$$

where $\mathcal{T}$ is the mesh and $h_{\mathcal{T}}|_T = |T|^{1/(d-1)}$ for every mesh element T.

(iii) $H^{1/2} - \textit{localization}$

$$\sum_{p=1}^{n} \underbrace{\langle V_p^{-1} u_h, u_h \rangle_{\widetilde{H}^{-1/2}(\Gamma_p) \times H^{1/2}(\Gamma_p)}}_{(\mathsf{M}_p \mathsf{V}_p^{-1} \mathsf{M}_p \mathbf{u}_h, \mathbf{u}_h) \leq} \overset{?}{\lesssim} \sum_{p=1}^{n} \|u_h|_{\Gamma_p}\|^2_{H^{1/2}(\Gamma)} \lesssim \|u_h\|^2_{\widetilde{H}^{1/2}(\Gamma)} \simeq \|\mathbf{u}_h\|^2_{\mathsf{A}_h},$$

[6] Aurada et al. 2017.

Numerical simulations

2D test case

- Problem: $-\Delta u + \kappa^2 u = 0$ with $\gamma_N u = 100 * (x - 1.5)^2$
- A_h is compressed using a $\mathcal{H}$-matrix
- Parameters: 24000 P1 elements, $\tau = 60$
- Supercomputer: Occigen (64th in `TOP500`)
- Solution for $\kappa = 1$

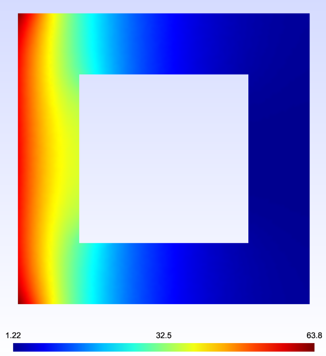


Figure 4: Solution of $-\Delta u + \kappa^2 u = 0$ with $\gamma_N u = 100 * (x - 1.5)^2$

Spectra

GenEO single layer
GenEO mass
GenEO stiffness
GenEO Slobodeckij
$|\lambda_k^p|$
Number of the subdomain p

Strong scaling 2D

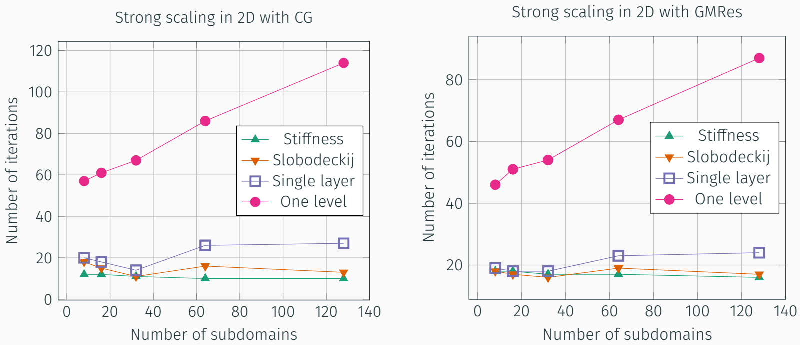
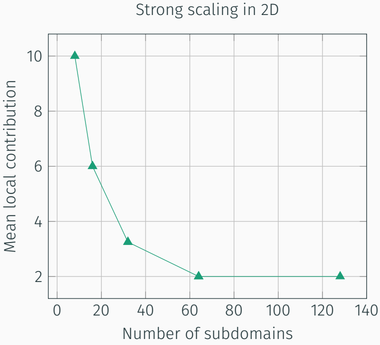


Figure 5: Number of iterations in function of the number of subdomain with CG (right) and GMRes (left) for $\kappa = 0.1$

Remark: 656 iterations without preconditioner for CG and 450 for GMRes.

Strong scaling 2D



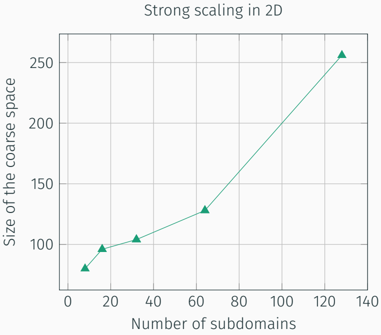


Figure 6: Mean local contribution (left) and size of the coarse space (right) in function of the number of subdomains for $\kappa = 0.1$

Implementation

C++ libraries available on GitHub:

- `https://github.com/xclaeys/BemTool` by X. Claeys,
  - BEM library for Laplace, Helmholtz, Yukawa and Maxwell matrices in 2D and 3D.
- `https://github.com/PierreMarchand20/htool` by P.-H. Tournier and P. M.,
  - Hierarchical matrices ($\mathcal{H}-$matrices),
  - Parallelized assembly, $\mathcal{H}-$matrice/vector and $\mathcal{H}-$matrice/matrice products using MPI and OpenMP,
  - DDM preconditioners with HPDDMs.
- `https://github.com/hpddm/hpddm` by P. Jolivet and F. Nataf,
  - Various iterative solvers and block solvers,
  - DDM preconditioners,
  - Support for dense/compressed matrices recently added.

Summary

- Theoretical results about a new coarse space for DDM preconditioner applied to BEM matrices for the hypersingular operator
- Implementation available using Htool and HPDDM (which can be interfaced with your own code)

$\Longrightarrow$ article to be submitted and Htool integrated in Freefem++

Outlook

- Numerical approximation of the fractional Laplacian[7]
- Other coarse spaces (SHEM)[8]: work with G. Ciaramella from University of Konstanz
- Bonus: try block solvers for problems with multiple right-hand side

[7] Ainsworth and Glusa 2018; Bonito et al. 2018.
[8] M. J. Gander, Loneland, and Rahman 2015.

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Thank you for your attention!

[7] Ainsworth and Glusa 2018; Bonito et al. 2018.
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Weak scaling 2D

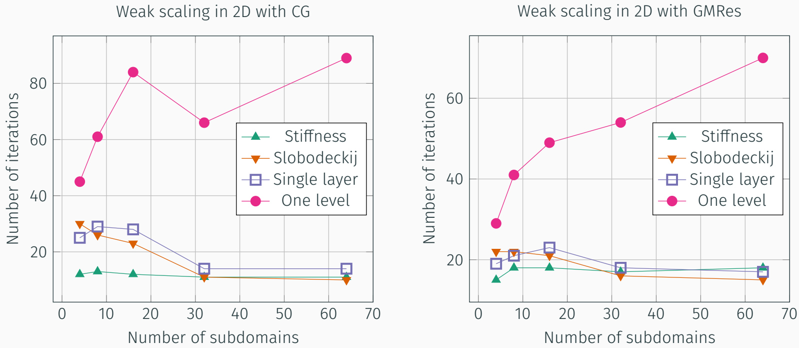

Figure 7: Number of iterations in function of the number of subdomain with CG (right) and GMRes (left) for $\kappa = 0.1$

Remark: We have 750 degrees of freedom for each subdomain.

Weak scaling 2D

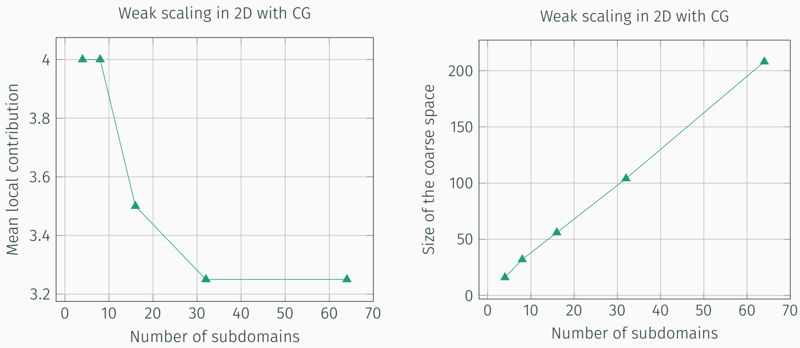

Figure 8: Mean local contribution (left) and size of the coarse space (right) in function of the number of subdomains for $\kappa = 0.1$

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